The Existence of Solution for the Nonstationary Two Dimensional Microflow Boundary Layer System

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Abstract: The paper concerns with the nonstationary two dimensional microflow boundary layer system. By posing some restrictions on the viscous function, the existence and the uniqueness of local solutions to the system are got. The main technique we used in the paper is Oleinik line method based on a successive approximation, which is used in the study of Prandtl system. However, the corresponding calculations in our paper are much more complicated.

Key–Words: Two dimensional microflow boundary layer system, Prandtl system, Existence, Local solution.

1 Introduction

As we known, the Prandtl system is a simplification of the Navier-Stokes system and describes the motion of a fluid with small viscosity about a solid body in a thin layer which is formed near its surface owing to the adhesion of the viscous fluid to the solid surface. Assume that the motion of a fluid occupying a two dimensional region is characterized by the velocity vector \( V = (u, v) \), where \( u, v \) are the projections of \( V \) onto the coordinate axes \( x, y \), respectively, the Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form

\[
\begin{align*}
\{ u_t + uu_x + vu_y = \nu u_{yy} - p_x, \\
u_x + v_y = 0, \\
u(0, x, y) = u_0(x, y), \\
u(t, 0, y) = u_1(t, y), \\
u(t, x, 0) = 0, \\
\lim_{y \to \infty} \nu(t, x, y) = U(t, x).
\end{align*}
\]

in a domain \( D = \{0 < t < T, 0 < x < X, 0 < y < \infty\} \), where \( \nu = \text{const} > 0 \) is the viscosity coefficient of the incompressible fluid. \( U_t + UU_x = -p_x(t, x), \\
U(t, x) > 0, u_0 > 0, u_1 > 0 \) for \( y > 0, u_{0y} > 0, u_{1y} > 0 \) for \( y \geq 0, U = U(t, x) \) is the velocity at the outer edge of the boundary layer, \( p = p(t, x) \) is the pressure. The density of the fluid \( \rho \) is equal to 1. Prandtl boundary theory does not consider both the influence of wall’s properties on the characteristic of the boundary layer and the interaction of the actual solid wall with the flow of water. If one considers these influences, the Prandtl system should be modified to the following system

\[
\begin{align*}
\{ u_t + uu_x + vu_y = (\nu(y)u_y)y - p_x, \\
u_x + v_y = 0,
\end{align*}
\]

with the conditions

\[
\begin{align*}
u(0, x, y) = u_0(x, y), \\
u(t, 0, y) = u_1(t, y), \\
u(t, x, 0) = 0, \\
\lim_{y \to \infty} \nu(t, x, y) = U(t, x).
\end{align*}
\]

where \( t, x, y \in D, \nu(y) \) is a boundary function, \( \nu \) and \( g(t, x) \) satisfies some other restrictions.

In recent decades, many scholars have been carrying out research in two dimensional boundary layer, achievements are abundant in literature on theoretical, numerical experimental aspects of the theory[1,2]. In particular, Oleinik had got the existence and the uniqueness of solutions for the Prandtl system by two different kinds of line methods, one of them is based on Rothe’s method[3], another one is based on a successive approximation[4]. If \( \nu \) is a sufficiently large positive function, which means that \( \nu(y) > \nu_0 > 0, \nu_0 \) is a constant, the system (1)-(3) is called the microflow boundary layer[4], and Li-Zhan [5] had got the local well posedness of the system by a similar method as [3], which is base on Rothe’s method. Also, there are many papers to deal with the other related problems in the boundary layer theory, such as the relation between the Navier-Stokes system and the Prandtl system and the long-time behavior of the solutions(see [6-15] and references therein).
In this paper, we used Oleinik’s successive approximation method to study the problem (1)-(3).

We use the following change of variables [3], which is known as Crocco transform,

$$\tau = t, \xi = x, \eta = u(t, x, y), w(\tau, \xi, \eta) = u_g.$$  \hspace{1cm} (4)

By calculation, we can get $u_g = w$, $u_{gg} = w_\eta w$, $u_{ggg} = w_{\eta\eta} w^2 + w_{\eta\eta}^2 w$, $u_{g\tau} = w_\eta u_t + w_\tau$, $u_{g\eta} = w_\eta u_x + \xi_\eta$, $u_{g\xi} = \nu_\eta w$, $u_{g\xi} = \nu_{\eta\eta} w^2 + \nu_{\eta\eta} w w$. From Eqs.(1)-(3) we obtain the following equation for $w$

$$L(w) = \nu w_{\eta\eta} w^2 - w_\tau - \eta w_\xi \eta + p_x w_\eta + \nu_{\eta\eta} \nu w^3$$
$$+ 2 \nu_\eta w_\eta w^2 = 0,$$ \hspace{1cm} (5)

in the domain $\Omega = \{ 0 < \tau < T, 0 < \xi < X, g(\tau, \xi) < \eta < U(\tau, \xi) \}$, with the conditions

$$\begin{align*}
w |_{\tau=0} &= w_0(\xi, \eta), \\
w |_{\eta=U(\tau, \xi)} &= 0.
\end{align*}$$ \hspace{1cm} (6)

$$l(w) = (\nu_\eta w_{\eta\eta} + \nu \nu_\eta w - p_x - \nu_\tau w - g_\tau$$
$$- g g_\xi) |_{\eta=g(\tau, \xi)} = 0,$$ \hspace{1cm} (7)

where $\nu(y)$, $g(t, x)$ turn into the corresponding functions of $\tau$, $\xi$ and $\eta$, but we still denoted them by $\nu(\tau, \xi, \eta)$, $g(\tau, \xi)$.

Clearly, if $w = c$, i.e. in the Prandtl boundary system, then (5) has the following simpler form

$$\nu w_{\eta\eta} w^2 - w_\tau - \eta w_\xi \eta + p_x w_\eta = 0.$$ 

Now, due to the nonlinear terms $\nu_{\eta\eta} w^3 + 2 \nu_\eta w_\eta w^2$, the problem becomes more difficult. In order to get the similar results as those of Prandtl boundary layer, some restrictions in $\nu$, $g$ have to be added. $0 < \nu_0 < \nu(y) < \nu_1$, where $\nu_i$, ($i = 0, 1$), are constants.

$\nu_{\eta\eta}$, $\nu_\eta$, $g_\tau$, $g_\xi$, $\nu_{\eta\eta}$ and $g_\xi$ all are bounded, $\nu_{\eta\eta} < 0, \nu_\eta < 0$ and $\eta(\tau, \xi) < \min \{ \frac{1}{4} U(\tau, \xi) \}$, where $U(\tau, \xi)$ is the press function of the flow outside the boundary layer.

2 Some important lemmas

**Definition 1** A function $w(\tau, \xi, \eta)$ is said to be a weak solution of problem (5)-(7), if $w$ has first order derivatives in equation (5) continuous in $\Omega$, and its derivative $w_{\eta\eta}$ continuous when $g(\tau, \xi) < \eta < U(\tau, \xi)$; $w$ satisfies equation (5) almost everywhere in $\Omega$, together with the conditions (6)(7).

The solution of problem (5)-(7) will be constructed as the limit of a sequence $w^n, n \to \infty$, which consists of solutions of the equations

$$L_n(w) \equiv \nu w_{\eta\eta} w^2 - w_\tau - \eta w_\xi \eta$$
$$+ p_x w_\eta + \nu_{\eta\eta} (w^{n-1})^3 + 2 \nu_{\eta\eta} w^n (w^{n-1})^2 = 0,$$ \hspace{1cm} (8)

supplemented by the conditions

$$\begin{align*}
w^n(0, \xi, \eta) &= w_0(\xi, \eta), \\
w^n(\tau, 0, \eta) &= w_1(\tau, \eta), \\
w^n(\tau, \xi, U(\tau, \xi)) &= 0.
\end{align*}$$ \hspace{1cm} (9)

At $w^0$ we take a function which is smooth in $\Omega$, satisfies the conditions (6), and is positive for $g(\tau, \xi) < \eta < U(\tau, \xi)$. We assume that there exists $\varphi_0(\tau, \xi, \eta)$ with the following properties: $\varphi_0$ is smooth in $\Omega$; $w_0 \geq \varphi_0(0, 0, \eta)$, $w_1 \geq \varphi_0(0, \eta)$, $\varphi_0 > 0$ for $g(\tau, \xi) < \eta < U(\tau, \xi)$; moreover,

$$\varphi_0 \equiv m_0(U(\tau, \xi) - \eta)^k,$$

for some $m_0 > 0$ and $k \geq 1$, provided that $U(\tau, \xi) - \eta < \delta_0$, where $\delta_0$ is a small positive constant.

Assuming that problem (8)(9) admits a solution $w^n(n = 1, 2, \ldots)$ with continuous third order derivatives in the closed domain $\Omega$, let us show that $w^n$ are convergent, as $n \to \infty$, to a solution of problem (5)-(7); after that we are going to show that the $w^n$ do exist, and we indicate a method for their approximation.

A solution will be constructed for problem (1)-(3) in the domain $\Omega$ for some $T = T_0$ and any $X$, as well as for some $X = X_0$ and any $T$. The constant $T_0$ and $X_0$ are determined by $u_0, u_1, v_0, p_x$.

**Lemma 2** Let $V$ be a smooth function such that $L_n(V) \geq 0$ in $\Omega$, $L_n(V) > 0$ for $\eta = g(\tau, \xi)$, and $V \leq w^n$ for $\tau = 0$ and $\xi = 0$. Assume that $w^{n-1} > 0$ for $\eta = g(\tau, \xi)$. Then $V \leq w^n$ everywhere in $\Omega$.

Let $V_1$ be a smooth function such that $L_n(V_1) \leq 0$ in $\Omega$, $L_n(V_1) < 0$ for $\eta = g(\tau, \xi)$, and $V_1 \geq w^n$ for $\tau = 0$ and $\xi = 0$. Assume that $w^{n-1} > 0$ for $\eta = g(\tau, \xi)$. Then $V_1 \geq w^n$ everywhere in $\Omega$.

**Proof:** Let us prove the first statement of Lemma 2. The difference $z = w^n - V$ satisfies the inequalities

$$L_n(z) = L_n(w^n) - L_n(V) \leq 0,$$
$$l_n(z) = l_n(w^n) - l_n(V) = \nu w^n \eta z_\eta < 0,$$

since $w^{n-1} > 0$ for $\eta = g(\tau, \xi)$. By assumption $V \leq w^n$ for $\tau = 0, \xi = 0$, we have $z \geq 0$ for $\tau = 0,$ and
z ≥ 0 for ξ = 0. Consider the function z₁ = ze−τ, clearly, z₁ ≥ 0 for τ = 0 and ξ = 0; z₁η < 0 for η = U(τ, ξ). It follow that z₁ can’t have a negative minimum at η = g(τ, ξ), since at the point of negative minimum z₁η > 0. The rest of the proof is similar as in [3]. The second statement of Lemma 2 can be proved in a similar fashion.

**Lemma 3** Suppose that νyy, η, ν, g, g, and g are all bounded. ψ(τ, ξ) < min U(τ, ξ)/2, g, k = eσ for 0 ≤ s ≤ 1, 1 ≤ k(s) ≤ 3 for s ≥ 1, and w = U(τ, ξ) − η such that for all ρ > 0 and all τ ≤ T₀ the inequalities

H₁(τ, ξ, η) ≥ w² ≥ h₁(τ, ξ, η),

hold in Ω, where H₁ and h₁ are continuous functions in Ω₁, h₁ ≥ 0 for η < η < U, τ < T₀.

**Proof:** Let us cut and construct functions V and V₁ satisfying the conditions of Lemma 2. To this end, we define a twice continuously differentiable function ψ(τ, ξ, η) such that ψ ≤ k(α₁(η − g)) for g < η < g + δ, 0 < δ < min U(τ, ξ)/2, g, k(s) = eσ for 0 ≤ s ≤ 1, 1 ≤ k(s) ≤ 3 for s ≥ 1, and w = U(τ, ξ) − η for U − η < δ; 0 < δ < 4 for δ < η < U − δ. Here δ₀ is a small constant. We define the functions V and V₁ by

V = mψe−ατ, V₁ = M(C − eβη)eατ,

where m, α, α₁, β, β₁, C, M are positive constants.

Let us show that T₀ and the constants in the definition of V and V₁ can be chosen independent of n, so that the inequality V ≤ w−¹ ≤ V₁ for τ ≤ T₀ implies that V ≤ w ≤ V₁ for τ ≤ T₀. Consider l₁(τ), lₙ(τ). For e−ατ ≥ 1/2, since w−¹ ≥ V = mψe−ατ and g, g, g, g, p, ν are bounded.

If we choose α₁ > 1 and β₁ > 1 large enough, we can get

lₙ(V₁) − νw−¹Vₙ₀ − νη(w−¹)²p−x − g−r − ggη ≥ me−ατ(νma₁e−ατ − v₀) − p−x + νη(w−¹)² − g−r − ggη > 0

lₙ(V₁) = νw−¹Vₙ₀ − νη(w−¹)²p−x − g−r − ggη ≤ me−ατ(−νβ₁Meατ − v₀) − p−x + νη(w−¹)² − g−r − ggη < 0,

due to ν₁ > ν > ν₀, νη < 0. The constant m, C and M should be chosen from the conditions

φ₀(τ, ξ, η) ≥ mψ(τ, ξ, η), C − eβη ≥ 1,

M ≥ max{w₀, w₁},

Let us choose β > 0 such that Lₙ(V₁) < 0 in Ω; Taking into account the inequality w−¹ ≥ V = mψe−ατ, νyy < 0, we find that for large positive β

Lₙ(V₁) = −ν(w−¹)²Mβ²eβηeατ − M(C − eβη)eατ − pβ Meατ − νη(w−¹)² + 2νβ Meατ(−β)βη(w−¹)² ≤ −eατνhψe−ατ²Mβ²eβη + Mβ + pβ Meβ¹η < 0.

For Lₙ(V), we have

Lₙ(V) = ν(w−¹)²mψηe−ατ + amψe−ατ

−mψe−ατ − νψηe−ατ + pψηe−ατ

+ νη(w−¹)² + 2νmψηe−ατ(w−¹)².

Since

0 ≤ w−¹ ≤ M(C − eβη)eβη,

and νyy, p, ν, νₙ are all bounded. the positive constant α can be chosen independent of n and so large that

Lₙ(V) > 0 in Ω for η < U(τ, ξ) − δ₀.

because of the inequality ψ ≥ min{δ₀, 1}. In the region η ≥ U(τ, ξ) − δ₀ where ψ = (U − η)k, we have

Lₙ(V) = me−ατν(w−¹)²k(k − 1)(U − η)k−2k(U − η)k Uξ + α(U − η)k − ηk(U − η)k−1Uξ

− pβ k(U − η)k−1 − 2νη(w−¹)²k(U − η)k−1

+ νη(w−¹)³.

It follows from the Bernoulli relation that

Uτ + ηUξ = −(U − η)Uξ.

Due to νyy, ν all are bounded, if we choose T₀ such that e−αT₀ ≤ 1/2, then for η ≤ T₀,

Lₙ(V) ≥ me−ατk(U − η)²Uξ + α(U − η)k

− 2νη(w−¹)² + νη(w−¹)³ > 0,

for large positive α. Thus, the conditions of Lemma 2 hold for V and V₁ in Ω. The constant α and T₀ depend only on the data of problem (5)–(7). Consequently, if by V ≤ w−¹ ≤ V₁ for τ ≤ T₀, it follow that V₁ ≥ w ≥ V for any n and τ ≤ T₀. Now, it remains to set h₁(τ, ξ, η) = V, H₁(τ, ξ, η) = V₁.
Lemma 4 Suppose that \( \nu_{\eta_0}, \nu_\eta, \nu, g_\tau, g \) and \( g_\xi \) are all bounded, \( \nu_{\eta_0} < 0, \nu_\eta < 0 \), there is a positive constant \( X_0 \) such that for all \( n \) and \( \xi \leq X_0 \) the inequalities
\[
H_2(\tau, \xi, \eta) \geq w^n - h_2(\tau, \xi, \eta),
\]
hold in \( \Omega \), where \( H_2, h_2 \) are continuous functions in \( \bar{\Omega} \) and \( h_2(\tau, \xi, \eta) > 0 \) for \( \eta < U, \xi \leq X_0 \).

Proof: Let us construct functions \( V \) and \( V_1 \) that satisfy the conditions of Lemma 2. Let \( \psi(\tau, \xi, \eta) \) be the function constructed in the proof of Lemma 3, and let \( \varphi(s) \) be a twice differentiable function for \( s \geq 0 \) and such that \( \varphi(s) = 3 - e^s \) for \( 0 \leq s \leq 1/2 \), \( 1 \leq \varphi(s) \leq 3 \), \( |\varphi'(s)| \leq 3 \), \( |\varphi''(s)| \leq 3 \) for all \( s \geq 0 \). Set
\[
V = mw^\alpha e^{-\alpha \xi}, \quad V_1 = M\varphi(\beta_1 \eta)e^{\beta_2}.
\]

Let us assume positive constant \( m, M, \alpha, \alpha_1, \beta, \beta_1 \) and \( X_0 \) can be chosen independent of \( n \), and \( V_1 \geq w^{n-1} \geq V \) for \( \xi \leq X_0, p_x, \nu_\eta, g_\tau, g, g_\xi \) are bounded, we have
\[
l_n(V) = \nu w^{n-1} m \alpha_1 e^{-\alpha \xi} - v_0 w^{n-1} - p_x + \nu_\eta (w^{n-1})^2
\]
\[
+ g_\tau - g \xi
\]
\[
\geq me^{-\alpha \xi} (\nu m \alpha_1 e^{-\alpha \xi} - v_0) - p_x > 0,
\]
for large enough \( \alpha_1 \), provided that \( e^{-\alpha \xi} \geq 1/2 \). If \( \beta_1 \) is sufficiently large and \( e^{-\alpha \xi} \geq 1/2 \), then
\[
l_n(V_1) \leq me^{-\alpha \xi} (\nu M \beta_1 e^{\beta_2} - v_0) - p_x + \nu_\eta (w^{n-1})^2
\]
\[
+ g_\tau - g \xi < 0.
\]

We have
\[
l_n(V_1) = \nu (w^{n-1})^2 M \beta_1^2 \varphi'(\alpha \xi) - \nu M \varphi e^{\beta_2}
\]
\[
\geq \nu [p_x + 2 \nu (w^{n-1})^2] M \beta_1 e^{\beta_2} + \nu_\eta (w^{n-1})^3.
\]
If \( \beta_1 \eta \leq 1/2, \varphi'' \leq -1 \). By assumption, \( w^{n-1} \geq m w^\alpha e^{-\alpha \xi} \), the function \( \psi \) has already been fixed, and the constant \( m \) is determined from the condition: \( m \varphi \leq \varphi_0 \), \( e^{-\alpha \xi} \geq 1/2 \) for \( \xi \leq X_0 \) and sufficiently small \( X_0 \). Therefore, \( \beta_1 \) is taken so large that \( l_n(V_1) < 0 \) for \( \beta_1 \eta \leq 1/2 \). Choosing \( \beta > 0 \) so large that \( l_n(V_1) < 0 \) for \( \beta_1 \eta \geq 1/2 \), choosing a suitable \( M \), we can ensure the inequality \( V_1 \geq w^{n-1} \) for \( \tau = 0 \) and \( \xi = 0 \). By Lemma 2, \( V_1 \geq w^n \) in \( \Omega \) for \( \xi \leq X_0 \).

For \( l_n(V) \) we have
\[
l_n(V) = \nu (w^{n-1})^2 m \psi_\eta e^{-\alpha \xi} + \alpha m w^\alpha e^{-\alpha \xi}
\]
\[
- m \psi_\eta e^{-\alpha \xi} - \eta m \psi_\xi e^{-\alpha \xi} + p_x m \psi_\eta e^{-\alpha \xi}
\]
\[
+ \nu_\eta (w^{n-1})^3 + 2 \nu \eta m \psi_\eta e^{-\alpha \xi} (w^{n-1})^2.
\]

since \( m \psi e^{-\alpha \xi} \leq w^{n-1} \), if \( \alpha_1 \eta \leq 1, e^{-\alpha \beta} \geq 1/2 \) and \( \alpha_1 \) large enough then
\[
l_n(V) \geq \nu m^3 \alpha_1^2 \beta_1^3 e^{3 \alpha_1 \eta} e^{-3 \alpha \xi}
\]
\[
+ [p_x + 2 \nu (w^{n-1})^2] \alpha_1 e^{\alpha_1 \eta} e^{-\alpha \xi} m + \nu_\eta (w^{n-1})^3 > 0,
\]
due to the boundedness of \( \nu_\eta, \nu_m \).

If \( 1/\alpha_1 < \eta < U - \delta_0, \psi \geq \alpha_0 \geq 0, 0 \leq w^{n-1} \leq M \varphi(\beta_1 \eta)e^{\beta_2} \) and \( \alpha \) is taken large enough, then
\[
l_n(V) > 0.
\]

If \( U(\tau, \xi, \eta) < \eta < \delta_0 \) from the Bernoulli law, as in the proof of Lemma 3, we take \( \alpha \) sufficiently large to assure \( l_n(V) > 0 \) for \( U - \eta < \delta_0 \). Therefore, \( l_n(V) > 0 \) in \( \Omega \) for \( 0 \leq \xi \leq X_0 \), if \( X_0 \) is chosen such that \( e^{-\alpha \eta}X_0 \leq 1/2 \). Since, owing to our choice of \( m \), we have \( V \leq w^n \) for \( \tau = 0 \) and \( \xi = 0 \), it follows from Lemma 2 that \( w^n \geq m \psi e^{-\alpha \xi} \) for \( \xi \leq X_0 \) and all \( \tau \). This completes the proof of Lemma 4, since it may be assumed that \( V \leq w^0 \leq V_1 \).

In what follows, it is assumed that the constants \( T_0 \) and \( X_0 \) in the definition of \( \Omega \) are the same as in Lemma 3 and Lemma 4. In order to estimate the first and the second order derivatives of \( w^n \), we pass to new unknown functions \( W^n = w^n e^{\alpha_1 \eta} \) in (8)(9), where \( \alpha \) is a positive constant to be chosen later. Thus, we have
\[
l_n(W) = \nu (w^{n-1})^2 W_{\eta \eta} - W_\tau - \eta W_\xi
\]
\[
+ [p_x + 2 \nu_\eta (w^{n-1})^2 - 2 \nu (w^{n-1})^2 \alpha] W_\eta
\]
\[
+ [\alpha^2 \nu (w^{n-1})^2 - p_x \alpha - 2 \nu_\eta \alpha e^{\alpha_1 \eta} (w^{n-1})^2]W^n
\]
\[
+ \nu_\eta (w^{n-1})^3 e^{\alpha_1 \eta} = 0.
\]
\[
l_n(W) = \nu w^{n-1} W_{\eta \eta} - \alpha \nu W^n - W_\tau
\]
\[
- W_\xi - v_0 + \nu (w^{n-1})^2 - p_x + g_\tau - g_\xi = 0.
\]

Set
\[
l_n^0(W) = \nu (w^{n-1})^2 W_{\eta \eta} - W_\tau - \eta W_\xi + A^n W_\eta,
\]
\[
A^n = p_x + 2 \nu_\eta (w^{n-1})^2 - 2 \nu (w^{n-1})^2 \alpha,
\]
\[
l_n^0(W) + B^n W^n + \nu_\eta (w^{n-1})^3 e^{\alpha_1 \eta} = 0,
\]
\[
B^n = \alpha^2 \nu (w^{n-1})^2 - p_x \alpha - 2 \nu_\eta \alpha (w^{n-1})^2.
\]

Consider the function
\[
\Phi_\eta = (W_\tau)^2 + (W_\eta)^2 + W_\eta (W^n - 2H^n)
\]
\[
+ [2g_\tau + g_\xi - \frac{2 \nu_\eta W^{n-1}}{\nu}] + k_0 + k_1 \eta,
\]
where

\[ H^n = \frac{\nu_0}{\nu} + \frac{p_x}{\nu W_n^{-1}} + \alpha W^n \chi(\eta) + \frac{g_r + g_g}{\nu W_n^{-1}} - \frac{\nu_0 W_n^{-1}}{\nu}. \]

We assume that \( H^n \) is defined in \( \Omega \), and \( v_0, p_x \) have been extended to the region \( \eta > g(\tau, \xi) \), so that \( v_0 = 0, p_x = 0 \) for \( \eta > \delta_2 = \min U(\tau, \xi)/2; v_0, p_x \) do not depend on \( \eta \) for \( \eta < \delta_2/2 \) and are sufficiently smooth for all \( \eta; \chi(\eta) \) is smooth function such that \( \chi(\eta) = 1 \) for \( \eta \leq \delta_2/2 \), and \( \chi(\eta) = 0 \) for \( \eta \geq \delta_2 \). Obviously, \( W^n = H^n \) for \( \eta = g(\tau, \xi) \).

**Lemma 5** Suppose that \( \nu_0, \nu_\eta, \nu, g_r, g \) and \( g_g \) all are bounded, \( \nu_\eta < 0, \nu < 0 \), and as before \( 0 < \nu_0 \geq \nu < \nu_1 \), then the constant \( k_0, k_1, \alpha \) can be chosen such that

\[ \frac{\partial^2 \Phi_n}{\partial \eta^2} \geq \alpha \Phi_n - \frac{\alpha}{2} \Phi_{n-1} \text{ for } \eta = g(\tau, \xi), \]

\[ L^0(\Phi_n) + R^n \Phi_n \geq 0 \text{ in } \Omega, \quad (11) \]

where \( R^n \) depends on \( w_n \) and its derivatives up to the second order.

**Proof:** For \( \frac{\partial^2 \Phi_n}{\partial \eta^2} \) at \( \eta = g(\tau, \xi) \),

\[ \frac{\partial^2 \Phi_n}{\partial \eta^2} = 2 W^n_\tau W^n_{\tau \eta} + 2 W^n_\xi W^n_{\xi \eta} + W^n_{\eta \eta}(W^n_\eta - 2 H^n_\eta) \]

\[ + W^n_{\eta \eta}(W^n_{\eta \eta} - 2 H^n_\eta) + 2 W^n_{\eta \eta}(g_r + g_g) + \frac{\nu_\eta W_n^{-1}}{\nu} \]

\[ + 2 W^n_\eta(g_r + g_g) \frac{\nu_\eta W_n^{-1}}{\nu^2}. \]

Using the boundary condition \( W^n_\eta = H^n \) at \( \eta = g(\tau, \xi) \), we obtain

\[ \frac{\partial^2 \Phi_n}{\partial \eta^2} = 2 W^n_\tau H^n_\tau + 2 W^n_\xi H^n_\xi - 2 H^n_\eta H^n_\eta \]

\[ + W^n_{\eta \eta}(g_r + g_g) + \frac{\nu_\eta W_n^{-1}}{\nu^2} \]

\[ + 2 H^n(g_r + g_g) \frac{\nu_\eta W_n^{-1}}{\nu^2}. \]

According to Lemma 3 and Lemma 4, \( W^n_n \geq h_0 > 0 \) for \( \eta = g(\tau, \xi) \). For \( \eta = g(\tau, \xi) \), we have

\[ H^n_\eta = -\frac{\nu_0 \nu_\eta}{\nu^2} + \frac{p_x + g_r + g_g}{(\nu W_n^{-1})^2} \]

\[ + \alpha W^n \chi(\eta) - \frac{\nu_\eta W_n^{-1}}{\nu W_n^{-1}}. \]

Let us express \( W^n_n \) and \( W^n_{n-1} \) from the condition \( W^n_\eta = H^n \). We find that \( H^n_\eta \) depends only on \( \nu, \nu_\eta, W^n_n, W^n_{n-1}, W^n_{n-2} \), and therefore, is uniformly bounded with respect to \( n \). Consequently, \( |2 H^n_\eta| \leq k_2, k_2 \) being independent of \( n \). Let us estimate \( W^n_\eta H^n_\eta \) and \( W^n_{n-1} H^n_\xi \).

For \( \eta = g(\tau, \xi) \), we have \( \chi(\eta) = 1 \),

\[ H^n_\eta = \frac{v_0}{\nu} + \frac{p_x}{\nu W_n^{-1}} - \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + \frac{\alpha W^n_\eta}{\nu} \]

\[ + (\frac{g_r + g_g}{\nu W_n^{-1}} - \frac{\alpha}{\nu W_n^{-1}})^2 \]

\[ + \frac{\alpha}{\nu W_n^{-1}} \]

\[ \geq \alpha(W^n)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]

\[ \geq \alpha(W^n)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]

\[ \geq \alpha(W^n)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]

Due to \( p_x, g_r, g \) and \( g_g \) all are bounded and \( \nu_\eta < \nu < \nu_1 \), we can choose a positive \( \alpha \) independent of \( n \) and such that

\[ \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \leq \alpha \frac{4}{4}. \]

Then

\[ W^n_\eta H^n_\xi \geq \frac{3 \alpha}{4}(W^n_\xi)^2 - \alpha(W_n^{-1})^2 - k_3, \]

\[ k_3 > \max \left\{ \frac{v_0}{\alpha} + \frac{p_x}{\nu W_n^{-1}} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right\} \]

and \( k_3 \) does not depend on \( n \). In a similar way, we find that

\[ W^n_\xi H^n_\xi \geq \frac{3 \alpha}{4}(W^n_\xi)^2 - \alpha(W_n^{-1})^2 - k_4, \]

\[ k_4 > \max \left\{ \frac{v_0}{\alpha} + \frac{p_x}{\nu W_n^{-1}} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right\} \]

\[ = 2(W^n_\tau + \eta W^n_\xi - [p_x - 2(\nu(W_n^{-1})^2)\alpha]W^n_\eta \]

\[ - [\alpha^2 \nu(W_n^{-1})^2 - p_x \nu]W^n_\eta \]

\[ \geq - \alpha \frac{4}{4}(W^n_\xi)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]

\[ \geq - \alpha \frac{4}{4}(W^n_\xi)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]

\[ \geq - \alpha \frac{4}{4}(W^n_\xi)^2 - \frac{1}{\alpha} \left[ \frac{v_0}{\nu} + \frac{p_x W_n^{-1}}{\nu(W_n^{-1})^2} + (\frac{g_r + g_g}{\nu W_n^{-1}})^2 \right] \]
\[-\frac{\alpha}{4}(W_{\xi})^2 - \frac{1}{\alpha}(2\eta(g_{\tau} + g_{\xi} + \nu)(W^{n-1})^2)\]
\[-2[p_{\eta} - 2\nu(w^{n-1})^2]W_\eta g_{\tau} + g_{\xi} + \nu(W^{n-1})^2\]
\[\frac{-\alpha}{4}(W_{\xi})^2 \]
\[\frac{-1}{\alpha}[(\alpha^2\nu(w^{n-1})^2 - p_{\eta}+\alpha)(g_{\tau} + g_{\xi} + \nu(W^{n-1})^2)]^2\]
\[\geq -\frac{\alpha}{4}(W_{\tau})^2 - \frac{\alpha}{4}(W_\xi)^2 - \frac{\alpha}{4}(W_{\eta})^2 - k\]

For \(\eta = g(\tau, \xi)\), we have
\[\frac{\partial \Phi_n}{\partial \eta} \geq \alpha \Phi_n - \frac{\alpha}{2} \Phi_{n-1} - k = \alpha \Phi_n - k\eta + k_1,\]
where \(k_1\) is a constant that does not depend on \(n\). Let us choose \(k_1 > k\). We have
\[\frac{\partial \Phi_n}{\partial \eta} \geq \alpha \Phi_n - \frac{1}{2} \alpha \Phi_{n-1} \text{ for } \eta = g(\tau, \xi).\]

Next we consider \(L_n^0(\Phi_n)\). Choosing a suitable \(k_0\), we may assume that \(\Phi_n \geq 1\) in \(\Omega\). Noting that
\[H_n = g_{\tau} + g_{\xi} + \nu(W^{n-1}) - \nu(W_n^{n-1})\]
for \(\eta \geq \delta\), we have
\[\Phi_n = \Phi_n^* = (W_{\eta}^2) + (W_\xi^2) + (W_\tau^2) + k_0 + k_1\eta.\]
Applying the operator
\[2W_{\tau} \frac{\partial}{\partial \tau} + 2W_{\xi} \frac{\partial}{\partial \xi} + 2W_{\eta} \frac{\partial}{\partial \eta}\]
to the equation
\[L_n^0(W_n) + B^nW_n = 0,\]
we can get the conclusion of (11). As the details of the proof, and we will give them in the appendix of the paper.

In order to estimate the second derivatives of \(w^n\) in \(\Omega\), consider the function
\[F_n = (W_{\tau})^2 + (W_{\xi})^2 + (W_{\eta})^2 + W_{\tau\eta}(W_{\eta} - 2H_{\tau})\]
\[+W_{\tau\tau}(W_{\tau} - 2H_{\tau}^n) + f(\eta)(W_{\eta}^n)^2 + N_0 + N_1\eta,\]
where \(N_0, N_1\) are constants, and \(f(\eta)\) is a smooth function such that \(f(g) = 0, f(g) = 0, f(\eta) > 0\) for \(\eta > g(\tau, \xi), f(\eta) = 1\) for \(\eta > \delta_2\).

**Lemma 6** The constant \(N_0\) and \(N_1\) can be chosen independent of \(w^n, w^{n-1}, w^{n-2}\) or their derivatives, so that
\[\frac{\partial F_n}{\partial \eta} \geq \alpha F_n - \frac{\alpha}{2} F_{n-1} \text{ for } \eta = g,\]
\[L_n^0(F_n) + C^nF_n + N_2 \geq 0 \text{ in } \Omega,\]
where the constant \(N_2\) depends only on the first derivatives of \(w^n, w^{n-1}, w^{n-2}\); the constant \(C^n\) depends on \(w^{n-1}\) and its derivatives up to the second order.

The proof is similar with the way of lemma 5.

### 3 The solution of the system (5)-(7)

**Theorem 7** Let \(w^n\) be solutions of problems (8)(9) (10). Then the derivatives of \(w^n\) up to the second order are uniformly bounded with respect to \(n\) in domain \(\Omega\) with a positive \(T\) depending on the data of problem (1)-(3).

**Proof:** Let us show that there exist constants \(M_1, M_2\) and \(T > 0\) such that the conditions \(\Phi_n \leq M_1, F_n \leq M_2\) for \(\tau \leq T, \mu \leq n - 1\), imply that \(\Phi_n \leq M_1, F_n \leq M_2\) for \(\tau \leq T\). According to Lemma 5, we have
\[L_n^0(\Phi_n) + R^n\Phi_n \geq 0,\]
where \(R^n\) depends on \(w^{n-1}\) and its derivatives up to the second order.

Consider the function \(\Phi_n^1 = \Phi_n e^{-\gamma \tau}\) with a positive constant \(\gamma\) to be chosen later. We have
\[L_n^0(\Phi_n^1) + (R^n - \gamma)\Phi_n^1 \geq 0 \text{ in } \Omega.\]

Let us choose \(\gamma\) in accordance with \(M_1\) and \(M_2\), so as to have \(R^n - \gamma \leq -1\) in \(\Omega\), as well as for \(\xi = X, \tau = T, \text{ or } \eta = U(\tau, \xi)\). If \(\Phi_n^1\) attains its largest value at \(\tau = 0\) or at \(\xi = 0\), we should have
\[\Phi_n^1 = \Phi_n e^{-\gamma \tau} \leq \Phi_n < k_{11},\]
where the constant \(k_{11}\) does not depend on \(n\) and is determined only by the data of problem (8)(9)(10). If \(\Phi_n^1\) attains its largest value at some point with \(\eta = g\), we must have \(\partial \Phi_n^1/\partial \eta \leq 0\) at that point, and it
follows from lemma 5 that $\Phi_n^1 \leq \frac{1}{2} \Phi_n^{1-1}$, i.e., $\Phi_n^1 \leq \frac{1}{2} M_1$. Thus we have

$$\Phi_n^1 \leq \max\{\frac{1}{2} M_1, k_{11}\},$$

$$\Phi_n \leq \max\{\frac{1}{2} M_1, k_{11}\} e^{\tau \tau} \text{ in } \Omega.$$

Let us take $T_1 \leq T$ such that $e^{\gamma_{T_1}} = 2$, and set $M_1 = 2k_{11}$. In this case, $\Phi_n \leq M_1$ for $\tau \leq T_1$.

We consider $F_n$. By Lemma 6, we have

$$L_0^0(F_n) + C^\sigma F_n + N_2 \geq 0 \text{ in } \Omega,$$

where $C^\sigma$ depends on the first and the second derivative of $u^{n-1}$, while $N_2$ depends on the first derivatives of $u^n, u^{n-1}, u^{n-2}$. Set $F_n^1 = F_n e^{\tau \tau}$. Then

$$L_0^0(F_n^1) + (C^\sigma - \gamma_{1}) F_n^1 \geq -N_2 e^{\tau \tau} \geq N_2 \text{ in } \Omega.$$

Let us choose $\gamma_1 > 0$ in accordance with $M_1$ and $M_2$, so as to have

$$C^\sigma - \gamma_1 \leq -1 \text{ in } \Omega_1 = \Omega \cap \{\tau \leq T_1\}.$$

Then, if $F_n^1$ attains its largest value inside $\Omega_1$, or at $\tau = T_1$, or at $\xi = X$, or at $\eta = U(\tau, \xi)$, we must have $F_n^1 \leq N_2(M_1)$.

If $F_n^1$ attains its largest value at $\tau = 0$ or at $\xi = 0$, then

$$F_n^1 = F_n e^{\tau \tau} \leq F_n \leq N_2,$$

where the constant $N_2$ depends on $M_1$. If $F_n^1$ attains its largest value at $\eta = g(\tau, \xi)$, then, according to Lemma 6, at the point of maximum we have

$$0 \geq \frac{\partial F_n^1}{\partial \eta} \geq \alpha F_n^1 - \frac{\alpha^2}{2} F_n^1 - 1.$$

and therefore

$$F_n^1 \leq \frac{1}{2} F_{n-1} \leq \frac{1}{2} F_{n-1} - \frac{1}{2} F_n \leq \frac{1}{2} M_2.$$

It follows that

$$F_n^1 \leq \max\{\frac{1}{2} M_2, N_1, N_2\} \text{ in } \Omega,$$

$$F_n \leq \max\{\frac{1}{2} M_2, N_1, N_2\} e^{\tau \tau}.$$

Let us take $T_2 \leq T$ such that $e^{\gamma_{T_2}} = 2$. Set $M_2 = \max\{2N_1, 2N_2\}$. Then $F_n \leq M_2$ for $\tau \leq T_2$ and $\tau \leq T_1$. The constant $T_2$, like $T_1$, depends only on $M_1$ and $M_2$ chosen above and determined only by the data of problem (1)(2)(3). It may be assumed that $w_0$ has been chosen such that $\Phi_0 \leq M_1$ and $F_0 \leq M_2$. The above results show that $\Phi_n$ and $F_n$ are uniformly bounded with respect to $n$ for $\tau \leq \min\{T_1, T_2\} = T$. The fact that $\Phi_n$ and $F_n$ are bounded with respect to $n$ allows us to conclude that the first and the second derivatives of $u^n$ are also bounded, since the boundedness of $u^n_\eta$ for $\eta \leq \beta_2$ follows from (8) and the boundedness of the first derivatives of $u^n$. Theorem 7 is proved.

By the last theorem, we obtain a solution of problem (8)(9)(10) for any $X$ and a sufficiently small $T$. The fact that derivatives of $u^n$ are bounded for an arbitrary $T$ and a sufficiently small $X$ is established by the following:

**Theorem 8** Let $\nu$ satisfy the conditions quoted before, and $u^n$ be solutions of problems (8)(9)(10). Then $u^n$ are uniformly bounded with respect to $n$ in domain $\Omega$ with $X$ depending on the data of problem (1)(2).

**Proof:** Let us show that there exist constants $M_1, M_2$, and $X > 0$ such that the conditions $\Phi_\mu \leq M_1$ and $F_\mu \leq M_2$ for $\xi \leq X$ and $\mu \leq n - 1$ imply that $\Phi_n \leq M_1$ and $F_n \leq M_2$ for $\xi \leq X$. By Lemma 5, we have $L_0^0(\Phi_n) + R^n \Phi_n \geq 0$, where $R^n$ depends on $w^{n-1}$ and its derivatives up to the second order.

Let $\Phi_n = \Phi_1 e^{\beta_1 \varphi_1(\beta_1 \eta)}$, where $\varphi_1(s)$ is a smooth function such that $\varphi_1(s) = 2 - e^{s/2}$ for $s \leq \ln(3/2), 1 \leq \varphi_1 \leq 3/2$ for all $s, \beta_1 \beta_1$ are positive constants that will be chosen later. We have

$$L_0^0(\Phi_n^1) + 2\nu(w^{n-1})^2 \beta_1 \frac{\varphi_1'}{\varphi_1} \Phi_n^1 +$$

$$(R^n - \eta \beta + A^n \beta_1 \frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2 \beta_1^2 \frac{\varphi_1''}{\varphi_1^2}) \Phi_n^1 \geq 0.$$

(12)

If $\beta_1 \eta \leq \ln(3/2)$, then $-3/4 \leq \varphi_1' \leq -1/2, \varphi_1'' \leq -1/2$. By Lemma 4, we have $(w^{n-1})^2 \geq \gamma_0 > 0$ for $\eta \leq \beta_2$ and $\xi \leq X_0$.

Let $\eta \leq \beta_2^2 \eta \leq \beta_2$. Due to $\nu$ is bounded, then we can find $\beta_1$ such that the coefficient of $\Phi_n$ in (12), for $\xi \leq X$, satisfies the inequality

$$R^n - \eta \beta + A^n \beta_1 \frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2 \beta_1^2 \frac{\varphi_1''}{\varphi_1^2} \leq -1.$$

In the region of $\eta > \min\{\beta_2, \beta_1 \ln(3/2)\}$ this inequality is valid if $\beta > 0$ has been chosen sufficiently large. Obviously, $\beta$ may be assumed independent of $M_1, M_2$. Then, according to (12), the function $\Phi_n^1$ can’t attain its largest value inside $\Omega$ for $\xi \leq X$ at any of the points $\tau = T, \xi = X, or \eta = U(\tau, \xi)$.

If $\Phi_n^1$ attains its largest value at $\tau = 0$ or $\xi = 0$, then

$$\Phi_n^1 = \frac{\Phi_n}{\varphi_1} e^{-\beta \xi} \leq \Phi_n \leq k_{12},$$
where \(k_{12}\) does not depend on \(n\), since \(\Phi_n\) \(|\tau=0\) and \(\Phi_n\) \(|\xi=0\) can be expressed through \(w_0\), \(w_1\) and their derivatives.

If \(\Phi^1_n\) attains its largest value at \(\eta = g(\tau, \xi)\), then 
\[
\frac{\partial \Phi^1_n}{\partial \eta} \leq 0
\]
and it follows from Lemma 5 that
\[
\Phi^1_n \leq \frac{1}{2} \Phi^1_{n-1} \quad \text{or} \quad \Phi^1_n \leq \frac{1}{2} \Phi^{-1}_n e^{-\beta \xi} \leq \frac{1}{2} M_1.
\]
by virtue of our assumption. Thus
\[
\Phi^1_n \leq \max\left\{ \frac{1}{2} M_1, k_{12} \right\} \quad \text{in} \quad \Omega \quad \text{for} \quad \xi \leq X,
\]
\[
\Phi_n \leq \max\left\{ \frac{1}{2} M_1, k_{12} \right\} \max[\eta \varphi_1(\beta \eta)]
\]
Since \(\varphi_1(\beta \eta) \leq 3/2\), we have 
\[
e^{\beta \xi} \varphi_1(\beta \eta) \leq 2, \text{if } e^{\beta \xi} \leq 4/3.
\]
Let us choose \(X_1 \leq X_0\) from the condition \(e^{\beta X_1} \leq 4/3\). Then
\[
\Phi_n \leq \max\{M_1, 2k_{12}\} \quad \text{for} \quad \xi \leq X_1.
\]
Set \(M_1 = 2k_{12}\). Then \(\Phi_n \leq M_1\) for \(\xi \leq X_1\), where \(X_1\) depends on \(M_1\) and \(M_2\).

Now, let us consider \(F_n\). By Lemma 6
\[
L^0_n(F_n) + C^n F_n \geq -N_2 \quad \text{in} \quad \Omega \quad \text{for} \quad \xi \leq X_1.
\]
Let \(F_n = F^1_n \varphi_1(\beta \eta)\) be the function defined above. We have
\[
\begin{align*}
L^0_n(F_n) &+ 2\nu(w^{n-1})^2 \beta_2 \frac{\varphi^1_n}{\varphi_1} F^n_1 + [C^n - \eta \beta_3] \\
&+ A^n \beta_2 \frac{\varphi^1_n}{\varphi_1} + \nu(w^{n-1})^2 \beta_2 \frac{\varphi^1_n}{\varphi_1} F^n_1 > - N_2 \frac{e^{-\beta \xi}}{\varphi_1}.
\end{align*}
\]
If \(\beta_2 \eta \leq \ln(3/2)\), then \(-3/4 \leq \varphi^1_n \leq -1/2, \varphi^1_n \leq -1/4, 1 \leq \varphi_1 \leq 3/2\).

It follows from Lemma 4 that \((w^{n-1})^2 \geq \gamma_0 > 0\) for \(\eta \leq \delta_2\). Let \(\eta \leq \min\{\delta_2, \beta_2^{-1} \ln(3/2)\}\). For these values of \(\eta\), the constant \(\beta_2\) can be chosen so as to make the coefficient by \(F^1_n\) in (13) satisfy the inequality
\[
C^n - \eta \beta_3 + A^n \beta_2 \frac{\varphi^1_n}{\varphi_1} + \nu(w^{n-1})^2 \beta_2 \frac{\varphi^1_n}{\varphi_1} \leq -1.
\]
This inequality will hold in the region of \(\eta > \min\{\delta_2, \beta_2^{-1} \ln(3/2)\}\) if \(\beta_3\) has been chosen sufficiently large. Clearly, \(\beta_3\) depends on \(M_1\) and \(M_2\). Just as in the proof of Theorem 7, we find that
\[
F^1_n \leq \max\left\{ \frac{1}{2} M_1, N_2, N_13 \right\} \quad \text{in} \quad \Omega \quad \text{for} \quad \xi \leq X_1.
\]
where \(N_13 = \max\{F_n\}\) for \(\tau = 0\) and for \(\xi = 0; N_13\) depends on \(M_1\). We have
\[
F_n \leq \max\{\frac{1}{2} M_2, N_2, N_13\} \max[e^{\beta \xi} \varphi_1(\beta \eta)] \leq \max\{\frac{1}{2} M_2, N_2, N_13\},
\]
provided that 
\[
e^{\beta \xi} \varphi_1(\beta \eta) \leq 2 \quad \text{or} \quad e^{\beta \xi} \leq 4/3.
\]
Let us take \(M_2 = \max\{2N_2, 2N_13\}\) and define \(X_2 \leq X_0\) from the inequality 
\[
e^{\beta \xi} X_2 \leq 4/3.
\]
Then 
\[
F_n \leq M_2 \quad \text{for} \quad \xi \leq X_1, \quad \text{where} \quad X = \min\{X_1, X_2\}.
\]
The fact that \(\Phi_n\) and \(F_n\) are bounded implies that the derivatives of \(w^n\) up to the second order are bounded uniformly in \(n\), since \(\max_n\), for \(\eta \leq \delta_2\), can be estimated from equation (8).

**Theorem 9** The function \(w^n (n \to \infty)\) are uniformly convergent in \(\Omega\) to a solution \(w^n\) of problem (5)(6)(7) in \(\Omega\), where \(T\) is defined in Theorem 7 and \(X\) may be taken arbitrarily, or \(X\) is that of Theorem 8 and \(T\) is arbitrary. The function \(w\) is continuously differentiable in \(\Omega\) and its derivative \(w_\eta\) is continuous for \(\eta < U(\tau, \xi)\).

**Proof:** Let \(v^n = w^n - w^{n-1}\). We obtain the following from equation (8)
\[
\begin{align*}
\nu(w^{n-1})^2 v^n_\eta &- \nu_\xi v^n_\xi + [p_x + 2\nu_\eta (w^{n-1})^2] v^n_\eta \\
+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\eta &+ 2\nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\xi \\
+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1} &+ \eta \nu_\eta (w^{n-1})^2 - (w^{n-2})^2 \beta^2 = 0,
\end{align*}
\]
and also the boundary condition:
\[
\begin{align*}
v^n |_{\tau=0} &= 0, \quad v^n |_{\xi=0} = 0, \quad v^n |_{\eta=U(\tau, \xi)} = 0, \\
\nu w^{-1}_\eta v^n_\eta &- \nu_\xi v^n_\xi + [p_x + 2\nu_\eta (w^{n-1})^2] v^n_\eta \\
+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\eta + 2\nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\xi &+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1} + \eta \nu_\eta (w^{n-1})^2 - (w^{n-2})^2 \beta^2 = 0.
\end{align*}
\]
Consider the function \(v^n_1\) defined by \(v^n = e^{\alpha \tau + \beta \eta} v^n_1\), \(\beta < 0\). We have
\[
\begin{align*}
\nu ((w^{n-1})^2 v^n_\eta_1 - v^n_1 \tau - \eta v^n_1 \xi + [p_x + 2\nu_\eta (w^{n-1})^2] v^n_\eta_1 \\
+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_1 &+ 2\nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\xi + \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1} \\
+ 2\nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\xi &+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_\xi + \eta \nu_\eta (w^{n-1})^2 - (w^{n-2})^2 \beta^2 = 0,
\end{align*}
\]
\[
\begin{align*}
\text{The boundary condition } \eta = g(\tau, \xi) \\
\nu w^{-1}_1 v^n_1 + \beta \nu w^{-1}_1 v^n_1 - \nu_\xi v^n_1 + \eta \nu_\eta v^n_1 &+ \nu_\eta [(w^{n-1})^2 - (w^{n-2})^2] v^{n-1}_1 = 0.
\end{align*}
\]
Due to $\nu$, $\nu_{\eta\eta}$ and $\nu_\eta$ are bounded, the constant $\beta < 0$ should be choose such that in the boundary condition for $v_1^\nu$ at $\eta = g(\tau, \xi)$ the coefficients by $v_1^\nu$ and $v_1^{\nu-1}$ satisfy the inequality
\[
\max |\nu w_{\eta\eta}^{\nu-1} - v_1 + \nu_\eta(w^{\nu-1} + w^{\nu-2})| \\
\leq q \min |\nu w_\eta^{\nu-1} |,
\]
for $q < 1$.

Having fixed $\beta$, let us choose $\alpha > 0$ such that
\[
\max |\nu w_{\eta\eta}^{\nu-1}(w^{\nu-1} + w^{\nu-2}) + 2\nu_\eta w_{\eta\eta}^{\nu-1}(w^{\nu-1} + w^{\nu-2}) + \nu_\eta [(w^{\nu-1})^2 + w^{\nu-1}w^{\nu-2} + (w^{\nu-2})^2]| \\
\leq q(\alpha - \max |\nu(w^{\nu-1})^2)| + 2\nu_\eta (w^{\nu-1})^2| |.
\]

Now, if $|v_{1\nu}^\nu|$ attains its largest value at an interior point of $\Omega$ or on its boundary, it follow equations (14), (15) that
\[
\max |v_{1\nu}^\nu| \leq q \max |v_1^{\nu-1}|,
\]
where means that the series $v_1^\nu + v_2^\nu + \cdots + v_n^\nu + \cdots$, whose partial sums have the form $w^\nu e^{-\alpha r - \beta \eta}$, is majorized by a geometrical progression, and therefore, is uniformly convergent. The fact $w^\nu$ and its derivatives up to the second order are bounded implies that the first derivatives of $w^\nu$ are uniformly convergent as $n \rightarrow \infty$.

It follows from equation (8) that $w_{\eta\eta}$ are also uniformly convergent as $n \rightarrow \infty$ for $\eta < U(\tau, \xi) - \delta_3$, where $\delta_3 < \min U(\tau, \xi) - \max g(\tau, \xi)$ and which is an arbitrary positive constant.

4 The existence of the solution of the system (8)-(9) and the main result

Now, let us establish the existence of the solution $w^\nu(\tau, \xi, \eta)$ for problem (8)(9). the way is similar as [3].

Consider the operator
\[
L^\varepsilon(w) = \varepsilon(w_{\tau\tau} + w_{\xi\xi} + w_{\eta\eta}) + a_1 w_{\tau\tau} + a_2 w_{\xi\xi} + a_3 w_{\eta\eta} + [\nu(w^{\nu-1})^2]_\varepsilon w_{\eta\eta} - \nu_\eta w_\eta + [p_\eta + 2\nu_\eta(w^{\nu-1})^2]_\varepsilon w_\eta + [\nu_\eta(w^{\nu-1})^2]_\varepsilon w_\eta - 2(a_1 + \varepsilon) w.
\]

Consider the following elliptic boundary value problem:
\[
L^\varepsilon(w) = (f)_\varepsilon \text{ in } Q, \quad (16)
\]
\[
\frac{\partial w}{\partial n} = (F)_\varepsilon \text{ on } S. \quad (17)
\]
where $\nu_{\eta\eta}$ is unit inward normal to $S$. The function $f$ in (14) is defined in $Q$ by
\[
f = \frac{v_0}{\nu} + \frac{p_\tau}{\nu} w^{\nu-1} + \frac{g_\tau + gg_\xi}{\nu} - \frac{\nu_\eta w^{\nu-1}}{\nu} \text{ on } S_0,
\]
\[
F = \frac{\partial w^*}{\partial \eta} \text{ on } \gamma,
\]
where $\gamma$ is the intersection of $S$ with the boundary of $Q \Omega_1$; on the rest of $S$, the function $F$ in (19) coincides with any smooth extension of $F$ just defined on $S_0$ and $\gamma$. Clearly, owing to the properties of $w^*$, we may assume that $f$ has bounded derivatives up to the fourth order in $Q$ and is infinitely differentiable outside a $\delta$-neighborhood of $\Omega$; $F$ has bounded derivatives up to the fourth order in a neighborhood of $S_0$ and is infinitely differentiable on the rest of $S$.

The boundary value problem (16)-(17) has a uniqueness solution $w_1^\nu$ in $Q$, one can see the fact in [3].

Let us show that the functions $w_1^\nu$ and their derivatives up to the fourth order are bounded uniformly with respect to $\varepsilon$.

**Lemma 10** The solutions $w_1^\nu$ of problem (16)(17) in $Q$ are bounded uniformly in $\varepsilon$.

**Proof:** Set $w_1^\nu = v_3^\psi_1$, where $\psi_3^\psi(\tau) = 1$ for $\tau \leq -1$, $\psi_3^\psi(\tau) = 1 + b(1 + \tau)^3$ for $-1 \leq \tau \leq T + 2$, choose suitable $b > 0$ such that $\psi_3^\tau \leq \psi_3^\tau$ in $Q$. Let $6b(T + 3) < 1$. Then $v_3$ satisfies the following equation in $Q$:
\[
(\frac{f_\varepsilon}{\psi_1})^\psi = \varepsilon(\Delta v_3) + a_1 v_{\tau\tau} + a_2 v_{\xi\xi} + a_3 v_{\eta\eta} \\
+ [\nu(w^{\nu-1})^2]_\varepsilon v_{\eta\eta} - \nu_\eta v_\eta + p_\eta + [2\nu_\eta(w^{\nu-1})^2]_\varepsilon v_\eta + [\nu_\eta(w^{\nu-1})^2]_\varepsilon v_\eta - 2(a_1 + \varepsilon) v.
\]

as well as the boundary conditions on $S$
\[
\frac{\partial v_3}{\partial n} = \frac{(F)_\varepsilon}{\psi_1} \text{ for } -2 \leq \tau \leq T + 1, \quad (19)
\]
\[
\frac{\partial v_\varepsilon}{\partial n} + \frac{1}{\psi^1} \frac{\partial \psi^1}{\partial n} v_\varepsilon = \frac{(F)_\varepsilon}{\psi^1} \quad \text{for} \quad \tau \geq T + 1. \quad (20)
\]

Since
\[
\frac{\partial \psi^1}{\partial n} = \psi^1 \frac{\partial \sigma}{\partial n} \leq 0 \quad \text{for} \quad \tau \geq T + 1 \quad \text{on} \quad S,
\]
the coefficient of \( v_\varepsilon \) in the boundary condition (20) is non-positive. (The domain \( Q \) may be assumed convex for \( \tau \geq T + 1 \).) The coefficient of \( v_\varepsilon \) in equation (18) is negative. Indeed, we have
\[
-(a_1 + \varepsilon) + (a_1 + \varepsilon) \frac{\psi^1_{\tau\tau}}{\psi^1} \leq 0.
\]

since \( \psi^1_{\tau\tau}/\psi^1 \leq 1 \), and \( \psi^1_{\tau} > 0 \) for \( \tau > -1 \), while \( a_1 > 0 \) for \( \tau < -1/2 \).

Applying to the solution of problem (18)(19)(20) the a priori estimate established by Theorem 4 of [4], we find that \( v_\varepsilon \) are bounded in \( Q \) by a constant independent of \( \varepsilon \) but depending only on the maximum moduli of the coefficients in equation (18), as well as on

\[
\max \left( \frac{(f)_\varepsilon}{\psi^1}, \frac{(F)_\varepsilon}{\psi^1} \right),
\]

\[
\min \left( (a_1 + \varepsilon) \frac{\psi^1_{\tau\tau}}{\psi^1} - \frac{\psi^1_{\tau}}{\psi^1} - 2(a_1 + \varepsilon) \right).
\]

**Lemma 11** The solution \( w_\varepsilon^0 \) of problem (16)(17) have their derivatives up to the fourth order in \( Q \) bounded by a constant independent of \( \varepsilon \).

**Proof:** First of all, we note that equation (16) is uniformly (with respect to \( \varepsilon \)) elliptic in \( Q \) for \( \tau > T + \delta + r_1 \), and for \( \tau < -1/2 - r_1 \), where \( r_1 \) is any positive constant. Therefore, according to the well-known estimates of Schauder type (see [3]), the \( m \)-th order derivatives of \( w_\varepsilon^0 \) have their absolute values bounded by a constant independent of \( \varepsilon \), for \( \tau > T + \delta + r_1 \) and for \( \tau < -1/2 - r_1 \), provided that \( w_\varepsilon^{n-1} \) have bounded \( (m-1) \)-th order derivatives (with \( m \geq 2 \)) in the same region.

Let \( \Gamma(\xi, \eta) \) be a point of \( \sigma_\delta \) such that \( |\xi| \geq 2\delta \), and let \( A_\delta \) be the intersection of its \( \delta \)-neighborhood on the plane \( \xi, \eta \) with the domain \( G \). Consider the cylinder
\[
B_\delta = [-\frac{1}{2} - r_1, T + \delta + r_1] \times A_\delta.
\]

Let us show that in this domain the functions \( w_\varepsilon^0 \) have their derivatives up to the fourth order bounded by a constant independent of \( \varepsilon \). We may assume that in \( B_\delta \) the coefficient \( a_1 \) depends merely on \( \tau \), whereas \( a_2 \) and \( a_3 \) depend only on \( \xi \) and \( \eta \).

In the domain \( A_\delta \), let us introduce new coordinates \( \xi' \) and \( \eta' \), so that the part of the boundary \( \sigma \) belonging to \( A_\delta \) would turn into a subset of the straight line \( \eta' = 0 \), and the direction of the normal \( n \) to \( \sigma \) would coincide with that of the \( \eta' \)-axis. In the new coordinates, again denoted by \( \xi, \eta \), the boundary condition (17) takes the form \( \partial w_\varepsilon^{n}/\partial \eta = F^*_\varepsilon \).

Let \( Y(\tau, \xi, \eta) \) be a function defined on \( B_\delta \) which satisfies the condition
\[
\frac{\partial Y}{\partial \eta} |_{\eta=0} = F^*_\varepsilon.
\]

The function \( z \equiv w_\varepsilon^n - Y \) satisfies the equation
\[
M(z) \equiv (\varepsilon + a_1) z_{\tau\tau} - z_\tau = a_{11} z_\xi + a_{12} z_\eta \equiv \psi^1_{\tau}\psi^1_{\eta} + b_1 z_\xi + b_2 z_\eta - 2(\varepsilon + a_1) z = f^*_\varepsilon, \quad (21)
\]

in \( B_\delta \) and also the condition \( z_\eta = 0 \) on \( S \), where
\[
a_{11} \sigma_1^2 + 2a_{12} \sigma_1 \sigma_2 + a_{22} \sigma_2^2 \geq \lambda_0 (\sigma_1^2 + \sigma_2^2),
\]

the constant \( \lambda_0 \) being positive and independent of \( \varepsilon \).

In order to estimate the first order derivatives of \( z \) with respect to \( \xi \) and \( \eta \), consider the function
\[
\Lambda_1 = \rho_3^2 (\xi, \eta) [z_\xi^2 + z_\eta^2] + c_1 z^2 + c_2 \eta, \quad c_2 > 0
\]

Here the constant \( c_1 \) is assumed sufficiently large and will be chosen later; \( \rho_3(\xi, \eta) = 1 \) in \( A_\delta/2 \), and \( \rho_3(\xi, \eta) = 0 \) in a small neighborhood of the boundary of \( A_\delta \) that does not belong to \( \sigma \); \( \rho_3 = 0 \) on \( \sigma \).

It is easy to see that \( \partial \Lambda_1/\partial \eta = c_2 > 0 \) on \( S \) and, therefore, \( \Lambda_1 \) cannot attain its largest value on \( S \). If the maximum of \( \Lambda_1 \) is reached on the boundary of \( B_\delta \) at a point where \( \rho_3 = 0 \), then
\[
\Lambda_1 \leq \max (c_1 z^2 + c_2 \eta) \leq c_3,
\]

where \( c_3 \) is a constant independent of \( \varepsilon \). It is easy to verify that for large enough \( c_1 \) we have \( M(\Lambda_1) - \Lambda_1 \geq -c_4 \) in \( B_\delta \), provided that \( c_4 \) is sufficiently large. Therefore, if \( \Lambda_1 \) takes its largest value inside \( B_\delta \), then \( \Lambda_1 \leq c_4 \).

As shown above, for \( \tau = T + \delta + r_1 \) and \( \tau = -1/2 - r_1 \), the function \( \Lambda_1 \) is uniformly bounded in \( \varepsilon \). Thus, \( \Lambda_1 \) in \( B_\delta \) is bounded by a constant independent of \( \varepsilon \) and, therefore, \( z_\xi, z_\eta \) are bounded in \( B_{\delta_1}, \delta_1 < \delta \).

Let us rewrite equation (21) in the firm of
\[
M(z) \equiv \Gamma(z) + M^1(z) = f^*_\varepsilon, \quad \Gamma(z) \equiv (\varepsilon + a_1) z_{\tau\tau} - z_\tau.
\]

The coefficients of the operator \( M^1 \) may be assumed independent of \( \tau \). Hence, it is easy to see that \( \Gamma \) satisfies the following equation and the boundary condition:
\[
M(\Gamma) \equiv \Gamma(\Gamma) + M^1(\Gamma) = \Gamma(f^*_\varepsilon) \quad \text{in} \quad B_\delta,
\]
\[ \Gamma_{\eta} \big|_{\eta=0} = 0 \quad \text{on} \ S. \quad (22) \]

In \( B_{\delta_1} \), consider the function
\[ \Lambda_2 \equiv \rho_{\delta_1}^2 \left[ z_{\xi}^2 + z_{\eta}^2 + \Gamma^2(z) \right] + c_5 (z_{\xi}^2 + z_{\eta}^2) + c_6 \eta. \]

Making use of equations (21) and (22), we easily find that
\[ M(\Lambda_2) - \Lambda_2 \geq c_7 \quad \text{in} \ B_{\delta_1}, \]
\[ \frac{\partial \Lambda_2}{\partial \eta} = c_6 > 0 \quad \text{on} \ S. \]

provided that \( c_5 > 0 \) is sufficiently large. Hence we see that \( \Lambda_2 \) is uniformly bounded in \( B_{\delta_1} \), and \( \Gamma(z), z_{\xi}, z_{\eta} \) are uniformly bounded in \( B_{\delta_2}, \delta_2 < \delta_1 \). It follows from (21) that \( z_{\eta\eta} \) is also bounded uniformly in \( \varepsilon \). If we consider an equation of the form
\[ (a_1 + \varepsilon)z_{\tau\tau} - z_{\varepsilon\tau} = \Gamma \quad \text{for} \ \tau \varepsilon \text{and use the fact that} \ \Gamma \text{is bounded in} B_{\delta_2} \text{and} z_{\varepsilon\tau} \text{is bounded for} \ \tau = -1/2 - r_1 \text{and} \ \tau = T + \delta + r_1, \]
we easily find that \( z_{\varepsilon\tau} \) is bounded in \( B_{\delta_2} \) uniformly with respect to \( \varepsilon \).

Since \( \Gamma(z) \) is bounded in \( B_{\delta_2} \) and satisfies equation (22), as well as the boundary condition \( \Gamma_{\eta} \big|_{\eta=0} = 0 \) on \( S \), we can consider functions similar to \( \Lambda_1 \) and \( \Lambda_2 \) for \( \Gamma \) in \( B_{\delta_2} \) (as we have done for \( z \)) and thus obtain uniform estimates in \( B_{\delta_3}, \delta_3 < \delta_2 \), for the derivatives
\[ \Gamma_{\varepsilon}, \ \Gamma_{\eta}, \ \Gamma_{\varepsilon\eta}, \ \Gamma_{\varepsilon\xi}, \ \Gamma_{\eta\eta}, \Gamma_{\varepsilon\xi}. \]

Differentiating equation (22) in \( \tau \), we obtain the following equation for \( \Gamma_{\tau\tau} \):\[
(a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a_1')\Gamma_{\tau\tau\tau} + M_1(\Gamma_{\tau}) = (\Gamma(f_{\tau}')_{\tau})_{\tau},
\]
as well as the boundary condition \( \Gamma_{\eta\eta} \big|_{\eta=0} = 0 \) on \( S \). By assumption, \( a_1'(\tau) \) is small in \( B_{\delta} \). Therefore, the equation for \( \Gamma_{\varepsilon} \) is similar to (22). Thus, for the derivatives of \( \Gamma \) of the form
\[ \Gamma_{\varepsilon\xi}, \ \Gamma_{\eta\xi}, \ \Gamma_{\varepsilon\eta}, \ \Gamma_{\xi\xi}, \ \Gamma_{\eta\eta}, \Gamma_{\xi\eta}, \Gamma_{\xi\xi}. \]
\[ (a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a_1')\Gamma_{\tau\tau\tau} + M_1(\Gamma_{\tau}) = (\Gamma(f_{\tau}')_{\tau})_{\tau}, \]
in \( B_{\delta_4}, \delta_4 < \delta_3 \), we obtain estimates uniform in \( \varepsilon \), just as we have done for \( z \).

Similar arguments applied to \( \Gamma_{\varepsilon\xi} \) allow us to show that the derivatives
\[ \Gamma_{\varepsilon\xi}, \ \Gamma_{\eta\xi}, \ \Gamma_{\varepsilon\eta}, \ \Gamma_{\xi\xi}, \ \Gamma_{\eta\eta}, \Gamma_{\xi\eta}, \Gamma_{\xi\xi}. \]
are bounded in \( B_{\delta_5}, \delta_5 < \delta_4 \), uniformly with respect to \( \varepsilon \).

These estimates show that in \( B_{\delta_5} \) the third and the fourth order derivatives of \( z \) involving more than one differentiation in \( \tau \) are bounded uniformly in \( \varepsilon \), while the first order derivatives of \( \Gamma_{\varepsilon} \) in \( \xi \) and \( \eta \) satisfy the Lipschitz condition in \( \xi, \eta \) with constants independent of \( \varepsilon, \tau \).

From the Schauder estimates for an elliptic equation of the form
\[ M(\Gamma) = -\Gamma_{\eta\eta} + \Gamma(f_{\tau}'), \]
it follows that the derivatives of \( \Gamma \) in \( \xi \) and \( \eta \) up to the third order are bounded and satisfy the Hölder condition in \( B_{\delta_5}, \delta_6 < \delta_5 \), uniformly with respect to \( \varepsilon \) and \( \tau \). Schauder estimates for the solution \( z \) of equation (21) written in the form
\[ M^1(\Gamma) = -\Gamma_{\eta\eta} + f_{\tau} \]
allow us to claim that \( z \) has its derivatives in \( \xi \) and \( \eta \) up to the forth order bounded in \( B_{\delta_7}, \delta_7 < \delta_6 \), uniformly with respect to \( \varepsilon \) and \( \tau \).

Thus, we have obtained estimate for the derivatives of \( w_{\eta_{\eta}^2} \) in \( \tau, \xi, \eta \) up to the forth order in a neighborhood of the entire boundary \( S \), except for a neighborhood of \( S_0 \) and a neighborhood \( w \) of the intersection \( S \cap \{ \xi = 0 \} \) which is a subset of the plane \( \{ \eta = \eta_1 \} \).

In equation (16) and the boundary condition (17), let us pass to another unknown function \( W \) given by
\[ w = W e^{\varphi_2(\eta)}, \quad \varphi_2(\eta) = -\frac{\alpha \eta - \eta_1}{\eta_2}, \quad \alpha = const > 0. \]

For \( W \) we obtain the following equations
\[ \frac{\partial W}{\partial \eta} - \alpha W = (F)_{\tau} \quad \text{for} \ \eta = 0, \]
\[ -\frac{\partial W}{\partial \eta} - \alpha W = (F)_{\tau} \quad \text{for} \ \eta = \eta_1. \]

In order to estimate the first order derivatives of \( w_{\eta_{\eta}^2} \) in \( Q \), consider the following function in \( Q_{r_1} \cap \{ -\frac{1}{2} - r_1 < \tau < T + \delta + r_1 \} \):
\[ X_1 = W_{\xi}^2 + W_{\tau}^2 + W_{\eta}(W_{\eta} - 2Y) + k(\eta), \]
where \( Y = (\alpha W + (F)_{\tau}) \kappa_1(\eta) \), we define function \( \kappa_1(\eta) \) such that \( \kappa_1(\eta) = 1 \) for \( \eta |< \delta, \kappa_1(\eta) = -1 \) for \( \eta - \eta_1 |< \delta, \kappa_1(\eta) = 0 \) for \( 2\delta < \eta < \eta_1 - 2\delta, k(\eta) \) is a positive function to chosen later. Clearly, \( \partial W/\partial \eta - Y = 0 \) on the part of the boundary \( S \) belonging to the planes \( \eta = 0 \) or \( \eta = \eta_1 \). We have
\[ \frac{\partial X_1}{\partial \eta} \big|_{\eta=0} = 2W_{\xi}W_{\xi\eta} + 2W_{\tau}W_{\eta\tau} - 2W_{\eta}Y_{\eta} + k'(0) \]
\[ = 2\alpha [W_{\xi}^2 + W_{\tau}^2] - 2Y_{\eta}Y_{\eta} + 2W_{\xi}(F)_{\xi\xi} + 2W_{\tau}(F)_{\tau\tau} + \]

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provided that \( k'(0) \) is positive and sufficiently large. Likewise, taking \( k(\eta) \) in \( X_1 \) such that \( k'(\eta) \) is negative and sufficiently large in absolute value, we find that
\[
\frac{\partial X_1}{\partial \eta} |_{\eta=\eta_0} < 0.
\]
By the methods already used in the proof of Lemma 5 we obtain
\[
L^o(X_1) + c_8 X_1 \geq -c_9, \quad (23)
\]
where
\[
L^o(W) \equiv L^o(W) + 2[(\varepsilon + a_3) + \nu (w^{n-1})^2] \frac{\partial W}{\partial \eta} + \{(\nu (w^{n-1})^2 + \varepsilon + a_3)[\varphi_{2}\varphi_{\eta} + (\varphi_{2\eta})^2] + (y + 2\nu \eta (w^{n-1})^{2}) \varepsilon \varphi_{2\eta}] W,
\]
the constant \( c_8 \) and \( c_9 \) do not depend on \( \varepsilon \). In \( Q_{r_1} \), consider the function
\[
X_1^* = X_1 e^{-\beta \varepsilon}, \quad \beta = \text{const} > 0.
\]
For sufficiently large \( \beta \), the coefficient of \( X_1^* \) in (23) is less than -1. It follows from (23) that if \( X_1^* \) takes its largest value inside \( Q_{r_1} \), then \( X_1^* \) is bounded by a constant independent of \( \varepsilon \).

Neither for \( \eta = 0 \) nor for \( \eta = \eta_1 \) can \( X_1^* \) attain its largest value. It follows from the above estimates that on the remaining part of the boundary of \( Q_{r_1} \) the function \( X_1^* \) is bounded uniformly in \( \varepsilon \). Likewise, we can estimate the second and the third order derivatives of \( w^n_\varepsilon \) by considering the functions
\[
X_2 = W_{2_\tau} + W_{2_\xi} + W_{2_\tau} + W_{\eta \xi}(W_{\eta \xi} - 2 Y_{\xi}) + 2 W_{\eta \tau}(W_{\eta \tau} - 2 Y_{\tau}) + g_0(\eta)W_{\eta \eta} + k(\eta),
\]
where \( (X_3)^* \) stands for the sum of third order derivatives of \( W \) in \( \xi \) and \( \tau \)
\[
g_1(\eta) = \begin{cases} 
0 & \text{for } \eta < \frac{\delta}{2}, \\
1 & \text{for } \eta_1 - \delta > \eta > \eta_1 - \delta/2.
\end{cases}
\]

The required estimates for \( X_2 \) and \( X_3 \) can be obtained by the method used above in relation to \( X_1 \); in order to establish the inequality of type (23) for \( X_2 \) and \( X_3 \), we can use the fact that in (16) the coefficient of \( W_{\eta \eta} \) is positive for \( \eta < \delta \) and \( \eta_1 - \eta < \delta \), as we have done in the proof of Lemma 6. While estimating the fourth order derivatives of \( W \), the following observations are useful.

Consider the function
\[
X_4 = (X_4)^* + g^2(\eta)(X_4)^* + W_{\eta \xi \xi}(W_{\eta \xi \xi} - 2 Y_{\xi \xi})
\]
where \( (X_4)^* \) is the sum of squared fourth order derivatives of \( W \) except those involving a differentiation in \( \eta \), and \( (X_4)^{**} \) is the sum of squared fourth order derivatives of \( W \) involving more than one differentiation in \( \eta \).

The expression for \( X_4 \) contains third order derivatives of \( Y \) and therefore, of \( (F)_\varepsilon \). The operator \( L^o(X_4) \) can be estimated through the expressions
\[
L^o(Y_{\tau \tau}), \quad L^o(Y_{\xi \xi}), \quad L^o(Y_{\tau \tau}), \quad L^o(Y_{\xi \xi}),
\]
which contain fifth order derivatives of \( (F)_\varepsilon \). By construction, \( F \) is infinitely differentiable outside the \( \delta \)-neighborhood of \( S_0 \) and has its fourth order derivatives bounded in \( \varepsilon \) on \( S \). In the intersection of the domain \( Q \) with the \( \delta \)-neighborhood of \( S_0 \), the operator \( L^o \) involves second order derivatives in \( \xi \) and \( \tau \) with the coefficient \( \varepsilon \), namely,
\[
\varepsilon \frac{\partial^2}{\partial \xi^2}, \quad \varepsilon \frac{\partial^2}{\partial \tau^2}.
\]
Since \( F \) has its fourth order derivatives bounded in \( \varepsilon \), the fifth order derivatives of its regularization \( (F)_\varepsilon \) can be written as \( O(\varepsilon^{-1}) \). Therefore, the operator \( L^o \) applied to the third order derivatives of \( (F)_\varepsilon \) results in a quantity uniformly bounded in \( \varepsilon \). For the rest, the proof of the estimate for \( X_4 \) literally follows the case of \( X_1, X_2, \) and \( X_3 \). Thus, we finally see that the derivatives of \( w^n_\varepsilon \) up to the fourth order are bounded uniformly in \( \varepsilon \).

**Theorem 12** The solutions \( w^n_\varepsilon \) of problem (16)/(17) in \( Q \) converge, as \( \varepsilon \to 0 \), to the function \( w^n \) which is a solution of problem (8)/(9) in \( \Omega \) and has its derivatives up to the fourth order bounded in \( \Omega \).

**Proof:** By Lemma 11, the derivatives of \( w^n_\varepsilon \) up to the fourth order are uniformly bounded in \( \varepsilon \). Therefore, there is a subsequence \( w^n_{\varepsilon_k} \) such that \( w^n_{\varepsilon_k} \), together with their derivatives up to the third order, are uniformly convergent to \( w^n \) in \( Q \) as \( \varepsilon_k \to 0 \). The limit function \( w^n(\tau, \xi, \eta) \) satisfies equation (8) in \( \Omega \), as well as the boundary condition (10). Let us show that the condition in (9) hold for \( w^n \). To this end, we prove
that \( w^n = w^* \) in \( Q \setminus \Omega_1 \). Set \( z = w^n - w^* \). By construction, we have

\[
a_1 z_{\tau\tau} + a_2 z_{\xi\xi} + a_3 z_{\eta\eta} + \nu(w^*)^2 z_{\eta\eta} - z_{\tau\tau} - \eta z_{\xi\xi}
\]

\[
+ [p_x + 2\nu(w^*)^2]w_{\eta\eta} - 2a_1 z = 0,
\]

in \( Q \setminus \Omega_1 \), and \( \partial z / \partial n = 0 \) on the part of the boundary of \( Q \setminus \Omega_1 \) that belongs to \( S \). In \( Q \setminus \Omega_1 \), consider the function \( z^* \) defined by \( z = z^* \psi_1(\tau) \), where \( \psi_1(\tau) \) is the function constructed in the proof of Lemma 10. For \( z^* \) we obtain an equation in \( Q \setminus \Omega_1 \) with the coefficient of \( z^* \) being strictly negative in the closure of \( Q \setminus \Omega_1 \).

Let \( E(\tau, \xi, \eta) \) be a smooth function in \( Q \) such that \( \partial E / \partial n < 0 \) on \( S \), and \( E > 1 \). Set \( z_1 = z^* (E + C) \), where \( C \) is a positive constant. It is easy to see that in the equation for \( z_1 \) the coefficient of \( z_1 \) is negative if \( C \) is sufficiently large. The boundary condition on \( S \) for \( z_1 \) will have the form

\[
\frac{\partial z_1}{\partial n} - \alpha_1 z_1 = 0, \quad \text{where} \quad \alpha_1 = - \frac{\partial E}{\partial n} > 0.
\]

Clearly, \( |z_1| \) cannot attain its largest value on \( S \), for at the point of maximum of \( |z_1| \) on \( S \) we must have

\[
\frac{\partial z_1}{\partial n} - \alpha_1 (z_1)^2 < 0,
\]

which is incompatible with the boundary condition on \( S \). The largest value of \( |z_1| \) cannot be attained inside \( Q \setminus \Omega_1 \), for at the point of its maximum we must have \( z_{1\tau} = 0, z_{1\xi} = 0, z_{1\eta} = 0, z_{1z_{1\eta\eta}} \leq 0, z_{1z_{1\xi\xi}} \leq 0 \), and \( z_1 z_{1\tau\tau} \leq 0 \), which is in contradiction with the equation obtained for \( z_1 \) at that point.

In a similar way, it can be shown that the maximum of \( |z_1| \) can be attained neither for \( \tau = 0 \) nor for \( \xi = 0 \) on the boundary of \( Q \setminus \Omega_1 \). It follows that \( z_1 \equiv 0 \) in \( Q \setminus \Omega_1 \), and therefore, \( w^n = w^* \) in \( Q \setminus \Omega_1 \). Hence, we see that

\[
w^n(0, \xi, \eta) = w_0, \quad w^n(\tau, 0, \eta) = w_1.
\]

Let us show that \( w^n = 0 \) on the surface \( \eta = U(\tau, \xi) \). It follows from the above results that \( w^n = 0 \) for \( \tau = 0 \) and \( \eta = U(0, \xi) \), as well as for \( \xi = 0 \) and \( \eta = U(\tau, 0) \). Since \( w^{n-1} = 0 \) on the surface \( \eta = U(\tau, \xi) \), the equation

\[
w^n + \eta w^n_{\eta\eta} - p_x w^n_{\eta\eta} = 0,
\]

holds for \( w^n \) on that surface. As indicated above, the vectors \( (1, \eta, -p_x) \) belong to planes tangential to the surface \( \eta = U(\tau, \xi) \) and form a vector field on that surface. Integral curves of that field, being extended for smaller values of \( \tau \), will cross the border of the surface either at \( \xi = 0 \) or at \( \tau = 0 \), where \( w^n = 0 \). Since \( w^n \) is constant on these integral curves, \( w^n = 0 \) on the entire surface \( \eta = U(\tau, \xi) \). Note that the function \( w^n \) constructed above has its third order derivatives in \( \Omega \) satisfying the Lipschitz condition.

**Theorem 13** Assume that \( p(t, x), v_0(t, x), u_0(x, y), u_1(t, y), w_0(\xi, \eta), w_1(\tau, \eta), v(y), g(t, x) \) are sufficiently smooth and satisfy the compatibility conditions which amount to the existence of the function \( w^* \) mentioned earlier. Then there is one and only one solution of problem (1)-(3) in the domain \( D \), with \( X \) being arbitrary and \( T \) depending on the data of problem (1)-(9), or \( T \) being arbitrary and \( X \) depending on the data. This solution has the following properties: \( u > 0 \) for \( y > 0 \), \( u_y > 0 \) for \( y \geq 0 \); the derivatives \( u_t, u_x, u_y, u_{yy}, v_y \) are continuous and bounded in \( D \); moreover, the ratios

\[
\frac{u_{yy}}{u_y}, \quad \frac{u_{yy} u_y - u_y^2}{u^3_y},
\]

are bounded in \( D \).

**Proof:** Let \( w \) be the solution of problem (5)-(7) constructed in the proof of Theorem 12. Let \( u \) be defined by the condition \( w = u_y \), or

\[
y = \int_0^u \frac{ds}{w(t, x, s)}.
\]

Since \( w(t, x, s) > 0 \) for \( s < U(t, x) \), and \( w = 0 \) for \( s = U(t, x) \), we have \( u \to U(t, x) \) as \( y \to \infty \), and \( 0 < u < U(t, x) \) for \( 0 < y < \infty \), \( w(t, x, 0) = 0 \). The condition \( u(0, x, y) = u_0 \) and \( u(t, 0, y) = u_1 \) are also valid, since \( w_0 = u_{yy} \) and \( w_1 = u_{yy} \). The function defined by (24) possesses the derivatives

\[
u_y = \nu_{\eta w}, \quad \nu_{yy} = \nu_{\eta w}^2 + \nu_{\eta w} w,
\]

The derivatives \( u_t \) and \( u_x \) are given by

\[
\begin{align*}
u_t &= -w \int_0^u \frac{w_t(t, x, s)}{w(t, x, s)} ds, \\
u_x &= -w \int_0^u \frac{w_x(t, x, s)}{w(t, x, s)} ds.
\end{align*}
\]

Set

\[
v = \frac{-u_t - u u_x - p_x + (\nu u_y)_y}{u_y}.
\]

Let us show that \( u \) and \( v \) defined by (24) and (25) satisfy system (1). Differentiating the relation \( u_y = w \), we find that there exist the derivatives

\[
u_{yt} = w_x + u_x w_\eta, \quad \nu_{yx} = w_x + u_x w_\eta.
\]
Therefore, \( v \) admits the derivatives in \( y \). Differentiating (25) in \( y \), we obtain

\[
v_y u_y + v u_{yy} = -u_y - u_y u_x - u_{xy} + v y u_y + 2v_y u_{yy} + \nu v u_{yy}
\]

(26)

The function \( w \) satisfies equation (5). Replacing in (5) the derivatives of \( w \) by their expressions in terms of derivatives of \( u \), we find that

\[
\nu u_y^2 w_y - u_y^2 w_y - u_y + u_t v_y u_y - u( u_y - \frac{u_x u_{yy}}{u_y} ) + p_x - \frac{u_{yy}}{u_y} + \nu v u_y u_y + v y u_{yy} = 0.
\]

(27)

It follows from (26) and (27) that

\[
u u_y^2 w_{yy} - u_y^2 w_{yy} - u_y + u_t v_y u_y - u( u_y - \frac{u_x u_{yy}}{u_y} ) + p_x - \frac{u_{yy}}{u_y} + \nu v u_y u_y + v y u_{yy} = 0.
\]

(28)

Let us show that \( v(t,x,0) = v_0(t,x) \). It follows from (7) that

\[
v_0 = \frac{\nu u w_y - p_x + \nu u w_2 - g_x - g g y_{\xi}}{w} |_{\eta = 0}.
\]

From (25) we find that

\[
v |_{y=0} = \frac{\nu u w_y - p_x + \nu u w_2 - g_x - g g y_{\xi}}{w} |_{y=0} = \frac{\nu u w_y - p_x + \nu u w_2 - g_x - g g y_{\xi}}{w} |_{\eta = 0} = v_0.
\]

Thus we have proved the existence of a solution for problem (1)-(3) in the class of smooth functions. Its uniqueness is able to been established by a similar way as Theorem 4.2.2 of [3], we omit the details here.

5 Appendix

At the last of the paper, we give the details of the proof to the inequality (11). As in the section 2, we have

\[
\Phi_n = \Phi_n^* \equiv (W_n^r)^2 + (W_n^\xi)^2 + (W_n^\eta)^2 + k_0 + k_1 \eta.
\]

Applying the operator

\[
2W_r^\eta \frac{\partial}{\partial \tau} + 2W_\xi^r \frac{\partial}{\partial \xi} + 2W_\eta^\eta \frac{\partial}{\partial \eta}
\]

to the equation

\[
L_0^\phi(W^n) + B^n W^n = 0,
\]

we find that

\[
0 = \nu(w^{n-1})^2 \Phi^*_{m\eta} - \Phi^*_{m\tau} - \eta \Phi^*_{m\xi} + A^n \Phi^*_{m\xi} + 2B^n \Phi^*_{n\tau}
\]

\[
-2\nu(w^{n-1})^2 \left\{ (W_{\eta}^n)^2 + (W_{\xi}^n)^2 + (W_{\eta}^n)^2 \right\}
\]

\[
+2\nu(w^{n-1})^2 \left\{ (W_{n\eta}^n)^2 + 2\nu((w^{n-1})^2 \xi)(W_{\eta}^n)(W_{\xi}^n) + 2\nu(w^{n-1})^2 \nu(W_{\eta}^n)(W_{\eta}^n) + [-2W_{\tau}^\xi W_{\eta}^n + 2A^n(W_{\eta}^n)^2]
\]

\[
+2A^n W_{\eta}^n W_{\xi}^r + 2A^n W_{\eta}^n W_{\xi}^r
\]

\[
+2W(B_{\eta}^n W_{\eta}^n + B_{\xi}^n W_{\xi}^n + B_{\eta}^n W_{\eta}^n)] - B^n(k_1 \eta + k_0) - A^n k_1 + 2W_n^r [\nu \Phi^*_{m\eta} \Phi^*_{m\xi}((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r] + 2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r.
\]

(29)

for the last three terms, we have

\[
2W_n^r [\nu \Phi^*_{m\eta} \Phi^*_{m\xi}((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r
\]

\[
= 2\alpha W_n^r [\nu \Phi^*_{m\eta} \Phi^*_{m\xi}((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r + 2\alpha W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r + 2\alpha W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r.
\]

\[
= 2W_n^r [\nu \Phi^*_{m\eta} \Phi^*_{m\xi}((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r.
\]

\[
\leq h_1(W_n^r)^2 + h_2(W_{\xi}^r)^2 + h_3(W_{\eta}^r)^2 + h(w^{n-1})^6,
\]

where \( h_1, h_2, h_3 \) are taken large enough, \( h = h_1 + h_2 + h_3 \)

\[
2W_n^r [\nu \Phi^*_{m\eta} \Phi^*_{m\xi}((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
\leq h_1(W_n^r)^2 + h_2(W_{\xi}^r)^2 + h_3(W_{\eta}^r)^2 + h(w^{n-1})^6
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]
\]

\[
+2W_n^r [\nu \Phi^*_{m\eta} ((w^{n-1})^3)_r]_r.
\]

\[
\text{Denote by } I_1 \text{ the terms in the first square brackets in (29), we obtain the following estimate from above}
\]

\[
I_1 \leq R_1(W_n^r)^2 + (W_{\xi}^r)^2 + (W_{\eta}^r)^2 + \nu \frac{\nu}{R_1} [((w^{n-1})^2)_r]^2 + ((w^{n-1})^2)_\xi^2
\]

\[
+((w^{n-1})^2)_\eta^2 (W_{\eta}^r)^2,
\]

\[
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\]
where $R_1$ is a constant.

It is well known that (see for instance [19]) any non-negative function $q(x)$ defined on interval $-\infty < x < +\infty$ and having bounded second derivatives on that interval satisfies the inequality

$$ (q_x)^2 \leq 2\max |q_{xx}| q(x). $$

The function $(w^{n-1})^2$ or $(w^{n-1})^2$ can be extended to the entire real axis with respect to any of its independent variables, so that its extension is a non-negative bounded function whose second derivative has its absolute value less than or equal to the maximum modulus of the second derivative of $(w^{n-1})^3$ or $(w^{n-1})^2$.

Therefore

$$ \nu_{\eta\eta}e^{\alpha(n-g)} h_4 \left\{ \left[ (w^{n-1})^3 \right]_{\eta}^2 + \left[ (w^{n-1})^2 \right]_{\eta}^2 \right\} \leq |\nu_{\eta\eta}| (w^{n-1})^3, $$

where $R_1$, $h_4$ is chosen sufficiently large. The constant $R_1$ depends on the second derivative of the functions $(w^{n-1})^2$ and $h_4$ depends on the second derivative of the functions $(w^{n-1})^3$.

Denote by $I_2$ the terms enclosed by the second square brackets in (29). By virtue of the inequality $2ab \leq a^2 + b^2$, these terms can be estimated from above by the expression $R_2 \Phi_n + k_8$, where the constant $R_2$ depends on the first order derivative of the functions $w^{n-1}$, $k_8$ is independent of $n$.

Therefore, in the region $n \geq \delta_2$, we have

$$ L_n^0(\Phi_n) + R_3 \Phi_n + k_9 \geq 0 \quad \text{or} \quad L_n^0(\Phi_n) + R^n \Phi_n \geq 0, $$

where the constant $k_9$ is independent of $n$, and the function $R^n$ depends on the first and the second derivatives of $w^{n-1}$.

In order to estimate $L_n^0(\Phi_n)$ in $\Omega$ for $n \leq 2$, we should also calculate $L_n^0(2W_n^0 H_n)$

$$ L_n^0(2W_n^0 H_n) = 2H_n L_n^0(W_n) + 2W_n^0 L_n^0(H_n) $$

$$ + 4\nu (w^{n-1})^2 W_{\eta\eta}^0 H_n $$

$$ = 2H_n \left\{ -\nu (w^{n-1})^2 W_{\eta\eta}^0 W_n + W_n^0 (W_n^2 - A_n^0 W_n^3) - B_n^0 W_n^3 - B_n^0 W_n^3 - \nu (w^{n-1})^3 e^{\alpha(n-g)} \eta \right\} $$

$$ + 2W_n^0 \left[ L_n^0(\nu_{\eta\eta}) \right] + L_n^0 \left( \frac{\nu_{\eta\eta}}{\nu W_n^3 + 1} \right) - \alpha (\eta) B_n^0 W_n^3 $$

$$ - \nu (\eta) \nu (w^{n-1})^3 e^{\alpha(n-g)} + \alpha W_n^0 L_n^0(\chi) $$

$$ + 2\alpha (\eta) (w^{n-1})^2 W_n^\eta \chi + L_n^0 \left( \frac{\nu_{\eta\eta} - g W_n^0}{\nu W_n^3 + 1} - \nu \right) $$

$$ + 4\nu (w^{n-1})^2 W_{\eta\eta}^0 H_n. $$

According to Lemma 3 and Lemma 4, $(w^{n-1})^2 > \gamma_0 > 0$ for $n \leq \delta_2$. There, the terms $I_1$ in (11) together with $2H_n \nu (w^{n-1})^2 W_{\eta\eta}$, can be estimated with the help of the inequality $2ab \leq a^2 + b^2$ as follows:

$$ I_1 + 2H_n \nu (w^{n-1})^2 W_{\eta\eta} \leq \frac{1}{2} \nu \gamma_0 (W_{\eta\eta}^2) + R_4 \Phi_n + k_{10}, $$

where the constant $R_4$ does not depend on $n$. It follows from (11) and $L_n^0(2W_n^0 H_n)$ that

$$ L_n^0(\Phi_n) + R_5 \Phi_n + R_6 \geq 0 \quad \text{for} \quad \eta \leq \delta_2, $$

where $R_5$ and $R_6$ are constant that depend neither on $w^{n-1}$ nor on its derivatives up to the second order. Since $\Phi_n \geq 1$, we have $R_5 \Phi_n \geq R_6$, Therefore

$$ L_n^0(\Phi_n) + R^n \Phi_n \geq 0. $$

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