Normal structure, slices and other properties in Banach spaces

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Abstract: In this paper, we introduce parameters $sl_\varepsilon(X)$ and $sl_0(X)$ based on slices of Banach space X. Using these parameters we describe some new properties of Banach spaces related to normal structure, uniformly non-squareness and others. In particular, we prove that if $sl_{\frac{2}{3}}(X) < 2$, then X has normal structure, and $sl_0(X) = \varepsilon_0(X)$ where $\varepsilon_0(X)$ is the characteristic of convexity of X. In addition, we give much more results about the modulus of NUC on X, and the modulus of UKK^* on the dual space X^* of X.

Key–Words: fixed point, modulus of NUC, modulus of UKK*, normal structure, Slices, super-reflexive, and ultraproduct space.

1 Introduction

Let X be a normed linear space, and let $B(X) = \{x \in X : \|x\| \le 1\}$, $S(X) = \{x \in X : \|x\| = 1\}$, and $B_\gamma(0) = \{x \in X : \|x\| < \gamma\}$ be the unit ball, the unit sphere, and open ball with radius γ of X respectively. Let X^* be the dual space of X.

A mapping T in a normed space X is called a non-expansive if $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in X$.

It is well known that a contractive mapping has unique one fixed point on Banach space X. However, a non-expansive mapping may have no fix point if X is anyone Banach space. One of the remaining unsolved questions is whether each non-expansive mapping on bounded, closed, convex subset in reflexive Banach space has a fixed point. The question for non-reflexive Banach space is false in general.

In order to study the existence of fixed point for the non-expansive mapping, one divided the Banach space X into the below types.

X is said to have the fixed point property if every non-expensive self-mapping of a nonempty closed convex subset of X has a fixed point.

X is said to have the weak fixed point property if every non-expensive self-mapping of a nonempty weak compact convex subset of X has a fixed point.

A dual space X^* is said to have the weak* fixed point property if every non-expensive self-mapping of a nonempty weak* compact convex subset of X^* has a fixed point.

Brodskiĭ and Mil'man [2] introduced the following geometric concepts in 1948 to study the fixed point properties under isometry which maps a weakly compact set to itself:

Definition 1 *A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that*

$$\sup\{\|x_0 - y\| : y \in H\} < d(H),$$

where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H.

A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure.

A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X has normal structure.

X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \cdot (d(K)).$$

For a reflexive Banach space, the normal structure and weak normal structure coincide.

In 1965, Kirk [19] proved that if a Banach space X has weak normal structure then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point. Since then much attention has been focused on normal structure. Whether or not a Banach space X has normal structure depends on the geometry of the unit sphere $S(X)$, or unit ball $B(X)$.

The use of parameters to study normal structure in a numerical manner is an important direction in this field. For instance, the modulus of convexity $\delta(\varepsilon)$ was introduced to study the relationship between convexity and normal structure, the modulus of smoothness $\rho(\tau)$ was introduced to study the smoothness and normal structure, the parameters $j(X), J(X), g(X)$ and $G(X)$ were introduced to study the relationship between squareness and normal structure, the coefficient $w(X)$ was introduced to study the relationship between weakly null sequences and normal structure, the modulus of noncompact convexity associated to the measure of non-compactness $\beta(A)$, where A is a bounded set in X was introduced to study normal structure, the modulus of U-convexity, $U(\varepsilon)$ was introduced to study the relationship between U-spaces and normal structure, the parameter $R(X)$ was introduced to study the relationship between arc length of $S(X)$ and normal structure, the normal structure was also studied by Pythagorean approach and so on in many literatures and articles in the last fifty years. We refer the interested readers to [3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28].

In this paper, we first introduce the concepts $sl_\varepsilon(X)$ and $sl_0(X)$ based on slices of X in section 2. The relationship between this parameter and the normal structure, uniformly non-squareness and other properties are obtained. Among these new results we prove that if $sl_{\frac{2}{3}}(X) < 2$, then X has normal structure; and $sl_0(X) = \varepsilon_0(X)$ where $\varepsilon_0(X)$ is the characteristic of convexity of X. Then, based on theorem 2.11 in [27], some corollaries about modulus of NUC on X, and modulus of UKK^* on the dual space X^* of X are also shown in section 3. Finally we consider uniform normal structure in section 4.

2 Slices and Normal structure

Definition 2 ([8]) *Let X be a Banach space. A hexagon H in X is called a normal hexagon if the length of each side is 1 and each pair of two opposite sides are parallel.*

Remark: The concept of normal hexagon is different from the concept of regular hexagon in Euclidean spaces. We may consider the normal hexagon as an image of a regular hexagon under a bounded linear mapping from an Euclidean space to a Banach space.

Lemma 3 ([8], [14]) *Let X be a Banach space without weak normal structure. Then for any $0 < \delta < 1$, there are x_1, x_2, and x_3 in $S(X)$ satisfying*

(i) $x_2 - x_3 = x_1$;

(ii) $\|\frac{x_1+x_2}{2}\| > 1 - \delta$; *and*

(iii) $\|\frac{x_3+(-x_1)}{2}\| > 1 - \delta$.

The geometric meaning of the lemma is that if a Banach space X fails to have weak normal structure then there is an inscribed normal hexagon with four sides are arbitrarily closed to the unit sphere $S(X)$.

Definition 4 *[6] Let D be a bounded subset of a Banach space X and suppose that $f \in X^*$, $f \neq 0$. Let*

$$M(D, f) = \sup\{\langle x, f\rangle : x \in D\}.$$

If $\alpha > 0$ then the set

$$S(D, f, \alpha) \equiv \{x \in D : \langle x, f\rangle > M(D, f) - \alpha\}$$

is called the slice of D determined by f and α, or more briefly, a slice of D.

Lemma 5 *[9] Let $x, y \in B(X)$ and $0 < \epsilon < 1$ such that $\frac{\|x+y\|}{2} > 1 - \epsilon$, then for all $0 \leq c \leq 1$ and $z = cx+(1-c)y \in [x, y]$, the line segment connecting x and y, $\|z\| > 1 - 2\epsilon$.*

Definition 6 *Let $B(X)$ the unit ball of X, and $f \in S(X^*)$, we define*

$$sl_\varepsilon(X) \equiv \sup\{d(S(B(X), f, \varepsilon)) : f \in S(X^*)\},$$

and

$$sl_0(X) \equiv \inf_{\varepsilon\to 0} sl_\varepsilon(X)$$

$$= \inf_{\varepsilon\to 0}\{\sup\{d(S(B(X), f, \varepsilon)) : f \in S(X^*)\}\},$$

where $d(H)$ is the diameter of set H.

It is clear that

(i) $sl_\varepsilon(X)$ is an increasing function. So, if $\varepsilon_1 \leq \varepsilon_2$, then $sl_{\varepsilon_1}(X) \leq sl_{\varepsilon_2}(X)$.

(ii) $0 \leq sl_\varepsilon(X) \leq 2$.

Theorem 7 *If a Banach space X fails to have weak normal structure, then $sl_0(X) \geq 1$.*

Proof: Suppose X fails to have weak normal structure. For $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$, and x_1, x_2, and x_3 in $S(X)$ satisfy three conditions in lemma 3 for $\delta > 0$.

We have for any $0 \leq t \leq 1$,

$$\|t(-x_1)+(1-t)x_3\| \geq 1-2\delta = 1-\varepsilon$$

by lemma 5.

Let X_2 be a two dimensional subspace of X spanned by x_1, x_2, and x_3. Let

$$x_3^* \in S(X_2^*) \text{ such that } \langle x_3, x_3^*\rangle = 1$$

and

$$-x_1^* \in S(X_2^*) \text{ such that } \langle -x_1, -x_1^*\rangle = 1.$$

Then

$$\langle x_2, x_3^*\rangle \geq 0 \text{ and } \langle x_2, -x_1^*\rangle \leq 0.$$

Therefore, there exists an

$$x \in \widehat{x_3, -x_1} \subseteq S(X_2) \text{ and an } x^* \in S(X^*)$$

such that

$$\langle x, x^*\rangle = 1 \text{ but } \langle x_2, x^*\rangle = 0,$$

where $\widehat{x_3, -x_1}$ be an arc on $S(X_2)$ from x_3 to $-x_1$ counter-clockwise.

Then for any $0 \leq t \leq 1$,

$$\begin{aligned}\langle t(-x_1)+(1-t)x_3, x^*\rangle &= \langle x_3 - t(x_1+x_3), x^*\rangle \\ &= \langle x_3 - tx_2, x^*\rangle = \langle x_3, x^*\rangle.\end{aligned}$$

There is a $0 \leq t_1 \leq 1$, such that

$$\alpha x = t_1(-x_1)+(1-t_1)x_3 \text{ for some } \alpha \geq 0.$$

We have

$$\begin{aligned}\langle t_1(-x_1)+(1-t_1)x_3, x^*\rangle &= \langle \alpha x, x^*\rangle \\ &= \|\alpha x\| \geq 1-2\delta = 1-\varepsilon.\end{aligned}$$

The geometrical meaning of the proof at this stage and the line $\langle y, x^*\rangle = 1, y \in X_2$ is shown in the figure 1.

By using Hahn-Banach extension theorem, we can extend $x^* \in S(X_2^*)$ to an $f \in S(X^*)$.

So,

$$[-x_1, x_3] \subseteq S(B(X), f, \varepsilon).$$

Therefore

$$d(S(B(X), f, \varepsilon)) \geq \|x_3 - (-x_1)\| = 1.$$

Since ε can be arbitrarily small, we have $sl_0(X) \geq 1$. □

Theorem 8 *If a Banach space X fails to have weak normal structure, then $sl_{\frac{2}{3}}(X) \geq 2$.*

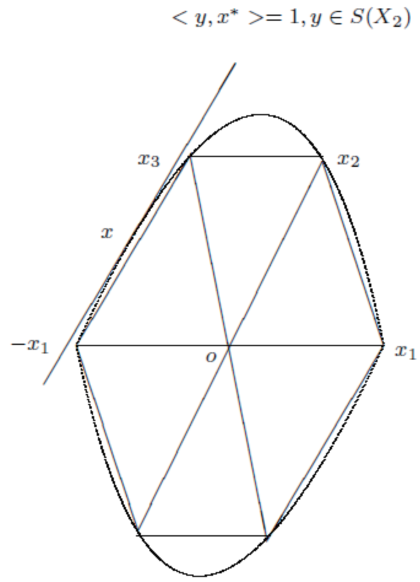

Figure 1: The geometrical meaning of the proof at this stage and the line $< y, x^* >= 1, y \in X_2$

Proof: Suppose X fails to have weak normal structure. For $\varepsilon > 0$, let δ, x_1, x_2, x_3 in $S(X)$ and X_2 be the same as in the proof of theorem 7.

Let

$$y(t) = -x_1 + t(x_2+x_1), 0 \leq t \leq 1,$$

then

$$\|\frac{-x_1+x_2}{2}\| = \|y(\frac{1}{2})\| = \frac{1}{2}.$$

Let

$$z = -x_1 + 2(x_3+x_1) = x_1 + 2(x_2 - x_1),$$

then $z \in X_2 \setminus B(X_2)$.

We first prove $\|y(t)\| \geq \|\frac{z}{3}\| \geq \frac{1}{3}$ for any $0 \leq t \leq 1$.

Since for any $0 \leq t \leq 1$,

$$\|t(-x_1)+(1-t)x_3\| \geq 1-\varepsilon,$$

the line segment

$$[\frac{-x_1}{1-\varepsilon}, \frac{x_3}{1-\varepsilon}] \subseteq X_2 \setminus B(X_2).$$

For $0 \leq t \leq \frac{1}{2}$, let $h > 0$ such that

$$\frac{y(t)}{h} \in [\frac{-x_1}{1-\varepsilon}, \frac{x_3}{1-\varepsilon}].$$

then

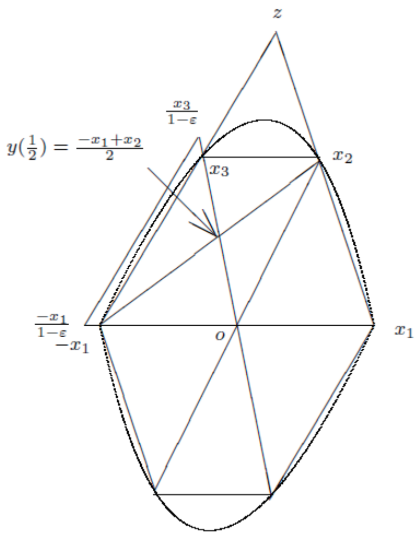

Figure 2: The geometrical meaning for $\|y(t)\| \geq \|\frac{z}{3}\| \geq \frac{1}{3}$

$$\frac{\|y(t)\|}{h} \geq 1, \|y(t)\| \geq h.$$

From the convexity of $B(X)$,

$$h = \frac{\|y(t)\|}{\|\frac{y(t)}{h}\|} \geq \frac{\|\frac{-x_1+x_2}{2}\|}{\|\frac{x_3}{1-\varepsilon}\|} = \frac{1-\varepsilon}{2}.$$

We have $\|y(t)\| \geq \frac{1-\varepsilon}{2}$, for $0 \leq t \leq \frac{1}{2}$.
We have

$$y(\frac{2}{3}) = -x_1 + \frac{2}{3}(x_2 + x_1) = \frac{1}{3}z.$$

For both $\frac{1}{2} \leq t \leq \frac{2}{3}$ and $\frac{2}{3} \leq t \leq 1$, from the convexity of $B(X)$, we have

$$\|y(t)\| \geq \|\frac{z}{3}\| \geq \frac{1}{3}.$$

The idea of the proof of $\|y(t)\| \geq \|\frac{z}{3}\| \geq \frac{1}{3}$ for any $0 \leq t \leq 1$ is shown in the Figure 2.
We have

$$\|x_1 + x_2\| \geq 2 - \varepsilon.$$

Let

$$x_2^* \in S(X_2^*) \text{ such that } \langle x_2, x_2^* \rangle = 1$$

and

$$-x_1^* \in S(X_2^*) \text{ such that } \langle -x_1, -x_1^* \rangle = 1.$$

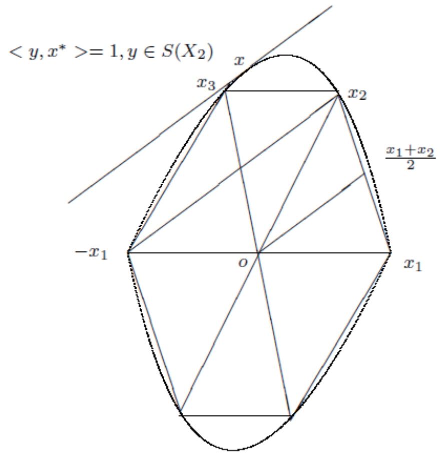

Figure 3: $< y, x^* >= 1, y \in X_2$

Then

$$\left\langle \frac{x_1 + x_2}{2}, x_2^* \right\rangle \geq 0$$

and

$$\left\langle \frac{x_1 + x_2}{2}, -x_1^* \right\rangle \leq 0.$$

Therefore, there exists an

$$x \in \widehat{x_2, -x_1} \subseteq S(X_2) \text{ and an } x^* \in S(X^*)$$

such that

$$\langle x, x^* \rangle = 1 \text{ but } \langle \tfrac{x_1+x_2}{2}, x^* \rangle = 0,$$

where $\widehat{x_2, -x_1}$ be an arc on $S(X_2)$ from x_2 to $-x_1$ counter-clockwise.
Then for any $0 \leq t \leq 1$,

$$\begin{aligned}\langle t(-x_1) + (1-t)x_2, x^* \rangle &= \langle x_2 - t(x_1 + x_2), x^* \rangle \\ &= \langle x_2, x^* \rangle.\end{aligned}$$

There is a $0 \leq t_1 \leq 1$, such that

$$\beta x = t_1(-x_1) + (1 - t_1)x_2 \text{ for some } \beta \geq 0.$$

We have

$$\begin{aligned}\langle t_1(-x_1) + (1-t_1)x_3, x^* \rangle = \langle \beta x, x^* \rangle = \beta \geq \tfrac{1}{3} \\ = 1 - \tfrac{2}{3}.\end{aligned}$$

The geometrical meaning of the proof at this stage and the line $< y, x^* >= 1, y \in X_2$ is shown in the Figure 3.
By using Hahn-Banach extension theorem, we can extend $x^* \in S(X_2^*)$ to an $f \in S(X^*)$.
So,

$$[-x_1, x_2] \subseteq S(B(X), f, \tfrac{2}{3}).$$

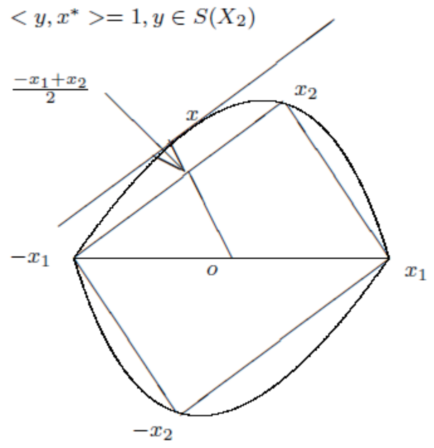

Figure 4: $< y, x^* >= 1, y \in X_2$

Therefore

$$d(S(B(X), f, \tfrac{2}{3})) \geq \|x_2 - (-x_1)\| \geq 2 - \varepsilon.$$

Since ε can be arbitrarily small, we have $sl_{\frac{2}{3}}(X) \geq 2$. □

Definition 9 *[18] A normed linear space is uniformly nonsquare if there exists a $\delta > 0$ such that for any $x, y \in S(X)$,*

$$\text{either } \|x+y\| \leq 2(1-\delta) \text{ or } \|x-y\| \leq 2(1-\delta).$$

Theorem 10 *If a Banach space X fails to be a uniformly nonsquare then $sl_0(X) = 2$.*

Proof: The proof is similar to the proof of theorem 7.

The geometrical meaning of the proof and the line $< y, x^* >= 1, y \in X_2$ is shown in the Figure 4. □

Since uniformly nonsquare implies super-reflexive, and super-reflexive implies reflexive, we have:

Corollary 11 *A Banach space X with $sl_0(X) < 2$ is uniformly nonsquare, therefore X is super-reflexive and then X is reflexive.*

Corollary 12 *A Banach space X with $sl_0(X) < 1$ has normal structure.*

Corollary 13 *A Banach space X with $sl_{\frac{2}{3}}(X) < 2$ has normal structure.*

Let

$$\delta_X(\epsilon) = \inf\{1 - \tfrac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \geq \epsilon\}$$

where $0 \leq \epsilon \leq 2$ be the modulus of convexity of X and

$$\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\epsilon) = 0\}.$$

be the characteristic of convexity of X.(For example, see [6].)

Theorem 14 *For a Banach space X, $sl_0(X) = \varepsilon_0(X)$.*

Proof: We first prove $sl_0(X) \geq \varepsilon_0(X)$.

For $\varepsilon_0(X) = 0$, it is true.

Let $\varepsilon_0(X) > 0$, for any $\eta > 0$, there are $x, y \in S(X)$, such that

$$\|x-y\| \geq \varepsilon_0(X) - \eta \text{ and}$$

$$1 - \tfrac{\|x+y\|}{2} \leq \eta.$$

We have

$$\|x-y\| \geq \varepsilon_0(X) - \eta \text{ and } \tfrac{\|x+y\|}{2} \geq 1 - \eta.$$

By using the same idea of proof of theorem 7, we get

$$sl_0(X) \geq \varepsilon_0(X) - \eta.$$

Since η can be arbitrarily small, we have $sl_0(X) \geq \varepsilon_0(X)$.

We next prove $\varepsilon_0(X) \geq sl_0(X)$.

For any $\eta > 0$, there is a slice $S(B(X), f, \eta)$ such that

$$d(S(B(X), f, \eta)) \geq sl_0(X), \text{ where } f \in S(X^*).$$

Let $x, y \in S(B(X), f, \eta)$ be such that $\|x-y\| \geq sl_0(X)$ and X_2 be a subspace of X spanned by x and y. Then,

$$\frac{x+y}{2} \in S(B(X), f, \eta)$$

and

$$\frac{\|x+y\|}{2} \geq \left\langle \frac{x+y}{2}, f \right\rangle \geq 1 - \eta.$$

We get $\|x-y\| \geq sl_0(X)$ and $1 - \frac{\|x+y\|}{2} \leq \eta$.

Since η can be arbitrarily small, we have $\varepsilon_0(X) \geq sl_0(X)$.

This complete the proof. □

Since X is uniformly convex if and only if $\varepsilon_0(X) = 0$. We have

Corollary 15 *X is uniformly convex if and only if $sl_0(X) = 0$.*

3 More results for UKK* and NUC spaces with Normal Structure

Let $co(x_n)$ be the convex hull of the sequence $\{x_n\}$, and let

$$sep(x_n) \equiv \inf\{\|x_n - x_m\| : n \neq m\}.$$

We also write $\xrightarrow{w}$ and $\xrightarrow{w^*}$ for the weak and the weak* convergence of the sequence respectively, and write $f(a^-)$ for $\lim_{\varepsilon \to a^-} f(\varepsilon)$ for a function $f(\varepsilon)$.

We say an infinite sequence (x_n) is an ε-separation sequence if

$$\|x_m - x_n\| \geq \varepsilon \text{ for } m \neq n.$$

Let

$$\mu(X) = \sup\{\varepsilon : B(X) \text{ contains an infinite } \varepsilon\text{-separation sequence}\}$$

([1], [20]).

Definition 16 ([17]) *A Banach space X is called an NUC space (Nearly Uniform Convex space) if for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for any sequence $\{x_n\} \subseteq B(X)$ with $sep(x_n) \geq \varepsilon$ it follows that $co(x_n) \bigcap B_\delta(0) \neq \emptyset$.*

Dowling et al. [7] introduced the concept of UKK* spaces in 2008.

Definition 17 ([7]) *Let X be a Banach space. The dual space X^* of X is called a UKK* space (Uniform Kadec–Klee* space) if for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for any sequence $\{f_n\} \subseteq B(X^*)$ with $f_n \xrightarrow{w^*} f$ and $sep(f_n) \geq \varepsilon$ it follows that $f \in B_\delta(0)$ for X^*.*

Based on the concepts NUC and UKK*, Saejung and Gao introduced the following two concepts into the Banach space X and its dual space X^* [27]:

Definition 18 *Let X be a Banach space.*

(i) For $0 < \varepsilon \leq \mu(X)$, let

$$NUC(\varepsilon) = 1 - \delta,$$

where δ is the smallest number in $(0, 1]$ such that

$$co(x_n) \bigcap B_\delta(0) \neq \emptyset,$$

whenever $\{x_n\} \subseteq B(X)$ satisfies $sep(x_n) \geq \varepsilon$.

(ii) For $\mu(X) < \varepsilon \leq 2$, let $NUC(\varepsilon) = 1$.

Then the function $NUC(\varepsilon)$ is called the NUC modulus of X.

Definition 19 *Let X^* be a dual of a Banach space X.*

(i) For $0 < \varepsilon \leq \mu(X^*)$, let

$$UKK^*(\varepsilon) = 1 - \delta,$$

where δ is the smallest number in $(0, 1]$ such that

$$f \in B_\delta(0),$$

whenever $\{f_n\} \subseteq B(X^*)$ satisfies $f_n \xrightarrow{w^*} f$ and $sep(f_n) \geq \varepsilon$.

(ii) For $\mu(X^*) < \varepsilon \leq 2$, let $UKK^*(\varepsilon) = 1$.

Then the function $UKK^*(\varepsilon)$ is called the UKK modulus of X^*.*

We obtain the following two results in [27]:

Theorem 20 *If X is a Banach space with $NUC(1^-) > 0$, then X is reflexive.*

Theorem 21 *Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If $UKK^*((\frac{1}{\mu(X^*)})^-) > 1 - \frac{1}{\mu(X^*)}$ for X^*, then X has weak normal structure.*

By using the above two theorems for $\mu(X^*) = 1$, it is easy to obtain the following results:

(i) Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If $UKK^*(1^-) > 0$ for X^* and $\mu(X^*) = 1$, then X has weak normal structure.

(ii) Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If X does not have weak normal structure then $UKK^*((\frac{1}{\mu(X^*)})^-) < 1 - \frac{1}{\mu(X^*)}$ for X^*.

(iii) Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If $UKK^*(1^-) > 0$ for X^* but X does not have weak normal structure then $\mu(X^*) > 1$.

(iv) Let X be a Banach space with $NUC((\frac{1}{\mu(X^*)})^-) > 1 - \frac{1}{\mu(X^*)}$ for X^*. Then X has normal structure.

(v) Let X be a Banach space with $NUC(1^-) > 0$ for X^* and $\mu(X^*) = 1$. Then X has normal structure.

(vi) Let X be a Banach space without normal structure. Then $NUC((\frac{1}{\mu(X^*)})^-) < 1 - \frac{1}{\mu(X^*)}$ for X^*.

(vii) Let X be a Banach space with $NUC(1^-) > 0$ for X^*. If X does not have normal structure, then $\mu(X^*) = 1$.

4 Uniform Normal Structure

Let F be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to converge to x with respect to F, denote by $\lim_F x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in F$.

A filter U on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion.

An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$.

Remark: We will use the fact that if U is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq U$ or $I \setminus A \subseteq U$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_U x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 22 *[29] Let U be an ultrafilter on I and let*

$$N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_U ||x_i|| = 0\}.$$

The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_U$ equipped with the quotient norm.

We will use $(x_i)_U$ to denote the element of the ultraproduct. It follows from remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_U\| = \lim_U \|x_i\| \tag{1}$$

In the following we will restrict our index set I to be $\mathbf{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbf{N}$ for some Banach space X. For an ultrafilter U on $\mathbf{N}$, we use X_U to denote the ultraproduct.

Lemma 23 ([29]) *Suppose U is an ultrafilter on $\mathbf{N}$ and X is a Banach space. Then*

(i) $(X^*)_U = (X_U)^*$ if and only if X is super-reflexive; and in this case,

(ii) the mapping J defined by

$$\langle (x_i)_U, J((f_i)_U)\rangle = \lim_U \langle x_i, f_i\rangle$$

for all $(x_i)_U \in X_U$, is the canonical isometric isomorphism from $(X^*)_U$ onto $(X_U)^*$.

Theorem 24 *For any Banach space X with $sl_0(X) < 2$, and for any nontrivial ultrafilter U on N, $sl_\varepsilon(X_U) = sl_\varepsilon(X)$.*

Proof: $sl_0(X) < 2$ implies X is uniformly non-square, so X is super-reflexive. We can use lemma 23.

Since X can be isometrically embedded onto X_U, we have $sl_\varepsilon(X) \le sl_\varepsilon(X_U)$.

To prove the reverse inequality, we may assume that $sl_\varepsilon(X) > 0$. For any $\eta > 0$ we choose an

$$f = (f_i)_U = S((X_U)^*),$$

and

$$(x_i)_U \in S(B(X_U), f, \eta), (y_i)_U \in S(B(X_U), f, \eta)$$

such that

$$\|(x_i)_U - (y_i)_U\| \ge sl_\varepsilon(X_U) - \eta.$$

We have

$$\langle (x_i)_U, f\rangle = \lim_U \langle x_i, f_i\rangle \ge 1 - \eta,$$

and

$$\langle (y_i)_U, f\rangle = \lim_U \langle y_i, f_i\rangle \ge 1 - \eta.$$

Without loss of generality, we may assume that $f_i \in S((X_i)^*)$ for all $i \in \mathbf{N}$.

From remark (i) and (ii) of ultrafilter, equation (1) and the paragraphs above, the sets:

$$J = \{i \in \mathbf{N} : \langle x_i, f_i\rangle \ge 1 - \eta\},$$

$$K = \{i \in \mathbf{N} : \langle y_i, f_i\rangle \ge 1 - \eta\},$$

and

$$M = \{i \in \mathbf{N} : \|x_i - y_i\| \ge sl_\varepsilon(X_U) - \eta\}$$

are all in U.

So the intersection $J \bigcap K \bigcap M$ is in U too, and is hence not empty.

Let $i \in J \bigcap K \bigcap M$ and $(X_i)_2$ be a two dimensional subspace of X spanned by x_i and y_i, we have

$$x_i, y_i \in S(B((X_i)_2), f_i, \eta)$$

and

$$\|x_i - y_i\| \ge sl_\varepsilon(X_U) - \eta.$$

Hence $sl_\varepsilon(X) \ge sl_\varepsilon(X_U) - \eta$.

Since η can be arbitrarily small, $sl_\varepsilon(X_U) \le sl_\varepsilon(X)$. □

Theorem 25 *If X is a Banach space with $sl_0(X) < 1$, then X has uniform normal structure.*

Proof: The idea of the proof is same as the proof of theorem 4.4 in [9]. Suppose $sl_0(X) < 1$,and X does not have uniform normal structure, we find a sequence $\{C_n\}$ of bounded closed convex subset of X such that for each n,

$$0 \in C_n, \quad d(C_n) = 1,$$

and

$$rad(C_n) = \inf_{x \in C_n} \sup_{y \in C_n} \|x - y\| > 1 - \frac{1}{n}.$$

Let U be any nontrivial ultrafilter on $\mathbf{N}$, and let

$$C = \{(x_n)_U : x_n \in C_n, n \in \mathbf{N}\},$$

then C is a nonempty bounded closed convex subset of X_U.

It follows from the properties of C_n above that $d(C) = rad(C) = 1$, so X_U does not have normal structure.

On the other hand, from theorem 24, $sl_0(X_U) = sl_0(X) < 1$.

This contradicts theorem 7, and X must have uniform normal structure. □

If X is super-reflexive, then $(X^*)_U = (X_U)^*$, and X has uniform normal structure if and if X_U has normal structure. Since X can be embedded into X_U, it is easy to see that $NUC(\varepsilon)$ of X is greater than or equal to $NUC(\varepsilon)$ of X_U. By using these facts we can prove the following result about uniform normal structure.

Theorem 26 *If X is a Banach space with $sl_{\frac{2}{3}}(X) < 2$, then X has uniform normal structure.*

Theorem 27 *Let X be a super-reflexive Banach space with $NUC(1^-) > 0$ for X_U. Then X has uniform normal structure.*

Proof: If X is a super-reflexive Banach space but fails to have uniform normal structure, then X_U fails to have normal structure. From theorem 2.14 of [27], we have $NUC(1^-) = 0$ for X_U. □

Acknowledgements: The research work of the first author was supported by the higher education research promotion and National Research University Project of Thailand, Office of Higher Education Commission, through the Cluster of Research to Enhance the Quality of Basic Education. The authors are very grateful to the referee for many constructive suggestions and improvements.

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