The Existence of Solution for the Nonstationary Two Dimensional Microflow Boundary Layer System

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Abstract: The paper concerns with the nonstationary two dimensional microflow boundary layer system. By posing some restrictions on the viscous function, the existence and the uniqueness of local solutions to the system are got. The main technique we used in the paper is Oleinik line method based on a successive approximation, which is used in the study of Prandtl system. However, the corresponding calculations in our paper are much more complicated.

Key–Words: Two dimensional microflow boundary layer system, Prandtl system, Existence, Local solution.

1 Introduction

As we known, the Prandtl system is a simplification of the Navier-Stokes system and describes the motion of a fluid with small viscosity about a solid body in a thin layer which is formed near its surface owing to the adhesion of the viscous fluid to the solid surface. Assume that the motion of a fluid occupying a two dimensional region is characterized by the velocity vector $V = (u, v)$, where u, v are the projections of V onto the coordinate axes x, y, respectively, the Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form

$$\begin{cases} u_t + uu_x + vu_y = \nu u_{yy} - p_x, \\ u_x + v_y = 0, \\ u(0, x, y) = u_0(x, y), \quad u(t, 0, y) = u_1(t, y), \\ u(t, x, 0) = 0, \quad v(t, x, 0) = v_0(t, x), \\ \lim_{y\to\infty} u(t, x, y) = U(t, x). \end{cases}$$

in a domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$, where $\nu = const > 0$ is the viscosity coefficient of the incompressible fluid. $U_t + UU_x = -p_x(t, x)$, $U(t, x) > 0$, $u_0 > 0, u_1 > 0$ for $y > 0$, $u_{0y} > 0, u_{1y} > 0$ for $y \geq 0$. $U = U(t, x)$ is the velocity at the outer edge of the boundary layer, $p = p(t, x)$ is the pressure. The density of the fluid ρ is equal to 1. Prandtl boundary theory does not consider both the influence of wall's properties on the characteristic of the boundary layer and the interaction of the actual solid wall with the flow of water. If one considers these influences, the Prandtl system should be modified to the following system

$$\begin{cases} u_t + uu_x + vu_y = (\nu(y)u_y)_y - p_x, \\ u_x + v_y = 0, \end{cases} \tag{1}$$

with the conditions

$$\begin{cases} u(0, x, y) = u_0(x, y), \quad u(t, 0, y) = u_1(t, y), \\ u(t, x, 0) = g(t, x), \quad v(t, x, 0) = v_0(t, x), \end{cases} \tag{2}$$

$$\lim_{y\to\infty} u(t, x, y) = U(t, x). \tag{3}$$

where $t, x, y \in D$, $\nu(y)$ is a boundary function, ν and $g(t, x)$ satisfies some other restrictions.

In recent decades, many scholars have been carrying out research in two dimensional boundary layer, achievements are abundant in literature on theoretical, numerical experimental aspects of the theory[1, 2]. In particular, Oleinik had got the existence and the uniqueness of solutions for the Prandtl system by two different kinds of line methods, one of them is based on Rothe's method[3], another one is based on a successive approximation[3]. If ν is a sufficiently large positive function, which means that $\nu(y) > \nu_0 > 0$, ν_0 is a constant, the system (1)-(3) is called the microflow boundary layer[4], and Li-Zhan [5] had got the local well posedness of the system by a similar method as [3], which is base on Rothe's method. Also, there are many papers to deal with the other related problems in the boundary layer theory, such as the relation between the Navier-Stockes system and the Prandtl system and the long-time behavior of the solutions(see [6-15] and references therein).

In this paper, we used Oleinik's successive approximation method to study the problem (1)-(3).

We use the following change of variables [3], which is known as Crocco transform,

$$\tau = t,\ \xi = x,\ \eta = u(t,x,y),\ w(\tau,\xi,\eta) = u_y. \quad (4)$$

By calculation, we can get $u_y = w$, $u_{yy} = w_\eta w$, $u_{yyy} = w_{\eta\eta}w^2 + w_\eta^2 w$, $u_{yt} = w_\eta u_t + w_\tau$, $u_{yx} = w_\eta u_x + w_\xi, \nu_y = \nu_\eta w,\quad \nu_{yy} = \nu_{\eta\eta}w^2 + \nu_\eta w_\eta w$.

From Eqs.(1)-(3) we obtain the following equation for w

$$L(w) = \nu w_{\eta\eta}w^2 - w_\tau - \eta w_\xi + p_x w_\eta + \nu_{\eta\eta}w^3$$

$$+2\nu_\eta w_\eta w^2 = 0, \quad (5)$$

in the domain $\Omega = \{0 < \tau < T, 0 < \xi < X, g(\tau,\xi) < \eta < U(\tau,\xi)\}$, with the conditions

$$\begin{cases} w\mid_{\tau=0} = w_0(\xi,\eta), \quad w\mid_{\xi=0} = w_1(\tau,\eta), \\ \quad w\mid_{\eta=U(\tau,\xi)} = 0, \end{cases} \quad (6)$$

$$l(w) = (\nu_\eta w^2 + \nu w_\eta w - p_x - v_0 w - g_\tau -$$

$$-g g_\xi)\mid_{\eta=g(\tau,\xi)} = 0, \quad (7)$$

where $\nu(y),\ g(t,x)$ turn into the corresponding functions of τ, ξ and η, but we still denoted them by $\nu(\tau,\xi,\eta),\ g(\tau,\xi)$.

Clearly, if $\nu = c$, i.e. in the Prandtl boundary system, then (5) has the following simpler form

$$\nu w_{\eta\eta}w^2 - w_\tau - \eta w_\xi + p_x w_\eta = 0.$$

Now, due to the nonlinear terms $\nu_{\eta\eta}w^3 + 2\nu_\eta w_\eta w^2$, the problem becomes more difficult. In order to get the similar results as those of Prandtl boundary layer, some restrictions in ν, g have to be added. $0 < \nu_0 < \nu(y) < \nu_1$, where ν_i, $(i = 0,1)$, are constants. $\nu_{\eta\eta}, \nu_\eta, g_\tau, g, \nu_{\eta\eta\eta}$ and g_ξ all are bounded, $\nu_{\eta\eta} < 0, \nu_\eta < 0$ and $g(\tau,\xi) < \min \frac{U(\tau,\xi)}{4}$, where $U(\tau,\xi)$ is the press function of the flow outside the boundary layer.

2 Some important lemmas

Definition 1 *A function $w(\tau,\xi,\eta)$ is said to be a weak solution of problem (5)-(7), if w has first order derivatives in equation (5) continuous in $\overline{\Omega}$, and its derivative $w_{\eta\eta}$ continuous when $g(\tau,\xi) < \eta < U(\tau,\xi)$; w satisfies equation (5) almost everywhere in Ω, together with the conditions (6)(7).*

The solution of problem (5)-(7) will be constructed as the limit of a sequence w^n, $n \to \infty$, which consists of solutions of the equations

$$L_n(w^n) \equiv \nu(w^{n-1})^2 w^n_{\eta\eta} - w^n_\tau - \eta w^n_\xi$$

$$+p_x w^n_\eta + \nu_{\eta\eta}(w^{n-1})^3 + 2\nu_\eta w^n_\eta (w^{n-1})^2 = 0, \quad (8)$$

supplemented by the conditions

$$\begin{cases} w^n(0,\xi,\eta) = w_0(\xi,\eta), \quad w^n(\tau,0,\eta) = w_1(\tau,\eta) \\ \quad w^n(\tau,\xi,U(\tau,\xi)) = 0, \end{cases} \quad (9)$$

$$l_n(w^n) = (\nu w^{n-1}w^n_\eta - v_0 w^{n-1} + \nu_\eta (w^{n-1})^2 -$$

$$-p_x - g_\tau - g g_\xi)\mid_{\eta=g(\tau,\xi)} = 0. \quad (10)$$

As w^0 we take a function which is smooth in $\overline{\Omega}$, satisfies the conditions (6), and is positive for $g(\tau,\xi) < \eta < U(\tau,\xi)$. We assume that there exists $\varphi_0(\tau,\xi,\eta)$ with the following properties: φ_0 is smooth in $\overline{\Omega}$; $w_0 \geq \varphi_0(0,\xi,\eta)$, $w_1 \geq \varphi_0(\tau,0,\eta)$, $\varphi_0 > 0$ for $g(\tau,\xi) < \eta < U(\tau,\xi)$; moreover,

$$\varphi_0 \equiv m_0(U(\tau,\xi) - \eta)^k$$

for some $m_0 > 0$ and $k \geq 1$, provided that $U(\tau,\xi) - \eta < \delta_0$, where δ_0 is a small positive constant.

Assuming that problem (8)(9) admits a solution $w^n (n = 1, 2...)$ with continuous third order derivatives in the closed domain $\overline{\Omega}$, let us show that w^n are convergent, as $n \to \infty$, to a solution of problem (5)-(7); after that we are going to show that the w^n do exist, and we indicate a method for their approximation. A solution will be constructed for problem (1)-(3) in the domain Ω for some $T = T_0$ and any X, as well as for some $X = X_0$ and any T. The constant T_0 and X_0 are determined by u_0, u_1, v_0, p_x.

Lemma 2 *Let V be a smooth function such that $L_n(V) \geq 0$ in Ω, $l_n(V) > 0$ for $\eta = g(\tau,\xi)$, and $V \leq w^n$ for $\tau = 0$ and $\xi = 0$. Assume that $w^{n-1} > 0$ for $\eta = g(\tau,\xi)$. Then $V \leq w^n$ everywhere in Ω.*

Let V_1 be a smooth function such that $L_n(V_1) \leq 0$ in Ω, $l_n(V) < 0$ for $\eta = g(\tau,\xi)$, and $V_1 \geq w^n$ for $\tau = 0$ and $\xi = 0$. Assume that $w^{n-1} > 0$ for $\eta = g(\tau,\xi)$. Then $V_1 \geq w^n$ everywhere in Ω.

Proof: Let us prove the first statement of Lemma 2. The difference $z = w^n - V$ satisfies the inequalities

$$L_n(z) = L_n(w^n) - L_n(V) \leq 0,$$

$$l_n(z) = l_n(w^n) - l_n(V) = \nu w^{n-1} z_\eta < 0.$$

since $w^{n-1} > 0$ for $\eta = g(\tau,\xi)$. By assumption $V \leq w^n$ for $\tau = 0, \xi = 0$, we have $z \geq 0$ for $\tau = 0$, and

$z \geq 0$ for $\xi = 0$. Consider the function $z_1 = ze^{-\tau}$, clearly, $z_1 \geq 0$ for $\tau = 0$ and $\xi = 0$; $z_{1\eta} < 0$ for $\eta = U(\tau, \xi)$. It follow that z_1 can't have a negative minimum at $\eta = g(\tau, \xi)$, since at the point of negative minimum $z_{1\eta} \geq 0$. The rest of the proof is similar as in [3].The second statement of Lemma 2 can be proved in a similar fashion.

Lemma 3 *Suppose that $\nu_{\eta\eta}, \nu_\eta, \nu, g_\tau, g$ and g_ξ all are bounded, $g(\tau, \xi) < \min \frac{U(\tau,\xi)}{4}$,$\nu_{\eta\eta} < 0, \nu_\eta < 0$. There is a positive constant T_0 such that for all n and all $\tau \leq T_0$ the inequalities*

$$H_1(\tau, \xi, \eta) \geq w^n \geq h_1(\tau, \xi, \eta),$$

hold in Ω, where H_1 and h_1 are continuous functions in $\overline{\Omega}$, $h_1 \geq 0$ for $g < \eta < U, \tau < T_0$.

Proof: Let us construct function V and V_1 satisfying the conditions of Lemma 2. To this end, we define a twice continuously differentiable function $\psi(\tau, \xi, \eta)$ such that $\psi \equiv \kappa(\alpha_1(\eta - g))$ for $g < \eta < g + \delta_1$, $0 < \delta_1 < \min U(\tau, \xi)/2 - g$, $\kappa(s) = e^s$ for $0 \leq s \leq 1$, $1 \leq \kappa(s) \leq 3$ for $s \geq 1$, and $\psi = (U(\tau, \xi) - \eta)^k$ for $U - \eta < \delta_0$; $0 < \alpha_0 \leq \psi \leq 4$ for $\delta_1 < \eta < U - \delta_0$. Here α_0 is a small constant. We define the functions V and V_1 by

$$V = m\psi e^{-\alpha\tau}, \quad V_1 = M(C - e^{\beta_1\eta})e^{\beta\tau},$$

where $m, \alpha, \alpha_1, \beta, \beta_1, C, M$ are positive constants.

Let us show that T_0 and the constants in the definition of V and V_1 can be chosen independent of n, so that the inequality $V \leq w^{n-1} \leq V_1$ for $\tau \leq T_0$ implies that $V \leq w^n \leq V_1$ for $\tau \leq T_0$. Consider $l_n(V), l_n(V_1)$. For $e^{-\alpha\tau} \geq 1/2$, since $w^{n-1} \geq V = m\psi e^{-\alpha\tau}$ and $g, g_\tau, g_\xi, p_x, \nu_\eta$ are bounded.

If we choose $\alpha_1 > 0$ and $\beta_1 > 0$ large enough, we can get

$$l_n(V) = \nu w^{n-1}V^n_\eta - v_0 w^{n-1} + \nu_\eta(w^{n-1})^2 - p_x - g_\tau - gg_\xi$$

$$\geq me^{-\alpha\tau}(\nu m\alpha_1 e^{-\alpha\tau} - v_0) - p_x + \nu_\eta(w^{n-1})^2$$

$$-g_\tau - gg_\xi > 0,$$

$$l_n(V_1) = \nu w^{n-1}V^n_{1\eta} - v_0 w^{n-1} + \nu_\eta(w^{n-1})^2 - p_x - g_\tau - gg_\xi$$

$$\leq me^{-\alpha\tau}(-\nu\beta_1 Me^{\beta\tau} - v_0) - p_x + \nu_\eta(w^{n-1})^2$$

$$-g_\tau - gg_\xi < 0,$$

due to $\nu_1 > \nu > \nu_0$, $\nu_\eta < 0$. The constant m, C and M should be chosen from the conditions

$$\varphi_0(\tau, \xi, \eta) \geq m\psi(\tau, \xi, \eta), \quad C - e^{\beta_1\eta} \geq 1,$$

$$M \geq \max\{w_0, w_1\},$$

Let us choose $\beta > 0$ such that $L_n(V_1) < 0$ in $\overline{\Omega}$. Taking into account the inequality $w^{n-1} \geq V = m\psi e^{-\alpha\tau}$, $\nu_{\eta\eta} < 0$, we find that for large positive β

$$L_n(V_1) = -\nu(w^{n-1})^2 M\beta_1^2 e^{\beta_1\eta}e^{\beta\tau} -$$

$$-M(C - e^{\beta_1\eta})\beta e^{\beta\tau} -$$

$$-p_x M\beta_1 e^{\beta_1\eta}e^{\beta\tau} + \nu_{\eta\eta}(w^{n-1})^3$$

$$+2\nu_\eta Me^{\beta\tau}(-\beta_1)e^{\beta_1\eta}(w^{n-1})^2$$

$$\leq -e^{\beta\tau}[\nu(m\psi e^{-\alpha\tau})^2 M\beta_1^2 e^{\beta_1\eta} + M\beta$$

$$+p_x M\beta_1 e^{\beta_1\eta}] < 0.$$

For $L_n(V)$, we have

$$L_n(V) = \nu(w^{n-1})^2 m\psi_{\eta\eta}e^{-\alpha\tau} + \alpha m\psi e^{-\alpha\tau}$$

$$-m\psi_\tau e^{-\alpha\tau} - \eta m\psi_\xi e^{-\alpha\tau} + p_x m\psi_\eta e^{-\alpha\tau}$$

$$+\nu_{\eta\eta}(w^{n-1})^3 + 2\nu_\eta m\psi_\eta e^{-\alpha\tau}(w^{n-1})^2.$$

Since

$$0 \leq w^{n-1} \leq M(C - e^{\beta_1\eta})e^{\beta\eta},$$

and $\nu_{\eta\eta}$, p_x, ν, ν_η are all bounded. the positive constant α can be chosen independent of n and so large that

$$L_n(V) > 0 \quad in \quad \Omega \quad for \quad \eta < U(\tau, \xi) - \delta_0.$$

because of the inequality $\psi \geq min\{\alpha_0, 1\}$. In the region $\eta \geq U(\tau, \xi) - \delta_0$ where $\psi = (U - \eta)^k$, we have

$$L_n(V) = me^{-\alpha\tau}[\nu(w^{n-1})^2 k(k-1)(U - \eta)^{k-2}$$

$$-k(U - \eta)^{k-1}U_\tau + \alpha(U - \eta)^k - \eta k(U - \eta)^{k-1}U_\xi$$

$$-p_x k(U - \eta)^{k-1} - 2\nu_\eta(w^{n-1})^2 k(U - \eta)^{k-1}]$$

$$+\nu_{\eta\eta}(w^{n-1})^3.$$

It follows from the Bernoulli relation that

$$U_\tau + \eta U_\xi + p_x = -(U - \eta)U_\xi.$$

Due to $\nu_{\eta\eta}, \nu_\eta$ all are bounded, if we choose T_0 such that $e^{-\alpha T_0} \leq 1/2$, then for $\tau \leq T_0$,

$$L_n(V) \geq me^{-\alpha\tau}[k(U - \eta)^k U_\xi + \alpha(U - \eta)^k$$

$$-2\nu_\eta(w^{n-1})^2] + \nu_{\eta\eta}(w^{n-1})^3 > 0,$$

for large positive α. Thus, the conditions of Lemma 2 hold for V and V_1 in Ω. The constant α and T_0 depend only on the data of problem (5)-(7). Consequently, if by $V \leq w^{n-1} \leq V_1$ for $\tau \leq T_0$, it follow that $V_1 \geq w^n \geq V$ for any n and $\tau \leq T_0$. Now, it remains to set $h_1(\tau, \xi, \eta) = V, H_1(\tau, \xi, \eta) = V_1$.

Lemma 4 *Suppose that $\nu_{\eta\eta}, \nu_\eta, \nu, g_\tau, g$ and g_ξ all are bounded, $\nu_{\eta\eta} < 0, \nu_\eta < 0$, there is a positive constant X_0 such that for all n and $\xi \le X_0$ the inequalities*

$$H_2(\tau, \xi, \eta) \ge w^n \ge h_2(\tau, \xi, \eta),$$

hold in Ω, where H_2, h_2 are continuous functions in $\overline{\Omega}$ and $h_2(\tau, \xi, \eta) > 0$ for $\eta < U$, $\xi \le X_0$.

Proof: Let us construct functions V and V_1 that satisfy the conditions of Lemma 2. Let $\psi(\tau, \xi, \eta)$ be the function constructed in the proof of Lemma 3, and let $\varphi(s)$ be a twice differentiable function for $s \ge 0$ and such that $\varphi(s) = 3 - e^s$ for $0 \le s \le 1/2$, $1 \le \varphi(s) \le 3$, $|\varphi'(s)| \le 3$, $|\varphi''(s)| \le 3$ for all $s \ge 0$. Set

$$V = m\psi e^{-\alpha\xi}, \quad V_1 = M\varphi(\beta_1\eta)e^{\beta\xi}.$$

Let us assume positive constant m, M, α, α_1, β, β_1 and X_0 can be chosen independent of n, and $V_1 \ge w^{n-1} \ge V$ for $\xi \le X_0$, p_x, ν_η, g_τ, g, and g_ξ are bounded, we have

$$l_n(V) = \nu w^{n-1} m\alpha_1 e^{-\alpha\xi} - v_0 w^{n-1} - p_x + \nu_\eta (w^{n-1})^2$$

$$-g_\tau - g g_\xi$$

$$\ge m e^{-\alpha\xi}(\nu m\alpha_1 e^{-\alpha\xi} - v_0) - p_x > 0,$$

for large enough α_1, provided that $e^{-\alpha\xi} \ge 1/2$. If β_1 is sufficiently large and $e^{-\alpha\xi} \ge 1/2$, then

$$l_n(V_1) \le m e^{-\alpha\xi}(-\nu M\beta_1 e^{\beta\xi} - v_0) - p_x + \nu_\eta (w^{n-1})^2$$

$$-g_\tau - g g_\xi < 0.$$

We have

$$L_n(V_1) = \nu (w^{n-1})^2 M\beta_1^2 \varphi'' e^{\beta\xi} - \eta M\varphi\beta e^{\beta\xi}$$

$$+[p_x + 2\nu_\eta (w^{n-1})^2] M\beta_1 \varphi' e^{\beta\xi} + \nu_{\eta\eta}(w^{n-1})^3.$$

If $\beta_1\eta \le 1/2$, $\varphi'' \le -1$. By assumption, $w^{n-1} \ge m\psi e^{-\alpha\xi}$, where the function ψ has already been fixed, and the constant m is determined from the condition: $m\varphi \le \varphi_0$, $e^{-\alpha\xi} \ge 1/2$ for $\xi \le X_0$ and sufficiently small X_0. Therefore, β_1 is taken so large that $L_n(V_1) < 0$ for $\beta_1\eta \le 1/2$. Choosing $\beta > 0$ so large that $L_n(V_1) < 0$ for $\beta_1\eta \ge 1/2$, choosing a suitable M, we can ensure the inequality $V_1 \ge w^n$ for $\tau = 0$ and for $\xi = 0$. By Lemma 2, $V_1 \ge w^n$ in Ω for $\xi \le X_0$.

For $L_n(V)$ we have

$$L_n(V) = \nu (w^{n-1})^2 m\psi_{\eta\eta} e^{-\alpha\xi} + \alpha m\psi e^{-\alpha\xi}$$

$$-m\psi_\tau e^{-\alpha\xi} - \eta m\psi_\xi e^{-\alpha\xi} + p_x m\psi_\eta e^{-\alpha\xi}$$

$$+\nu_{\eta\eta}(w^{n-1})^3 + 2\nu_\eta m\psi_\eta e^{-\alpha\xi}(w^{n-1})^2.$$

since $m\psi e^{-\alpha\xi} \le w^{n-1}$, if $\alpha_1\eta \le 1, e^{-\alpha\beta} \ge 1/2$ and α_1 large enough then

$$L_n(V) \ge \nu m^3 \alpha_1^2 e^{3\alpha_1\eta} e^{-3\alpha\xi}$$

$$+[p_x + 2\nu_\eta (w^{n-1})^2]\alpha_1 e^{\alpha_1\eta} e^{-\alpha\xi} m + \nu_{\eta\eta}(w^{n-1})^3 > 0,$$

due to the boundedness of $\nu_\eta, \nu_{\eta\eta}$.

If $1/\alpha_1 < \eta < U - \delta_0, \psi \ge \alpha_0 > 0, 0 \le w^{n-1} \le M\varphi(\beta_1\eta)e^{\beta\xi}$ and α is taken large enough, then

$$L_n(V) > 0.$$

If $U(\tau, \xi) - \eta < \delta_0$ from the Bernoulli law, as in the proof of Lemma 3, we take α sufficiently large to assure $L_n(V) > 0$ for $U - \eta < \delta_0$. Therefore, $L_n(V) > 0$ in Ω for $0 \le \xi \le X_0$, if X_0 is chosen such that $e^{-\alpha X_0} \le 1/2$. Since, owing to our choice of m, we have $V \le w^n$ for $\tau = 0$ and $\xi = 0$, it follows from Lemma 2 that $w^n \ge m\psi e^{-\alpha\xi}$ for $\xi \le X_0$ and all τ. This completes the proof of Lemma 4, since it may be assumed that $V \le w^0 \le V_1$.

In what follows, it is assumed that the constants T_0 and X_0 in the definition of Ω are the same as in Lemma 3 and Lemma 4. In order to estimate the first and the second order derivatives of w^n, we pass to new unknown functions $W^n = w^n e^{\alpha\eta}$ in (8)(9), where α is a positive constant to be chosen later. Thus, we have

$$L_n(w^n) = \nu (w^{n-1})^2 W^n_{\eta\eta} - W^n_\tau - \eta W^n_\xi$$

$$+[p_x + 2\nu_\eta (w^{n-1})^2 - 2\nu (w^{n-1})^2\alpha] W^n_\eta$$

$$+[\alpha^2 \nu (w^{n-1})^2 - p_x\alpha - 2\nu_\eta \alpha e^{\alpha\eta}(w^{n-1})^2] W^n$$

$$+\nu_{\eta\eta}(w^{n-1})^3 e^{\alpha(\eta - g)} = 0.$$

$$l_n(w^n) = \nu W^{n-1} W^n_\eta - \alpha\nu W^{n-1} W^n$$

$$-W^{n-1} v_0 + \nu_\eta (W^{n-1})^2 - p_x - g_\tau - g g_\xi = 0.$$

Set

$$L_n^0(W) \equiv \nu (w^{n-1})^2 W_{\eta\eta} - W_\tau - \eta W_\xi + A^n W_\eta,$$

$$A^n = p_x + 2\nu_\eta (w^{n-1})^2 - 2\nu (w^{n-1})^2 \alpha,$$

$$L_n^0(W^n) + B^n W^n + \nu_{\eta\eta}(w^{n-1})^3 e^{\alpha\eta} = 0,$$

$$B^n = \alpha^2 \nu (w^{n-1}) - p_x\alpha - 2\nu_\eta \alpha (w^{n-1})^2.$$

Consider the function

$$\Phi_n = (W^n_\tau)^2 + (W^n_\xi)^2 + W^n_\eta (W^n_\eta - 2H^n$$

$$+2\frac{g_\tau + g g_\xi}{\nu W^{n-1}} - \frac{2\nu_\eta W^{n-1}}{\nu}) + k_0 + k_1\eta,$$

where

$$H^n \equiv \frac{v_0}{\nu} + \frac{p_x}{\nu W^{n-1}} + \alpha W^n \chi(\eta) + \frac{g_\tau + g g_\xi}{\nu W^{n-1}}$$

$$-\frac{\nu_\eta W^{n-1}}{\nu}.$$

We assume that H^n is defined in Ω, and v_0, p_x have been extended to the region $\eta > g(\tau, \xi)$, so that $v_0 = 0, p_x = 0$ for $\eta > \delta_2 = \min U(\tau, \xi)/2$; v_0, p_x do not depend on η for $\eta < \delta_2/2$ and are sufficiently smooth for all η; $\chi(\eta)$ is smooth function such that $\chi(\eta) = 1$ for $\eta \le \delta_2/2$, and $\chi(\eta) = 0$ for $\eta \ge \delta_2$. Obviously, $W^n_\eta = H^n$ for $\eta = g(\tau, \xi)$.

Lemma 5 *Suppose that $\nu_{\eta\eta}, \nu_\eta, \nu, g_\tau, g$ and g_ξ all are bounded, $\nu_{\eta\eta} < 0, \nu_\eta < 0$, and as before $0 < \nu_0 \ge \nu < \nu_1$, then the constant k_0, k_1, α can be chosen such that*

$$\frac{\partial \Phi_n}{\partial \eta} \ge \alpha \Phi_n - \frac{\alpha}{2}\Phi_{n-1} \quad for \quad \eta = g(\tau, \xi),$$

$$L^0_n(\Phi_n) + R^n \Phi_n \ge 0 \quad in \quad \Omega, \tag{11}$$

where R^n depends on w^{n-1} and its derivatives up to the second order.

Proof: For $\frac{\partial \Phi_n}{\partial \eta}$ at $\eta = g(\tau, \xi)$,

$$\frac{\partial \Phi_n}{\partial \eta} = 2W^n_\tau W^n_{\tau\eta} + 2W_\xi W^n_{\xi\eta} + W^n_{\eta\eta}(W^n_\eta - 2H^n)$$

$$+W^n_\eta(W^n_{\eta\eta} - 2H^n_\eta) + 2W^n_{\eta\eta}(\frac{g_\tau + g g_\xi}{\nu W^{n-1}}$$

$$+\frac{\nu_\eta W^{n-1}}{\nu}) + 2W^n_\eta(\frac{g_\tau + g g_\xi}{\nu W^{n-1}} - \frac{\nu_\eta W^{n-1}}{\nu})_\eta + k_1.$$

Using the boundary condition $W^n_\eta = H^n$ at $\eta = g(\tau, \xi)$, we obtain

$$\frac{\partial \Phi_n}{\partial \eta} = 2W^n_\tau H^n_\tau + 2W^n_\xi H^n_\xi - 2H^n H^n_\eta$$

$$+2W^n_{\eta\eta}(\frac{g_\tau + g g_\xi}{\nu W^{n-1}} + \frac{\nu_\eta W^{n-1}}{\nu})$$

$$+2H^n(\frac{g_\tau + g g_\xi}{\nu W^{n-1}} - \frac{\nu_\eta W^{n-1}}{\nu})_\eta + k_1.$$

According to Lemma 3 and Lemma 4, $W^n_\eta \ge h_0 > 0$ for $\eta = g(\tau, \xi)$. For $\eta = g(\tau, \xi)$, we have

$$H^n_\eta = -\frac{v_0 \nu_\eta}{\nu^2} - \frac{(p_x + g_\tau + g g_\xi)(\nu_\eta W^{n-1} + \nu W^{n-1}_\eta)}{(\nu W^{n-1})^2}$$

$$+\alpha W^n_\eta \chi(\eta) - (\frac{\nu_\eta}{\nu})_\eta W^{n-1} - \frac{\nu_\eta}{\nu} W^{n-1}_\eta.$$

Let us express W^n_η and W^{n-1}_η from the condition $W^n_\eta = H^n$. We find that $H^n H^n_\eta$ depends only on $\nu, \nu_\eta, \nu_{\eta\eta}, W^n, W^{n-1}, W^{n-2}$, and therefore, is uniformly bounded with respect to n. Consequently, $| 2H^n H^n_\eta | \le k_2$, k_2 being independent of n. Let us estimate $W^n_\tau H^n_\tau$ and $W^n_\xi H^n_\xi$.

For $\eta = g(\tau, \xi)$, we have $\chi(\eta) = 1$,

$$H^n_\tau = \frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}} - \frac{p_x W^{n-1}_\tau}{\nu (W^{n-1})^2} + \alpha W^n_\tau$$

$$+\frac{(g_\tau + g g_\xi)_\tau}{\nu W^{n-1}} - \frac{(g_\tau + g g_\xi) W^{n-1}_\tau}{\nu (W^{n-1})^2},$$

$$W^n_\tau H^n_\tau = W^n_\tau [\frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}} - \frac{p_x W^{n-1}_\tau}{\nu (W^{n-1})^2}$$

$$+\alpha W^n_\tau + \frac{(g_\tau + g g_\xi)_\tau}{\nu W^{n-1}} - \frac{(g_\tau + g g_\xi) W^{n-1}_\tau}{\nu (W^{n-1})^2}]$$

$$\ge \alpha (W^n_\tau)^2 - \frac{1}{\alpha}[\frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}} + \frac{(g_\tau + g g_\xi)_\tau}{\nu W^{n-1}}]^2$$

$$-\frac{1}{\alpha}[\frac{p_x + g_\tau + g g_\xi}{\nu (W^{n-1})^2}]^2 (W^{n-1}_\tau)^2 - \frac{\alpha}{4}(W^n_\tau)^2.$$

Due to p_x, g_τ, g and g_ξ all are bounded,$\nu_0 < \nu < \nu_1$, we can choose a positive α independent of n and such that

$$\frac{1}{\alpha}[\frac{p_x + g_\tau + g g_\xi}{\nu (W^{n-1})^2}]^2 \le \frac{\alpha}{4},$$

Then

$$W^n_\tau H^n_\tau \ge \frac{3\alpha}{4}(W^n_\tau)^2 - \frac{\alpha}{4}(W^{n-1}_\tau)^2 - k_3,$$

$$k_3 > \max \frac{1}{\alpha}[\frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}} + \frac{(g_\tau + g g_\xi)_\tau}{\nu W^{n-1}}]^2,$$

and k_3 does not depend on n. In a similar way, we find that

$$W^n_\xi H^n_\xi \ge \frac{3\alpha}{4}(W^n_\xi)^2 - \frac{\alpha}{4}(W^{n-1}_\xi)^2 - k_4,$$

$$k_4 \ge \max \frac{1}{\alpha}[\frac{v_{0\xi}}{\nu} + \frac{p_{x\xi}}{\nu W^{n-1}} + \frac{(g_\tau + g g_\xi)_\xi}{\nu W^{n-1}}]^2$$

$$2W^n_{\eta\eta}\frac{g_\tau + g g_\xi + \nu_\eta (W^{n-1})^2}{\nu W^{n-1}}$$

$$= 2\{W^n_\tau + \eta W^n_\xi - [p_x - 2\nu (w^{n-1})^2 \alpha] W^n_\eta$$

$$-[\alpha^2 \nu (w^{n-1})^2 - p_x \alpha] W^n\} \frac{g_\tau + g g_\xi + \nu_\eta (W^{n-1})^2}{\nu W^{n-1}}$$

$$\ge -\frac{\alpha}{4}(W^n_\tau)^2 - \frac{1}{\alpha}(\frac{g_\tau + g g_\xi + \nu_\eta (W^{n-1})^2}{\nu W^{n-1}})^2$$

$$-\frac{\alpha}{4}(W_\xi^n)^2-\frac{1}{\alpha}(\frac{2\eta(g_\tau+gg_\xi+\nu_\eta(W^{n-1})^2)}{\nu W^{n-1}})^2$$

$$-2[p_x-2\nu(w^{n-1})^2\alpha]W_\eta^n\frac{g_\tau+gg_\xi+\nu_\eta(W^{n-1})^2}{\nu W^{n-1}}$$

$$-\frac{\alpha}{4}(W^n)^2$$

$$-\frac{1}{\alpha}[\frac{(\alpha^2\nu(w^{n-1})^2-p_x\alpha)(g_\tau+gg_\xi+\nu_\eta(W^{n-1})^2)}{\nu W^{n-1}}]^2$$

$$\geq-\frac{\alpha}{4}(W_\tau^n)^2-\frac{\alpha}{4}(W_\xi^n)^2-\frac{\alpha}{4}(W^n)^2-k_5.$$

For $\eta=g(\tau,\xi)$, we have

$$\frac{\partial\Phi_n}{\partial\eta}\geq\alpha[(W_\tau^n)^2+(W_\xi^n)^2]$$

$$-\frac{\alpha}{2}[(W_\tau^n)^2+(W_\xi^{n-1})^2]-k_6+k_1,$$

where $k_6=k_2+2k_3+2k_4+k_5$.

The function $W_\eta^n(W_\eta^n-2H^n)\mid_{\eta=g(\tau,\xi)}$ is uniformly bounded with respect to n, according to the boundary condition $W_\eta^n=H^n$. Therefore

$$\frac{\partial\Phi_n}{\partial\eta}\geq\alpha\Phi_n-\frac{\alpha}{2}\Phi_{n-1}-k_7+k_1,$$

where k_7 is a constant that does not depend on n. Let us choose $k_1>k_7$. We have

$$\frac{\partial\Phi_n}{\partial\eta}\geq\alpha\Phi_n-\frac{1}{2}\alpha\Phi_{n-1}\quad for\quad \eta=g(\tau,\xi).$$

Next we consider $L_n^0(\Phi_n)$. Choosing a suitable k_0, we may assume that $\Phi_n\geq1$ in Ω. Noting that

$$H^n=\frac{g_\tau+gg_\xi}{\nu W^{n-1}}-\frac{\nu_\eta W^{n-1}}{\nu},$$

for $\eta\geq\delta_2$, we have

$$\Phi_n=\Phi_n^*\equiv(W_\tau^n)^2+(W_\xi^n)^2+(W_\eta^n)^2+k_0+k_1\eta.$$

Applying the operator

$$2W_\tau^n\frac{\partial}{\partial\tau}+2W_\xi^n\frac{\partial}{\partial\xi}+2W_\eta^n\frac{\partial}{\partial\eta}$$

to the equation

$$L_n^0(W^n)+B^nW^n=0,$$

we can get the conclusion of (11). As the details of the proof, and we will give them in the appendix of the paper.

In order to estimate the second derivatives of w^n in Ω, consider the function

$$F_n=(W_{\tau\tau}^n)^2+(W_{\xi\xi}^n)^2+(W_{\tau\xi}^n)^2+W_{\xi\eta}^n(W_{\xi\eta}^n-2H_\xi^n)$$

$$+W_{\tau\eta}^n(W_{\tau\eta}^n-2H_\tau^n)+f(\eta)(W_{\eta\eta}^n)^2+N_0+N_1\eta,$$

where N_0,N_1 are constants, and $f(\eta)$ is a smooth function such that $f(g)=0, f'(g)=0$, $f(\eta)>0$ for $\eta>g(\tau,\xi)$, $f(\eta)=1$ for $\eta>\delta_2$.

Lemma 6 *The constant N_0 and N_1 can be chosen independent of w^n,w^{n-1},w^{n-2} or their derivatives, so that*

$$\frac{\partial F_n}{\partial\eta}\geq\alpha F_n-\frac{\alpha}{2}F_{n-1}\quad for\quad \eta=g,$$

$$L_n^0(F_n)+C^nF_n+N_2\geq0\quad in\quad\Omega,$$

where the constant N_2 depends only on the first derivatives of w^n,w^{n-1},w^{n-2};the constant C^n depends on w^{n-1} and its derivatives up to the second order.

The proof is similar with the way of lemma 5.

3 The solution of the system (5)-(7)

Theorem 7 *Let w^n be solutions of problems(8)(9)(10). Then the derivatives of w^n up to the second order are uniformly bounded with respect to n in domain Ω with a positive T depending on the data of problem (1)-(3).*

Proof: Let us show that there exist constants M_1,M_2 and $T>0$ such that the conditions $\Phi_\mu\leq M_1, F_\mu\leq M_2$ for $\tau\leq T,\mu\leq n-1$, imply that $\Phi_n\leq M_1,F_n\leq M_2$ for $\tau\leq T$. According to Lemma 5, we have

$$L_n^0(\Phi_n)+R^n\Phi_n\geq0,$$

where R^n depends on w^{n-1} and its derivatives up to the second order.

Consider the function $\Phi_n^1=\Phi_ne^{-\gamma\tau}$ with a positive constant γ to be chosen later. We have

$$L_n^0(\Phi_n^1)+(R^n-\gamma)\Phi_n^1\geq0\quad in\quad\Omega.$$

Let us choose γ in accordance with M_1 and M_2, so as to have $R^n-\gamma\leq-1$ in Ω, as well as for $\xi=X,\tau=T$, or $\eta=U(\tau,\xi)$. If Φ_n^1 attains its largest value at $\tau=0$ or at $\xi=0$, we should have

$$\Phi_n^1=\Phi_ne^{-\gamma\tau}\leq\Phi_n<k_{11},$$

where the constant k_{11} does not depend on n and is determined only by the data of problem (8)(9)(10). If Φ_n^1 attains its largest value at some point with $\eta=g$, we must have $\partial\Phi_n^1/\partial\eta\leq0$ at that point, and it follows from lemma 5 that $\Phi_n^1 \le \frac{1}{2}\Phi_{n-1}^1$, i.e, $\Phi_n^1 \le \frac{1}{2}M_1$. Thus we have

$$\Phi_n^1 \le \max\{\frac{1}{2}M_1, k_{11}\},$$

$$\Phi_n \le \max\{\frac{1}{2}M_1, k_{11}\}e^{\gamma\tau} \quad in \quad \Omega.$$

Let us take $T_1 \le T$ such that $e^{\gamma T_1} = 2$, and set $M_1 = 2k_{11}$. In this case, $\Phi_n \le M_1$ for $\tau \le T_1$.

We consider F_n. By Lemma 6, we have

$$L_n^0(F_n) + C^nF_n + N_2 \ge 0 \quad in \quad \Omega,$$

where C^n depends on the first and the second derivatives of w^{n-1}, while N_2 depends on the first derivatives of w^n, w^{n-1}, w^{n-2}. Set $F_n^1 = F_ne^{-\gamma_1\tau}$. Then

$$L_n^0(F_n^1)+(C^n-\gamma_1)F_n^1 \ge -N_2e^{-\gamma_1\tau} \ge N_2 \quad in \quad \Omega.$$

Let us choose $\gamma_1 > 0$ in accordance with M_1 and M_2, so as to have

$$C^n - \gamma_1 \le -1 \quad in \quad \Omega_1 = \Omega \cap \{\tau \le T_1\}.$$

Then, if F_n^1 attains its largest value inside Ω_1, or at $\tau = T_1$, or at $\xi = X$, or at $\eta = U(\tau, \xi)$, we must have $F_n^1 \le N_2(M_1)$.

If F_n^1 attains its largest value at $\tau = 0$ or at $\xi = 0$, then

$$F_n^1 = F_ne^{-\gamma_1\tau} \le F_n \le N_{12},$$

where the constant N_{12} depends on M_1. If F_n^1 attains its largest value at $\eta = g(\tau, \xi)$, then, according to Lemma 6, at the point of maximum we have

$$0 \ge \frac{\partial F_n^1}{\partial\eta} \ge \alpha F_n^1 - \frac{\alpha}{2}F_{n-1}^1.$$

and therefore

$$F_n^1 \le \frac{1}{2}F_{n-1}^1 \le \frac{1}{2}F_{n-1}e^{-\gamma_1\tau} \le \frac{1}{2}M_2.$$

It follows that

$$F_n^1 \le \max\{\frac{1}{2}M_2, N_{12}, N_2\} \quad in \quad \Omega,$$

$$F_n \le \max\{\frac{1}{2}M_2, N_{12}, N_2\}e^{\gamma_1\tau}.$$

Let us take $T_2 \le T$ such that $e^{\gamma_1 T_2} = 2$. Set $M_2 = \max\{2N_{12}, 2N_2\}$. Then $F_n \le M_2$ for $\tau \le T_2$ and $\tau \le T_1$. The constant T_2, like T_1, depends only on M_1 and M_2 chosen above and determined only by the data of problem (1)(2)(3). It may be assumed that w^0 has been chosen such that $\Phi_0 \le M_1$ and $F_0 \le M_2$. The above results show that Φ_n and F_n are uniformly bounded with respect with respect to n for $\tau \le \min\{T_1, T_2\} = T$. The fact that Φ_n and F_n are bounded with respect to n allows us to conclude that the first and the second derivatives of w^n are also bounded, since the boundedness of $w_{\eta\eta}^n$ for $\eta \le \delta_2$ follows from (8) and the boundedness of the first derivatives of w^n. Theorem 7 is proved.

By the last theorem, we obtain a solution of problem (8)(9)(10) for any X and a sufficiently small T. The fact that derivatives of w^n are bounded for an arbitrary T and a sufficiently small X is established by the following:

Theorem 8 *Let ν satisfy the conditions quoted before, and w^n be solutions of problems(8)(9) (10). Then w^n are uniformly bounded with respect to n in domain Ω with X depending on the data of problem (1)-(3).*

Proof: Let us show that there exist constants M_1, M_2, and $X > 0$ such that the conditions $\Phi_\mu \le M_1$ and $F_\mu \le M_2$ for $\xi \le X$ and $\mu \le n-1$ imply that $\Phi_n \le M_1$ and $F_n \le M_2$ for $\xi \le X$. By Lemma 5, we have $L_n^0(\Phi_n) + R^n\Phi_n \ge 0$, where R^n depends on w^{n-1} and its derivatives up to the second order.

Let $\Phi_n = \Phi_n^1 e^{\beta\xi}\varphi_1(\beta_1\eta)$, where $\varphi_1(s)$ is a smooth function such that $\varphi_1(s) = 2 - e^s/2$ for $s \le \ln(3/2)$, $1 \le \varphi_1 \le 3/2$ for all s; β, β_1 are positive constants that will be chosen later. We have

$$L_n^0(\Phi_n^1) + 2\nu(w^{n-1})^2\beta_1\frac{\varphi_1^{'}}{\varphi_1}\Phi_{n\eta}^1 +$$

$$+(R^n - \eta\beta + A^n\beta_1\frac{\varphi_1^{'}}{\varphi_1} + \nu(w^{n-1})^2\beta_1^2\frac{\varphi_1^{''}}{\varphi_1})\Phi_n^1 \ge 0. \tag{12}$$

If $\beta_1\eta \le \ln(3/2)$, then $-3/4 \le \varphi_1^{'} \le -1/2$, $\varphi_1^{''} \le -1/2$. By Lemma 4, we have $(w^{n-1})^2 \ge \gamma_0 > 0$ for $\eta \le \delta_2$ and $\xi < X_0$.

Let $\eta \le \beta_1^{-1}\ln(3/2)$ and $\eta \le \delta_2$. Due to ν is bounded, then we can find β_1 such that the coefficient of Φ_n^1 in (12), for $\xi \le X$, satisfies the inequality

$$R^n - \eta\beta + A^n\beta_1\frac{\varphi_1^{'}}{\varphi_1} + \nu(w^{n-1})^2\beta_1^2\frac{\varphi_1^{''}}{\varphi_1} \le -1.$$

In the region of $\eta > \min\{\delta_2, \beta_1^{-1}ln(3/2)\}$ this inequality is valid if $\beta > 0$ has been chosen sufficiently large. Obviously, β may be assumed independent of M_1, M_2. Then, according to (12), the function Φ_n^1 can't attain its largest value inside Ω for $\xi < X$ at any of the points $\tau = T$, $\xi = X$, or $\eta = U(\tau, \xi)$.

If Φ_n^1 attains its largest value at $\tau = 0$ or $\xi = 0$, then

$$\Phi_n^1 = \frac{\Phi_n}{\varphi_1}e^{-\beta\xi} \le \Phi_n \le k_{12},$$

where k_{12} does not depend on n, since $\Phi_n \mid_{\tau=0}$ and $\Phi_n \mid_{\xi=0}$ can be expressed through w_0, w_1 and their derivatives.

If Φ_n^1 attains its largest value at $\eta = g(\tau, \xi)$, then $\partial\Phi_n^1/\partial\eta \le 0$ at the point of maximum, and it follows from Lemma 5 that

$$\Phi_n^1 \le \frac{1}{2}\Phi_{n-1}^1 \quad or \quad \Phi_n^1 \le \frac{1}{2}\frac{\Phi_{n-1}}{\varphi_1}e^{-\beta\xi} \le \frac{1}{2}M_1.$$

by virtue of our assumption. Thus

$$\Phi_n^1 \le \max\{\frac{1}{2}M_1, k_{12}\} \quad in \quad \Omega \ for \ \xi \le X,$$

$$\Phi_n \le \max\{\frac{1}{2}M_1, k_{12}\}\max[e^{\beta\xi}\varphi_1(\beta_1\eta)].$$

Since $\varphi_1(\beta_1\eta) \le 3/2$, we have $e^{\beta\xi}\varphi_1(\beta_1\eta) \le 2$, if $e^{\beta\xi} \le 4/3$. Let us choose $X_1 \le X_0$ from the condition $e^{\beta X_1} \le 4/3$. Then

$$\Phi_n \le \max\{M_1, 2k_{12}\} \quad for \quad \xi \le X_1.$$

Set $M_1 = 2k_{12}$. Then $\Phi_n \le M_1$ for $\xi \le X_1$, where X_1 depends on M_1 and M_2.

Now, let us consider F_n. By Lemma 6

$$L_n^0(F_n) + C^n F_n \ge -N_2 \quad in \quad \Omega \quad for \quad \xi \le X_1.$$

Let $F_n = F_n^1\varphi_1(\beta_2\eta)e^{\beta_3\xi}$ and $\varphi_1(s)$ be the function defined above. We have

$$L_n^0(F_n^1) + 2\nu(w^{n-1})^2\beta_2\frac{\varphi_1'}{\varphi_1}F_{n\eta}^1 + [C^m - \eta\beta_3$$

$$+A^n\beta_2\frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2\beta_1^2\frac{\varphi_1''}{\varphi_1}]F_n^1 > -N_2\frac{e^{-\beta_3\xi}}{\varphi_1}, \tag{13}$$

If $\beta_2\eta \le \ln(3/2)$, then $-3/4 \le \varphi_1' \le -1/2$, $\varphi_1'' \le -1/2$, $1 \le \varphi_1 \le 3/2$.

It follows from Lemma 4 that $(w^{n-1})^2 \ge \gamma_0 > 0$ for $\eta \le \delta_2$. Let $\eta \le \min\{\delta_2, \beta_2^{-1}\ln(3/2)\}$. For these values of η, the constant β_2 can be chosen so as to make the coefficient by F_n^1 in (13) satisfy the inequality

$$C^n - \eta\beta_3 + A^n\beta_2\frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2\beta_1^2\frac{\varphi_1''}{\varphi_1} \le -1.$$

This inequality will hold in the region of $\eta > \min\{\delta_2, \beta_2^{-1}\ln(3/2)\}$ if β_3 has been chosen sufficiently large. Clearly, β_3 depends on M_1 and M_2. Just as in the proof of Theorem 7, we find that

$$F_n^1 \le \max\{\frac{1}{2}M_1, N_2, N_{13}\} \quad in \quad \Omega \quad for \quad \xi \le X,$$

where $N_{13} = \max\{F_n\}$ for $\tau = 0$ and for $\xi = 0$; N_{13} depends on M_1. We have $F_n \le \max\{\frac{1}{2}M_2, N_2, N_{13}\}\max[e^{\beta_3\xi}\varphi_1(\beta_2\eta)] \le \max\{M_2, 2N_2, 2N_{13}\}$, provided that $e^{\beta_3\xi}\varphi_1(\beta_2\eta) \le 2$ and $e^{\beta_3\xi} \le 4/3$.

Let us take $M_2 = \max\{2N_2, 2N_{13}\}$ and define $X_2 \le X_0$ from the inequality $e^{\beta_3 X_2} \le 4/3$. Then $F_n \le M_2$ for $\xi \le X$, where $X = \min\{X_1, X_2\}$. The fact that Φ_n and F_n are bounded implies that the derivatives of w^n up to the second order are bounded uniformly in n, since $w_{\eta\eta}^n$, for $\eta \le \delta_2$, can be estimated from equation (8).

Theorem 9 *The function w^n $(n \to \infty)$are uniformly convergent in Ω to a solution w^n of problem (5)(6)(7) in Ω, where T is defined in Theorem 7 and X may be taken arbitrarily, or X is that of Theorem 8 and T is arbitrary. The function w is continuously differentiable in $\overline{\Omega}$ and its derivative $w_{\eta\eta}$ is continuous for $\eta < U(\tau, \xi)$.*

Proof: Let $v^n = w^n - w^{n-1}$. We obtain the following equation from (8)

$$\nu(w^{n-1})^2 v_{\eta\eta}^n - v_\tau^n - \eta v_\xi^{n-1} + [p_x + 2\nu_\eta(w^{n-1})^2]v_\eta^n$$

$$+\nu[(w^{n-1})^2 - (w^{n-2})^2]w_{\eta\eta}^{n-1}$$

$$+2\nu_\eta[(w^{n-1})^2 - (w^{n-2})^2]w_\eta^{n-1}$$

$$+\nu_{\eta\eta}[(w^{n-1})^2 - (w^{n-2})^3] = 0,$$

and also the boundary condition:

$$v^n \mid_{\tau=0} = 0, \qquad v^n \mid_{\xi=0} = 0, \qquad v^n \mid_{\eta=U(\tau,\xi)} = 0,$$

$$\nu w^{n-1}v_\eta^n - v_0 v^{n-1} + \nu w_\eta^{n-1}v^{n-1}$$

$$+\nu_\eta(w^{n-1} + w^{n-2})v^{n-1} = 0 \quad for \quad \eta = g.$$

Consider the function v_1^n defined by $v^n = e^{\alpha\tau + \beta\eta}v_1^n$, $\beta < 0$. We have

$$\nu(w^{n-1})^2 v_{1\eta\eta}^n - v_{1\tau}^n - \eta v_{1\xi}^n + [p_x + 2\nu_\eta(w^{n-1})^2]v_{1\eta}^n$$

$$+\nu w_{\eta\eta}^{n-1}(w^{n-1} + w^{n-2})v_1^{n-1}$$

$$+2\nu_\eta w_\eta^{n-1}(w^{n-1} + w^{n-2})v_1^{n-1} + \nu_{\eta\eta}[(w^{n-1})^2$$

$$+w^{n-1}w^{n-2} + (w^{n-2})^2]v_1^{n-1}$$

$$+2\nu(w^{n-1})^2\beta v_{1\eta}^n + [\nu(w^{n-1})^2\beta^2 + p_x\beta$$

$$+2\nu_\eta(w^{n-1})^2\beta - \alpha]v_1^n = 0, \tag{14}$$

The boundary condition $\eta = g(\tau, \xi)$

$$\nu w^{n-1}v_{1\eta}^n + \beta\nu w^{n-1}v_1^n - v_0 v_1^{n-1} + \nu w_\eta^{n-1}v_1^{n-1} +$$

$$+\nu_\eta(w^{n-1} + w^{n-2})v_1^{n-1} = 0. \tag{15}$$

Due to ν, $\nu_{\eta\eta}$ and ν_η are bounded, the constant $\beta < 0$ should be choose such that in the boundary condition for v_1^n at $\eta = g(\tau, \xi)$ the coefficients by v_1^n and v_1^{n-1} satisfy the inequality

$$\max \mid \nu w_\eta^{n-1} - v_0 + \nu_\eta (w^{n-1} + w^{n-2}) \mid$$
$$\leq q \min \mid \nu w^{n-1} \beta \mid,$$

for $q < 1$.

Having fixed β, let us choose $\alpha > 0$ such that

$$\max \mid \nu w_{\eta\eta}^{n-1} (w^{n-1} + w^{n-2})$$
$$+ 2\nu_\eta w_\eta^{n-1} (w^{n-1} + w^{n-2}) + \nu_{\eta\eta} [(w^{n-1})^2$$
$$+ w^{n-1} w^{n-2} + (w^{n-2})^2] \mid$$
$$\leq q(\alpha - \max \mid \nu (w^{n-1})^2 \beta^2 + p_x \beta + 2\nu_\eta (w^{n-1})^2 \beta \mid).$$

Now, if $\mid v_1^n \mid$ attains its largest value at an interior point of $\overline{\Omega}$ or on its boundary, it follow equations (14), (15) that

$$\max \mid v_1^n \mid \leq q \max \mid v_1^{n-1} \mid,$$

where means that the series $v_1^1 + v_1^2 + \cdots + v_1^n + \cdots$, whose partial sums have the form $w^n e^{-\alpha\tau - \beta\eta}$, is majorized by a geometrical progression, and therefore, is uniformly convergent. The fact w^n and its derivatives up to the second order are bounded implies that the first derivatives of w^n are uniformly convergent as $n \to \infty$.

It follows from equation (8) that $w_{\eta\eta}$ are also uniformly convergent as $n \to \infty$ for $\eta < U(\tau, \xi) - \delta_3$, where $\delta_3 < \min U(\tau, \xi) - \max g(\tau, \xi)$ and which is an arbitrary positive constant.

4 The existence of the solution of the system (8)-(9) and the main result

Now, let us establish the existence of the solution $w^n(\tau, \xi, \eta)$ for problem (8)(9). the way is the similar as [3].

Consider the operator

$$L^\varepsilon(w) \equiv \varepsilon(w_{\tau\tau} + w_{\xi\xi} + w_{\eta\eta}) + a_1 w_{\tau\tau} + a_2 w_{\xi\xi} + a_3 w_{\eta\eta}$$
$$+ [\nu (w^{n-1})^2]_\varepsilon w_{\eta\eta} - w_\tau - \eta w_\xi + [p_x + 2\nu_\eta (w^{n-1})^2]_\varepsilon w_\eta$$
$$+ [\nu_{\eta\eta} (w^{n-1})^3]_\varepsilon - 2(a_1 + \varepsilon) w.$$

Consider the following elliptic boundary value problem:

$$L^\varepsilon(w) = (f)_\varepsilon \quad in \quad Q, \tag{16}$$
$$\frac{\partial w}{\partial n} = (F)_\varepsilon \quad on \quad S. \tag{17}$$

where n is unit inward normal to S. The function f in (14) is defined in Q by

$$f = L(w^*) + a_1 w_{\tau\tau}^* + a_2 w_{\xi\xi}^* + a_3 w_{\eta\eta}^* - 2a_1 w^*,$$

in $Q \backslash \Omega_1$, $f = 0$ in Ω, f coincides with an arbitrary smooth extension of this function on the rest of Q. The function F is given by

$$F = \frac{v_0}{\nu} + \frac{p_x}{\nu W^{n-1}} + \frac{g_\tau + g g_\xi}{\nu W^{n-1}} - \frac{\nu_\eta W^{n-1}}{\nu} \quad on \quad S_0,$$
$$F = \frac{\partial w^*}{\partial n} \quad on \quad \gamma,$$

where γ is the intersection of S with the boundary of $Q\ \Omega_1$; on the rest of S, the function F in (19) coincides with any smooth extension of F just defined on S_0 and γ. Clearly, owing to the properties of w^*, we may assume that f has bounded derivatives up to the fourth order in Q and is infinitely differentiable outside a δ-neighborhood of Ω; F has bounded derivatives up to the fourth order in a neighborhood of S_0 and is infinitely differentiable on the rest of S.

The boundary value problem (16)-(17) has a uniqueness solution w_ε^n in Q, one can see the fact in [3].

Let us show that the functions w_ε^n and their derivatives up to the fourth order are bounded uniformly with respect to ε.

Lemma 10 *The solutions w_ε^n of problem (16)(17) in Q are bounded uniformly in ε.*

Proof: Set $w_\varepsilon^n = v_\varepsilon \psi^1$, where $\psi^1(\tau) = 1$ for $\tau \leq -1$, $\psi^1(\tau) = 1 + b(1+\tau)^3$ for $-1 \leq \tau \leq T + 2$, choose suitable $b > 0$ such that $\psi_{\tau\tau}^1 \leq \psi^1$ in Q. Let $6b(T+3) < 1$. Then v_3 satisfies the following equation in Q:

$$\frac{(f)_\varepsilon}{\psi^1} = \varepsilon(\triangle v_\varepsilon) + a_1 v_{\varepsilon\tau\tau} + a_2 v_{\varepsilon\xi\xi} + a_3 v_{\varepsilon\eta\eta}$$
$$+ [\nu (w^{n-1})^2]_\varepsilon v_{\varepsilon\eta\eta} - v_{\varepsilon\tau} - \eta w_{\varepsilon\xi}$$
$$+ [p_x + 2\nu_\eta (w^{n-1})^2]_\varepsilon v_{\varepsilon\eta} + 2(a_1 + \varepsilon) \frac{\psi_\tau^1}{\psi^1} v_{\varepsilon\tau}$$
$$+ [(a_1 + \varepsilon) \frac{\psi_\tau^1}{\psi^1} -$$
$$- \frac{\psi_\tau^1}{\psi^1} - 2(a_1 + \varepsilon)] v_\varepsilon + \frac{[\nu_{\eta\eta} (w^{n-1})^3]_\varepsilon}{\psi^1}, \tag{18}$$

as well as the boundary conditions on S

$$\frac{\partial v_\varepsilon}{\partial n} = \frac{(F)_\varepsilon}{\psi^1} \quad for \quad -2 \leq \tau \leq T + 1, \tag{19}$$

$$\frac{\partial v_\varepsilon}{\partial n}+\frac{1}{\psi^1}\frac{\partial\psi^1}{\partial n}v_\varepsilon=\frac{(F)_\varepsilon}{\psi^1}\quad for\quad \tau\geq T+1. \quad (20)$$

Since

$$\frac{\partial\psi^1}{\partial n}=\psi^1_\tau\frac{\partial\tau}{\partial n}\leq 0\quad for\quad \tau\geq T+1\quad on\quad S,$$

the coefficient of v_ε in the boundary condition (20) is non-positive. (The domain Q may be assumed convex for $\tau\geq T+1$.) The coefficient of v_ε in equation (18) is negative. Indeed, we have

$$-(a_1+\varepsilon)+(a_1+\varepsilon)\frac{\psi^1_{\tau\tau}}{\psi^1}\leq 0.$$

since $\psi^1_{\tau\tau}/\psi^1\leq 1$, and $\psi^1_\tau>0$ for $\tau>-1$, while $a_1>0$ for $\tau<-1/2$.

Applying to the solution of problem (18)(19)(20) the a priori estimate established by Theorem 4 of [4], we find that v_ε are bounded in Q by a constant independent of ε but depending only on the maximum moduli of the coefficients in equation (18), as well as on

$$\max\frac{(f)_\varepsilon}{\psi^1},\quad \max\frac{(F)_\varepsilon}{\psi^1},$$

$$\min[(a_1+\varepsilon)\frac{\psi^1_{\tau\tau}}{\psi^1}-\frac{\psi^1_\tau}{\psi^1}-2(a_1+\varepsilon)].$$

Lemma 11 *The solution w^n_ε of problem (16)(17) have their derivatives up to the fourth order in Q bounded by a constant independent of ε.*

Proof: First of all, we note that equation (16) is uniformly (with respect to ε) elliptic in Q for $\tau>T+\delta+r_1$, and for $\tau<-1/2-r_1$, where r_1 is any positive constant. Therefore, according to the well-known estimates of Schauder type (see [3]), the m-th order derivatives of w^n_ε have their absolute values bounded by a constant independent of ε, for $\tau>T+\delta+r_1$ and for $\tau<-1/2-r_1$, provided that w^{n-1} have bounded $(m-1)$-th order derivatives (with $m\geq 2$) in the same region.

Let $P(\xi,\eta)$ be a point of σ_δ such that $\mid\xi\mid\geq 2\delta$, and let A_δ be the intersection of its $\delta-$neighborhood on the plane ξ,η with the domain G. Consider the cylinder

$$B_\delta=[-\frac{1}{2}-r_1,T+\delta+r_1]\times A_\delta.$$

Let us show that in this domain the functions w^n_ε have their derivatives up to the fourth order bounded by a constant independent of ε. We may assume that in B_δ the coefficient a_1 depends merely on τ, whereas a_2 and a_3 depend only on ξ and η.

In the domain A_δ, let us introduce new coordinates ξ' and η', so that the part of the boundary σ belonging to A_δ would turn into a subset of the straight line $\eta'=0$, and the direction of the normal n to σ would coincide with that of the η'-axis. In the new coordinates, again denoted by ξ, η, the boundary condition (17) takes the form $\partial w^n_\varepsilon/\partial\eta=F^*_\varepsilon$.

Let $Y(\tau,\xi,\eta)$ be a function defined on B_δ which satisfies the condition

$$\frac{\partial Y}{\partial\eta}\mid_{\eta=0}=F^*_\varepsilon.$$

The function $z\equiv w^n_\varepsilon-Y$ satisfies the equation

$$M(z)\equiv(\varepsilon+a_1)z_{\tau\tau}-z_\tau+a_{11}z_{\xi\xi}+2a_{12}z_{\xi\eta}$$
$$+a_{22}z_{\eta\eta}+b_1z_\xi+b_2z_\eta-2(\varepsilon+a_1)z=f^*_\varepsilon,\quad (21)$$

in B_δ and also the condition $z_\eta=0$ on S, where

$$a_{11}\alpha_1^2+2a_{12}\alpha_1\alpha_2+a_{22}\alpha_2^2\geq\lambda_0(\alpha_1^2+\alpha_2^2),$$

the constant λ_0 being positive and independent of ε.

In order to estimate the first order derivatives of z with respect to ξ and η, consider the function

$$\Lambda_1=\rho^2_\delta(\xi,\eta)[z^2_\xi+z^2_\eta]+c_1z^2+c_2\eta,\quad c_{2>0}$$

Here the constant c_1 is assumed sufficiently large and will be chosen later; $\rho_\delta(\xi,\eta)=1$ in $A_{\delta/2}$, and $\rho_\delta(\xi,\eta)=0$ in a small neighborhood of the boundary of A_δ that does not belong to σ; $\rho_{\delta\eta}=0$ on σ.

It is easy to see that $\partial\Lambda_1/\partial\eta=c_2>0$ on S and, therefore, Λ_1 cannot attain its largest value on S. If the maximum of Λ_1 is reached on the boundary of B_δ at a point where $\rho_\delta=0$, then

$$\Lambda_1\leq\max(c_1z^2+c_2\eta)\leq c_3,$$

where c_3 is a constant independent of ε. It is easy to verify that for large enough c_1 we have $M(\Lambda_1)-\Lambda_1\geq -c_4$ in B_δ, provided that c_4 is sufficiently large. Therefore, if Λ_1 takes its largest value inside B_δ, then $\Lambda_1\leq c_4$.

As shown above, for $\tau=T+\delta+r_1$ and $\tau=-1/2-r_1$, the function Λ_1 is uniformly bounded in ε. Thus, Λ_1 in B_δ is bounded by a constant independent of ε and, therefore, z_ξ, z_η are bounded in B_{δ_1},$\delta_1<\delta$.

Let us rewrite equation (21) in the firm

$$M(z)\equiv\Gamma(z)+M^1(z)=f^*_\varepsilon,\ \Gamma(z)\equiv(\varepsilon+a_1)z_{\tau\tau}-z_\tau.$$

The coefficients of the operator M^1 may be assumed independent of τ. Hence, it is easy to see that Γ satisfies the following equation and the boundary condition:

$$M(\Gamma)\equiv\Gamma(\Gamma)+M^1(\Gamma)=\Gamma(f^*_\varepsilon)\quad in\quad B_\delta,$$

$$\Gamma_\eta \mid_{\eta=0}= 0 \quad on \quad S. \tag{22}$$

In B_{δ_1} consider the function

$$\Lambda_2 \equiv \rho^2_{\delta_1}[z^2_{\xi\xi} + z^2_{\xi\eta} + \Gamma^2(z)] + c_5(z^2_\xi + z^2_\eta) + c_6\eta.$$

Making use of equations (21) and (22), we easily find that

$$M(\Lambda_2) - \Lambda_2 \geq c_7 \quad in \quad B_{\delta_1},$$

$$\frac{\partial \Lambda_2}{\partial \eta} = c_6 > 0 \quad on \quad S.$$

provided that $c_5 > 0$ is sufficiently large. Hence we see that Λ_2 is uniformly bounded in B_{δ_1}, and $\Gamma(z)$, $z_{\xi\xi}$,$z_{\xi\eta}$ are uniformly bounded in B_{δ_2}, $\delta_2 < \delta_1$. It follows from (21) that $z_{\eta\eta}$ is also bounded uniformly in ε. If we consider an equation of the form $(a_1 + \varepsilon)z_{\tau\tau} - z_\tau = \Gamma$ for z_τ and use the fact that Γ is bounded in B_{δ_2} and z_τ is bounded for $\tau = -1/2 - r_1$ and $\tau = T + \delta + r_1$, we easily find that z_τ is bounded in B_{δ_2} uniformly with respect to ε.

Since $\Gamma(z)$ is bounded in B_{δ_2} and satisfies equation (22), as well as the boundary condition $\Gamma_\eta \mid_{\eta=0}= 0$ on S , we can consider functions similar to Λ_1 and Λ_2 for Γ in B_{δ_2} (as we have done for z) and thus obtain uniform estimates in B_{δ_3}, $\delta_3 < \delta_2$, for the derivatives

$$\Gamma_\xi, \quad \Gamma_\eta, \quad \Gamma_{\xi\xi}, \Gamma_{\xi\eta}, \Gamma_{\eta\eta}, \Gamma_\tau.$$

Differentiating equation (22) in τ, we obtain the following equation for Γ_τ:

$$(a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a'_1)\Gamma_{\tau\tau} + M_1(\Gamma_\tau) = (\Gamma(f^*_\varepsilon))_\tau,$$

as well as the boundary condition $\Gamma_{\tau\eta} \mid_{\eta=0}= 0$ on S. By assumption, $a'_1(\tau)$ is small in B_δ. Therefore, the equation for Γ_τ is similar to (22). Thus, for the derivatives of Γ of the form

$$\Gamma_{\tau\xi}, \quad \Gamma_{\tau\eta}, \quad \Gamma_{\tau\xi\xi}, \quad \Gamma_{\tau\eta\xi},$$

$$(a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a'_1)\Gamma_{\tau\tau}, \Gamma_{\tau\eta\eta}, \Gamma_{\tau\tau}.$$

in B_{δ_4}, $\delta_4 < \delta_3$, we obtain estimates uniform in ε, just as we have done for z.

Similar arguments applied to $\Gamma_{\tau\tau}$ allow us to show that the derivatives

$$\Gamma_{\tau\tau\xi}, \quad \Gamma_{\tau\tau\eta}, \quad \Gamma_{\tau\tau\xi\xi}, \quad \Gamma_{\tau\tau\eta\xi},$$

$$(a_1 + \varepsilon)\Gamma_{\tau\tau\tau\tau} - (1 - a'_1)\Gamma_{\tau\tau\tau}, \Gamma_{\tau\tau\eta\eta}, \Gamma_{\tau\tau\tau}.$$

are bounded in B_{δ_5}, $\delta_5 < \delta_4$, uniformly with respect to ε.

These estimates show that in B_{δ_5} the third and the fourth order derivatives of z involving more than one differentiation in τ are bounded uniformly in ε, while the first order derivatives of $\Gamma(\Gamma)$ in ξ and η satisfy the Lipschitz condition in ξ, η with constants independent of ε, τ.

From the Schauder estimates for an elliptic equation of the form

$$M^1(\Gamma) = -\Gamma(\Gamma) + \Gamma(f^*_\varepsilon).$$

it follows that the derivatives of Γ in ξ and η up to the third order are bounded and satisfy the Hölder condition in B_{δ_6}, $\delta_6 < \delta_5$, uniformly with respect to ε and τ. Schauder estimates for the solution z of equation (21) written in the form

$$M^1(z) = -\Gamma(z) + f^*_\varepsilon,$$

allow us to claim that z has its derivatives in ξ and η up to the forth order bounded in B_{δ_7}, $\delta_7 < \delta_6$, uniformly with respect to ε and τ.

Thus, we have obtained estimate for the derivatives of w^n_ε in τ, ξ, η up to the fourth order in a neighborhood of the entire boundary S, except for a neighborhood of S_0 and a neighborhood w of the intersection $S \cap \{\xi = 0\}$ which is a subset of the plane $\{\eta = \eta_1\}$.

In equation (16) and the boundary condition (17), let us pass to another unknown function W given by

$$w = We^{\varphi_2(\eta)}, \quad \varphi_2(\eta) = -\alpha\eta\frac{\eta_1 - \eta}{\eta_1},$$

$$\alpha = const > 0.$$

For W we obtain the following equations

$$\frac{\partial W}{\partial \eta} - \alpha W = (F)_\varepsilon \quad for \quad \eta = 0,$$

$$-\frac{\partial W}{\partial \eta} - \alpha W = (F)_\varepsilon \quad for \quad \eta = \eta_1.$$

In order to estimate the first order derivatives of w^n_ε in Q, consider the following function in $Q_{r_1} \cap \{-\frac{1}{2} - r_1 < \tau < T + \delta + r_1\}$:

$$X_1 = W^2_\xi + W^2_\tau + W_\eta(W_\eta - 2Y) + k(\eta),$$

where $Y = (\alpha W + (F)_\varepsilon)\kappa_1(\eta)$, we define function $\kappa_1(\eta)$ such that $\kappa_1(\eta) = 1$ for $\mid \eta \mid < \delta$; $\kappa_1(\eta) = -1$ for $\mid \eta - \eta_1 \mid < \delta$; $\kappa_1(\eta) = 0$ for $2\delta < \eta < \eta_1 - 2\delta$, $k(\eta)$ is a positive function to chosen later. Clearly, $\partial W/\partial\eta - Y = 0$ on the parts of the boundary S belonging to the planes $\eta = 0$ or $\eta = \eta_1$. We have

$$\frac{\partial X_1}{\partial \eta} \mid_{\eta=0}= 2W_\xi W_{\xi\eta} + 2W_\tau W_{\tau\eta} - 2W_\eta Y_\eta + k'(0)$$

$$= 2\alpha[W^2_\xi + W^2_\tau] - 2YY_\eta + 2W_\xi(F)_{\varepsilon\xi} + 2W_\tau(F)_{\varepsilon\tau} +$$

$$+k'(0) > 0,$$

provided that $k'(0)$ is positive and sufficiently large. Likewise, taking $k(\eta)$ in X_1 such that $k'(\eta_1)$ is negative and sufficiently large in absolute value, we find that

$$\frac{\partial X_1}{\partial \eta} \mid_{\eta=\eta_1} < 0.$$

By the methods already used in the proof of Lemma 5 we obtain

$$L^{0\varepsilon}(X_1) + c_8 X_1 \geq -c_9, \tag{23}$$

where

$$L^{0\varepsilon}(W) \equiv L^{\varepsilon}(W) + 2[(\varepsilon + a_3) + \nu(w^{n-1})^2_{\varepsilon}]\varphi_{2\eta}\frac{\partial W}{\partial \eta}$$

$$+\{(\nu(w^{n-1})^2_{\varepsilon} + \varepsilon + a_3)[\varphi_{2\eta\eta} + (\varphi_{2\eta})^2]$$

$$+(p_x + 2\nu_{\eta}(w^{n-1})^2)_{\varepsilon}\varphi_{2\eta}\}W,$$

the constant c_8 and c_9 do not depend on ε. In Q_{r_1}, consider the function

$$X_1^* = X_1 e^{-\beta t}, \quad \beta = const > 0.$$

For sufficiently large β, the coefficient of X_1^* in (23) is less than -1. It follows from (23) that if X_1^* takes its largest value inside Q_{r_1}, then X_1^* is bounded by a constant independent of ε.

Neither for $\eta = 0$ nor for $\eta = \eta_1$ can X_1^* attain its largest value. It follows from the above estimates that on the remaining part of the boundary of Q_{r_1} the function X_1^* is bounded uniformly in ε. Likewise, we can estimate the second and the third order derivatives of w^n_{ε} by considering the functions

$$X_2 = W^2_{\tau\tau} + W^2_{\xi\xi} + W^2_{\tau\xi} + W_{\eta\xi}(W_{\eta\xi} - 2Y_{\xi})$$

$$+W_{\eta\tau}(W_{\eta\tau} - 2Y_{\tau}) + g_1^2(\eta)W^2_{\eta\eta} + k(\eta),$$

$$X_3 = (X_3)' + g_1^2(\eta)[W^2_{\eta\eta\eta} + W^2_{\eta\eta\xi} + W^2_{\eta\eta\tau}]$$

$$+W_{\eta\xi\xi}(W_{\eta\xi\xi} - 2Y_{\xi\xi}) + W_{\eta\tau\tau}(W_{\eta\tau\tau} - 2Y_{\tau\tau})$$

$$+W_{\eta\xi\tau}(W_{\eta\xi\tau - 2Y_{\xi\tau}}) + k(\eta),$$

where $(X_3)'$ stands for the sum of third order derivatives of W in ξ and τ

$$g_1(\eta) = \begin{cases} 0 & for \quad \eta < \frac{\delta}{2} \quad or \quad \eta > \eta_1 - \frac{\delta}{2}, \\ 1 & for \quad \eta_1 - \delta > \eta > \delta. \end{cases}$$

The required estimates for X_2 and X_3 can be obtained by the method used above in relation to X_1; in order to establish the inequality of type (23) for X_2 and X_3, we can use the fact that in (16) the coefficient of $W_{\eta\eta}$ is positive for $\eta < \delta$ and $\eta_1 - \eta < \delta$, as we have done in the proof of Lemma 6. While estimating the fourth order derivatives of W, the following observations are useful.

Consider the function

$$X_4 = (X_4)' + g_1^2(\eta)(X_4)'' + W_{\eta\xi\xi\xi}(W_{\eta\xi\xi\xi} - 2Y_{\xi\xi\xi})$$

$$+W_{\eta\tau\tau\tau}(W_{\eta\tau\tau\tau} - 2Y_{\tau\tau\tau}) + W_{\eta\xi\xi\tau}(W_{\eta\xi\xi\tau} - 2Y_{\xi\xi\tau})$$

$$+W_{\eta\tau\tau\xi}(W_{\eta\tau\tau\xi} - 2Y_{\tau\tau\xi}) + k(\eta),$$

where $(X_4)'$ is the sum of squared fourth order derivatives of W except those involving a differentiation in η, and $(X_4)''$ is the sum of squared fourth order derivatives of W involving more than one differentiation in η.

The expression for X_4 contains third order derivatives of Y and, therefore, of $(F)_{\varepsilon}$. The operator $L^{0\varepsilon}(X_4)$ can be estimated through the expressions

$$L^{0\varepsilon}(Y_{\tau\tau\tau}), \quad L^{0\varepsilon}(Y_{\xi\xi\xi}), \quad L^{0\varepsilon}(Y_{\tau\tau\xi}), \quad L^{0\varepsilon}(Y_{\tau\xi\xi}).$$

which contain fifth order derivatives of $(F)_{\varepsilon}$. By construction, F is infinitely differentiable outside the δ-neighborhood of S_0 and has its fourth order derivatives bounded in ε on S. In the intersection of the domain Q with the δ-neighborhood of S_0, the operator $L^{0\varepsilon}$ involves second order derivatives in ξ and τ with the coefficient ε, namely,

$$\varepsilon\frac{\partial^2}{\partial \tau^2}, \quad \varepsilon\frac{\partial^2}{\partial \xi^2}.$$

Since F has its fourth order derivatives bounded in ε, the fifth order derivatives of its regularization $(F)_{\varepsilon}$ can be written as $O(\varepsilon^{-1})$. Therefore, the operator $L^{0\varepsilon}$ applied to the third order derivatives of $(F)_{\varepsilon}$ results in a quantity uniformly bounded in ε. For the rest, the proof of the estimate for X_4 literally follows the case of X_1, X_2, and X_3. Thus, we finally see that the derivatives of w^n_{ε} up to the fourth order are bounded uniformly in ε.

Theorem 12 *The solutions w^n_{ε} of problem (16)(17) in Q converge, as $\varepsilon \to 0$, to the function w^n which is a solution of problem (8)(9) in Ω and has its derivatives up to the fourth order bounded in Ω.*

Proof: By Lemma 11, the derivatives of w^n_{ε} up to the fourth order are uniformly bounded in ε. Therefore, there is a subsequence $w^n_{\varepsilon k}$ such that $w^n_{\varepsilon k}$, together with their derivatives up to the third order, are uniformly convergent to w^n in Q as $\varepsilon_k \to 0$. The limit function $w^n(\tau, \xi, \eta)$ satisfies equation (8) in Ω, as well as the boundary condition (10). Let us show that the condition in (9) hold for w^n. To this end, we prove that $w^n = w^*$ in $Q \setminus \Omega_1$. Set $z = w^n - w^*$. By construction, we have

$$a_1 z_{\tau\tau} + a_2 z_{\xi\xi} + a_3 z_{\eta\eta} + \nu(w^*)^2 z_{\eta\eta} - z_\tau - \eta z_\xi$$

$$+[p_x + 2\nu_\eta(w^*)^2]w_\eta - 2a_1 z = 0,$$

in $Q \setminus \Omega_1$, and $\partial z/\partial n = 0$ on the part of the boundary of $Q \setminus \Omega_1$ that belongs to S. In $Q \setminus \Omega_1$, consider the function z^* defined by $z = z^*\psi_1(\tau)$, where $\psi_1(\tau)$ is the function constructed in the proof of Lemma 10. For z^* we obtain an equation in $Q \setminus \Omega_1$ with the coefficient of z^* being strictly negative in the closure of $Q \setminus \Omega_1$.

Let $E(\tau, \xi, \eta)$ be a smooth function in Q such that $\partial E/\partial n < 0$ on S, and $E > 1$. Set $z_1 = z^*(E + C)$, where C is a positive constant. It is easy to see that in the equation for z_1 the coefficient of z_1 is negative if C is sufficiently large. The boundary condition on S for z_1 will have the form

$$\frac{\partial z_1}{\partial n} - \alpha_1 z_1 = 0, \quad where \quad \alpha_1 = -\frac{\partial E}{\partial n} > 0.$$

Clearly, $\mid z_1 \mid$ cannot attain its largest value on S, for at the point of maximum of $\mid z_1 \mid$ on S we must have

$$z_1 \frac{\partial z_1}{\partial n} - \alpha_1 (z_1)^2 < 0,$$

which is incompatible with the boundary condition on S. The largest value of $\mid z_1 \mid$ cannot be attained inside $Q \setminus \Omega_1$, for at the point of its maximum we must have $z_{1\tau} = 0, z_{1\xi} = 0, z_{1\eta} = 0, z_1 z_{1\eta\eta} \le 0, z_1 z_{1\xi\xi} \le 0$, and $z_1 z_{1\tau\tau} \le 0$, which is in contradiction with the equation obtain for z_1 at that point.

In a similar way, it can be shown that the maximum of $\mid z_1 \mid$ can be attained neither for $\tau = 0$ nor for $\xi = 0$ on the boundary of $Q \setminus \Omega_1$. It follows that $z_1 \equiv 0$ in $Q \setminus \Omega_1$ and, therefore, $w^n \equiv w^*$ in $Q \setminus \Omega_1$. Hence, we see that

$$w^n(0, \xi, \eta) = w_0, \qquad w^n(\tau, 0, \eta) = w_1.$$

Let us show that $w^n = 0$ on the surface $\eta = U(\tau, \xi)$. It follows from the above results that $w^n = 0$ for $\tau = 0$ and $\eta = U(0, \xi)$, as well as for $\xi = 0$ and $\eta = U(\tau, 0)$. Since $w^{n-1} = 0$ on the surface $\eta = U(\tau, \xi)$, the equation

$$w^n_\tau + \eta w^n_\xi - p_x w^n_\eta = 0,$$

holds for w^n on that surface. As indicated above, the vectors $(1, \eta, -p_x)$ belong to planes tangential to the surface $\eta = U(\tau, \xi)$ and form a vector field on that surface. Integral curves of that field, being extended for smaller values of τ, will cross the border of the surface either at $\xi = 0$ or at $\tau = 0$, where $w^n = 0$. Since w^n is constant on these integral curves, $w^n = 0$ on the entire surface $\eta = U(\tau, \xi)$. Note that the function w^n constructed above has its third order derivatives in Ω satisfying the Lipschitz condition.

Theorem 13 *Assume that $p(t, x)$, $v_0(t, x)$, $u_0(x, y)$, $u_1(t, y)$, $w_0(\xi, \eta)$, $w_1(\tau, \eta)$, $\nu(y)$, $g(t, x)$ are sufficiently smooth and satisfy the compatibility conditions which amount to the existence of the function w^* mentioned earlier. Then there is one and only one solution of problem (1)-(3) in the domain D, with X being arbitrary and T depending on the data of problem (1)-(9), or T being arbitrary and X depending on the data. This solution has the following properties: $u > 0$ for $y > 0$, $u_y > 0$ for $y \ge 0$; the derivatives $u_t, u_x, u_y, u_{yy}, v_y$ are continuous and bounded in $\bar{D}$; moreover, the ratios*

$$\frac{u_{yy}}{u_y}, \frac{u_{yyy}u_y - u_{yy}^2}{u_y^3},$$

are bounded in D.

Proof: Let w be the solution of problem (5)-(7) constructed in the proof of Theorem 12. Let u be defined by the condition $w = u_y$, or

$$y = \int_0^u \frac{ds}{w(t, x, s)}. \tag{24}$$

Since $w(t, x, s) > 0$ for $s < U(t, x)$, and $w = 0$ for $s = U(t, x)$, we have $u \to U(t, x)$ as $y \to \infty$, and $0 < u < U(t, x)$ for $0 < y < \infty$, $u(t, x, 0) = 0$. The condition $u(0, x, y) = u_0$ and $u(t, 0, y) = u_1$ are also valid, since $w_0 = u_{0y}$ and $w_1 = u_{1y}$. The function defined by (24) possesses the derivatives

$$u_y = w, \quad u_{yy} = w_\eta w, \quad u_{yyy} = w_{\eta\eta} u_y^2 + w_\eta u_{yy},$$

$$\nu_y = \nu_\eta w, \quad \nu_{yy} = \nu_{\eta\eta} w^2 + \nu_\eta w_\eta w.$$

The derivatives u_t and u_x are given by

$$u_t = -w \int_0^u \frac{w_t(t, x, s)}{w^2(t, x, s)} ds,$$

$$u_x = -w \int_0^u \frac{w_x(t, x, s)}{w^2(t, x, s)} ds.$$

Set

$$v = \frac{-u_t - u u_x - p_x + (\nu u_y)_y}{u_y}. \tag{25}$$

Let us show that u and v defined by (24) and (25) satisfy system (1). Differentiating the relation $u_y = w$, we find that there exist the derivatives

$$u_{yx} = w_\xi + u_x w_\eta, \quad u_{yt} = w_\tau + u_t w_\eta.$$

Therefore, v admits the derivatives in y. Differentiating (25) in y, we obtain

$$v_y u_y + v u_{yy} = -u_{ty} - u_y u_x -$$

$$u u_{xy} + \nu_{yy} u_y + 2\nu_y u_{yy} + \nu u_{yyy}. \qquad (26)$$

The function w satisfies equation (5). Replacing in (5) the derivatives of w by their expressions in terms of derivatives of u, we find that

$$\nu u_y^2 \frac{u_y u_{yyy} - u_{yy}^2}{u_y^3} - u_{yt} + u_t \frac{u_{yy}}{u_y} - u(u_{yx} - \frac{u_x u_{yy}}{u_y}) +$$

$$+ p_x \frac{u_{yy}}{u_y} + \nu_{yy} u_y + \nu_y u_{yy} = 0. \qquad (27)$$

It follows from (26) and (27) that

$$u_x + v_y = 0. \qquad (28)$$

Let us show that $v(t,x,0) = v_0(t,x)$. It follows from (7) that

$$v_0 = (\frac{\nu w w_\eta - p_x + \nu_\eta w^2 - g_\tau - g g_\xi}{w}) \mid_{\eta=g} .$$

From (25) we find that

$$v \mid_{y=0} = (\frac{-u_t - u u_x - p_x + (\nu u_y)_y}{u_y}) \mid_{y=0}$$

$$= (\frac{\nu w w_\eta - p_x + \nu_\eta w^2 - g_\tau - g g_\xi}{w}) \mid_{\eta=g} = v_0.$$

Thus we have proved the existence of a solution for problem (1)-(3) in the class of smooth functions. Its uniqueness is able to been established by a similar way as Theorem 4.2.2 of [3], we omit the details here.

5 Appendix

At the last of the paper, we give the details of the proof to the inequality (11). As in the section 2, we have

$$\Phi_n = \Phi_n^* \equiv (W_\tau^n)^2 + (W_\xi^n)^2 + (W_\eta^n)^2 + k_0 + k_1\eta.$$

Applying the operator

$$2W_\tau^n \frac{\partial}{\partial \tau} + 2W_\xi^n \frac{\partial}{\partial \xi} + 2W_\eta^n \frac{\partial}{\partial \eta}$$

to the equation

$$L_n^0(W^n) + B^n W^n = 0,$$

we find that

$$\begin{aligned}
&0 = \nu(w^{n-1})^2 \Phi^*_{n\eta\eta} - \Phi^*_{n\tau} - \eta\Phi^*_{n\xi} + A^n \Phi^*_{n\xi} + 2B^n \Phi^*_n \\
&-2\nu(w^{n-1})^2\{(W^n_{\tau\eta})^2 + (W^n_{\xi\eta})^2 + (W^n_{\eta\eta})^2\} \\
&+[2\nu((w^{n-1})^2)_\tau W^n_{\eta\eta} W^n_\tau + 2\nu((w^{n-1})^2)_\xi W^n_{\eta\eta} W^n_\xi \\
&+2\nu((w^{n-1})^2)_\eta W^n_{\eta\eta} W^n_\eta] + [-2W^n_\xi W^n_\eta + 2A^n_\eta (W^n_\eta)^2 \\
&+2A^n_\xi W^n_\eta W^n_\xi + 2A^n_\tau W^n_\eta W^n_\tau \\
&+2W(B^n_\eta W^n_\eta + B^n_\xi W^n_\xi + B^n_\tau W^n_\tau)] \\
&-B^n(k_1\eta + k_0) - A^n k_1 + 2W^n_\tau[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\tau \\
&+2W^n_\xi[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\xi + 2W^n_\eta[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\eta.
\end{aligned} \qquad (29)$$

for the last three terms, we have

$$\begin{aligned}
&2W^n_\tau[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\tau + 2W^n_\xi[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\xi \\
&+2W^n_\eta[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\eta \\
&= 2\alpha W^n_\tau \nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3 \\
&+2W^n_\tau \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\tau + 2\alpha W^n_\xi \nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3 \\
&+2W^n_\xi \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\xi + 2W^n_\eta \nu_{\eta\eta\eta} e^{\alpha\eta}(w^{n-1})^3 \\
&+2\alpha W^n_\eta \nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3 + 2W^n_\eta \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\eta.
\end{aligned}$$

according to $ab \le \frac{a^2+b^2}{2}$,ν,$\nu_{\eta\eta}$,$\nu_{\eta\eta\eta}$ are bounded, we have

$$\begin{aligned}
&2\alpha W^n_\tau \nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3 + 2\alpha W^n_\xi \nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3 \\
&+(2\alpha\nu_{\eta\eta} + 2\nu_{\eta\eta\eta}) W^n_\eta e^{\alpha\eta}(w^{n-1})^3 \\
\le\ & h_1[(W^n_\tau)^2 + (w^{n-1})^6] + h_2[(W^n_\xi)^2 + (w^{n-1})^6] \\
&+h_3[(W^n_\eta)^2 + (w^{n-1})^6] \\
\le\ & h[(W^n_\tau)^2 + (W^n_\xi)^2 + (W^n_\tau)^2] + h(w^{n-1})^6,
\end{aligned}$$

where h_1, h_2, h_3 are taken large enough, $h = h_1 + h_2 + h_3$

$$\begin{aligned}
&2W^n_\tau \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\tau + 2W^n_\xi \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\xi \\
&+2W^n_\eta[\nu_{\eta\eta} e^{\alpha\eta}(w^{n-1})^3]_\eta \\
\le\ & h[(W^n_\tau)^2 + (W^n_\xi)^2 + (W^n_\tau)^2] + h(w^{n-1})^6 \\
+\ & 2W^n_\tau \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\tau + 2W^n_\xi \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\xi \\
&+2W^n_\eta \nu_{\eta\eta} e^{\alpha\eta}[(w^{n-1})^3]_\eta \\
\le\ & h[(W^n_\tau)^2 + (W^n_\xi)^2 + (W^n_\tau)^2] + h(w^{n-1})^6 \\
&+h_4[(W^n_\tau)^2 + (W^n_\xi)^2 + (W^n_\eta)^2] \\
&+\frac{\nu_{\eta\eta} e^{\alpha\eta}}{h_4}\{[((w^{n-1})^3)_\tau]^2 + [((w^{n-1})^3)_\xi]^2 \\
&+[((w^{n-1})^3)_\eta]^2\}.
\end{aligned}$$

Denote by I_1 the terms in the first square brackets in (29), we obtain the following estimate from above

$$\begin{aligned}
I_1 \ \le\ & R_1[(W^n_\tau)^2 + (W^n_\xi)^2 + (W^n_\eta)^2] \\
&+\frac{\nu}{R_1}\{[((w^{n-1})^2)_\tau]^2 + [((w^{n-1})^2)_\xi]^2 \\
&+[((w^{n-1})^2)_\eta]^2\}(W^n_{\eta\eta})^2,
\end{aligned}$$

where R_1 is a constant.

It is well known that (see for instance [19]) any non-negative function $q(x)$ defined on interval $-\infty < x < +\infty$ and having bounded second derivatives on that interval satisfies the inequality

$$(q_x)^2 \leq 2\{\max \mid q_{xx} \mid\} q(x).$$

The function $(w^{n-1})^3$ or $(w^{n-1})^2$ can be extended to the entire real axis with respect to any of its independent variable, so that its extension is a non-negative bounded function whose second derivative has its absolute value less than or equal to the maximum modulus of the second derivative of $(w^{n-1})^3$ or $(w^{n-1})^2$. Therefore

$$\frac{\nu_{\eta\eta} e^{\alpha(\eta-g)}}{h_4}\{[((w^{n-1})^3)_\tau]^2 + [((w^{n-1})^3)_\xi]^2 + [((w^{n-1})^3)_\eta]^2\} \leq \mid \nu_{\eta\eta} \mid (w^{n-1})^3,$$

$$\frac{\nu^2}{R_1}\{[((w^{n-1})^2)_\tau]^2 + [((w^{n-1})^2)_\xi]^2 + [((w^{n-1})^2)_\eta]^2\}(W^n_{\eta\eta})^2 \leq \nu (w^{n-1})^2 (W^n_{\eta\eta})^2,$$

where R_1, h_4 is chosen sufficiently large. The constant R_1 depends on the second derivative of the functions $(w^{n-1})^2$ and h_4 depends on the second derivative of the functions $(w^{n-1})^3$.

Denote by I_2 the terms enclosed by the second square brackets in (29). By virtue of the inequality $2ab \leq a^2 + b^2$, these terms can be estimated from above by the expression $R_2\Phi^*_n + k_8$, where the constant R_2 depends on the first order derivative of the functions w^{n-1}, k_8 is independent of n.

Therefore, in the region $\eta \geq \delta_2$, we have

$$L^0_n(\Phi_n) + R_3\Phi_n + k_9 \geq 0 \quad or \quad L^0_n(\Phi_n) + R^n\Phi_n \geq 0,$$

where the constant k_9 is independent of n, and the function R^n depends on the first and the second derivatives of w^{n-1}.

In order to estimate $L^0_n(\Phi_n)$ in Ω for $\eta \leq \delta_2$, we should also calculate $L^0_n(2W^n_\eta H^n)$

$$\begin{aligned} L^0_n(2W^n_\eta H^n) &= 2H^n L^0_n(W^n_\eta) + 2W^n_\eta L^0_n(H^n) \\ &\quad + 4\nu(w^{n-1})^2 W^n_{\eta\eta} H^n_\eta \\ &= 2H^n\{-\nu(w^{n-1})^2_\eta W^n_{\eta\eta} + W^n_\xi - A^n_\eta W^n_\eta \\ &\quad - B^n_\eta W^n - B^n W^n_\eta - [\nu_{\eta\eta}(w^{n-1})^3 e^{\alpha(\eta-g)}]_\eta\} \\ &\quad + 2W^n_\eta[L^0_n(\frac{v_0}{\nu}) + L^0_n(\frac{p_x}{\nu W^{n-1}}) - \alpha\chi(\eta)B^n W^n \\ &\quad - \alpha\chi(\eta)\nu_{\eta\eta}(w^{n-1})^3 e^{\alpha(\eta-g)} + \alpha W^n L^0(\chi) \\ &\quad + 2\alpha\nu(w^{n-1})^2 W^n_\eta \chi' + L^0_n(\frac{g_\tau - g g_\xi}{\nu W^{n-1}} - \frac{\nu_\eta W^{n-1}}{\nu})] \\ &\quad + 4\nu(W^{n-1})^2 W^n_{\eta\eta} H^n_\eta. \end{aligned}$$

According to Lemma 3 and Lemma 4, $(w^{n-1})^2 \geq \gamma_0 > 0$ for $\eta \leq \delta_2$. There, the terms I_1 in (11) together with $2H^n\nu(w^{n-1})^2_\eta W^n_{\eta\eta}$, can be estimated with the help of the inequality $2ab \leq a^2/h + hb^2$ as follows:

$$I_1 + 2H^n\nu(w^{n-1})^2_\eta W^n_{\eta\eta} \leq \frac{1}{2}\nu\gamma_0(W^n_{\eta\eta})^2 + R_4\Phi_n + k_{10},$$

where the constant R_4 does not depend on n, It follows from (11) and $L^0_n(2W^n_\eta H^n)$ that

$$L^0_n(\Phi_n) + R_5\Phi_n + R_6 \geq 0 \quad for \quad \eta \leq \delta_2,$$

where R_5 and R_6 are constant that depend neither on w^{n-1} nor on its derivatives up to the second order. Since $\Phi_n \geq 1$, we have $R_6\Phi_n \geq R_6$, Therefore

$$L^0_n(\Phi_n) + R^n\Phi_n \geq 0.$$

Acknowledgements: The research was supported by the NSF of Fujian Province of China (grant No. 2012J01011) and supported by SF of Xiamen University of Technology, China.

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