Multistability of competitive neural networks with time-delay in the saturation region

Yunquan Ke
Department of Mathematics,
Shaoxing University
Shaoxing, Zhejiang, 312000
P. R. China
keyunquan@usx.edu.cn

Chunfang Miao
Department of Mathematics,
Shaoxing University
Shaoxing, Zhejiang, 312000
P. R. China
miaochf@usx.edu.cn

Abstract: In this paper, the multistability analysis of the competitive neural networks with time-delay are investigated based on the stability theory, by selecting properly the system parameters, using solution matrix property and inequality technique, several novel sufficient conditions ensuring the existence and exponential stability of equilibria are obtained when the equilibrium point located in the saturation region. These conditions can be directly derived from the synaptic weights, the strength of the external stimulus and the external input of the competitive neural networks, and be verified easily. In addition, four examples are given to show the effectiveness of the results.

Key-words: competitive neural networks; saturation regions; multistability; exponential stable

1 Introduction

Competitive neural networks with different time scales are proposed in [1], which model the dynamics of cortical cognitive maps with unsupervised synaptic modifications. In this model, there are two types of state variables, that of the short-term memory (STM) describing the fast neural activity and that of the long-term memory (LTM) describing the slow unsupervised synaptic modifications. The competitive neural networks with different time scales are extensions of Grossberg's shunting network [2] and Amari's model for primitive neuronal competition [3]. For neural network models without considering the synaptic dynamics, their stability has been extensively analyzed. Cohen and Grossberg [4] found a Lyapunov functional for such a neural network, and derived some sufficient conditions ensuring absolute stability. The theory on the dynamics of the networks has been developed according to the purposes of the applications. On the one hand, in the applications to parallel computation and optimization problem, the existence of a computable solution for all possible initial states is the best situation. Mathematically, this means that an equilibrium of the networks exists and any state in the neighborhood converges to the equilibrium, which is called "monostability" of networks. On the other hand, existence of many equilibria is a necessary feature in the applications of neural networks to associative memory storage, pattern recognition, and decision making. The notion of "multistability" of networks describes coexistence of multiple stable patterns such as equilibria or periodic orbits. In the past few years, the monostability analysis of competitive neural networks with time-varying and/or distributed delays has been developed [5-10]. Recently, the multistability analysis of neural networks has attracted the attention of many researchers [11-18].

In this paper, the Multistability analysis of the competitive neural networks with time-delay when the equilibrium point located in the saturation region are investigated based on the stability theory.

We consider the following competitive neural networks with time-delay:

$$\begin{cases} STM: \epsilon\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum\limits_{j=1}^{N} b_{ij} f_j(x_j(t)) \\ + \sum\limits_{j=1}^{N} c_{ij} f_j(x_j(t-\tau_{ij}(t))) + B_i \sum\limits_{j=1}^{p} m_{ij}(t) y_j \\ \quad +I_i, i = 1, 2, \cdots, N, \\ LTM: \frac{dm_{ij}(t)}{dt} = -m_{ij}(t) + y_j f_i(x_i(t)), \\ \quad i = 1, 2, \cdots, N, j = 1, 2, \cdots, p, \end{cases} \tag{1}$$

where $x_i(t)$ is the neuron current activity level, $a_i > 0$ is the time constant of the neuron, $f_j(x_j(t))$ is the output of neurons, $m_{ij}(t)$ is the synaptic efficiency, y_j is the constant external stimulus, B_i is the strength of the external stimulus, ϵ is the time scale of STM state, b_{ij} and c_{ij} represent the synaptic weight of delayed feedback, I_i is the constant input, $\tau_{ij}(t)$ corresponds to the transmission delay and satisfies $0 < \tau_{ij}(t) < \tau_{ij}$($\tau_{ij}$ is a positive constant).

After setting

$$s_i(t) = \sum_{j=1}^{p} m_{ij}(t) y_j = y^T m_i(t),$$

where $y = (y_1, y_2, \cdots, y_P)^T$,

$$m_i(t) = (m_{i1}(t), m_{i2}(t), \cdots, m_{iP}(t))^T$$

and summing up the LTM over j and the fast time-scale parameter ϵ is also assumed to be unit, the networks (1) can be rewritten in the following form

$$\begin{cases} STM: \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum\limits_{j=1}^{N} b_{ij} f_j(x_j(t)) \\ + \sum\limits_{j=1}^{N} c_{ij} f_j(x_j(t-\tau_{ij}(t))) + B_i s_i(t) + I_i, \\ LTM: \frac{ds_i(t)}{dt} = -s_i(t) + \alpha f_i(x_i(t)), \end{cases} \tag{2}$$

where $i = 1, 2, \cdots, N, \alpha = \sum\limits_{j=1}^{p} y_j^2 > 0$.

The initial conditions associated with system (2) are of the form

$$\begin{cases} x_i(t) = \varphi_i(t), t \in (-\infty, 0], i = 1, 2, \cdots, N, \\ s_i(t) = \psi_i(t), t \in (-\infty, 0], i = 1, 2, \cdots, N, \end{cases} \tag{3}$$

where $\varphi_i(t)$ and $\psi_i(t) (i = 1, 2, \cdots, N)$ are bounded and continuous function on $t \in (-\infty, 0]$.

In the following discussion, we assume that the output function is a piecewise linear function

$$f_j(x) = \frac{1}{2}[|x+1| - |x-1|], j = 1, 2, \cdots, N.$$

2 Preliminaries

For the convenience, we introduce some notations.

$$P_{ij} = \begin{cases} \frac{b_{ij}+c_{ij}}{a_i}, & i \neq j \\ \frac{b_{ij}+c_{ij}+\alpha B_i}{a_i}, & i = j \end{cases}, \ i, j = 1, 2, \cdots, N,$$

$$P = \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{N1} \\ P_{12} & P_{22} & \cdots & P_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ P_{1N} & P_{2N} & \cdots & P_{NN} \end{bmatrix},$$

$$P(-k) = \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{k-11} & -P_{k1} & P_{k+11} & \cdots & P_{N1} \\ P_{12} & P_{22} & \cdots & P_{k-12} & -P_{k2} & P_{k+12} & \cdots & P_{N2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1k} & P_{2k} & \cdots & P_{k-1k} & -P_{kk} & P_{k+1k} & \cdots & P_{Nk} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1N} & P_{2N} & \cdots & P_{k-1N} & -P_{kN} & P_{k+1N} & \cdots & P_{NN} \end{bmatrix},$$

$$P(-k;-k) =$$

$$\begin{bmatrix} P_{11} & \cdots & P_{k-11} & -P_{k1} & P_{k+11} & \cdots & P_{N1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1k-1} & \cdots & P_{k-1k-1} & -P_{kk-1} & P_{k+1k-1} & \cdots & P_{Nk-1} \\ -P_{1k} & \cdots & -P_{k-1k} & P_{kk} & -P_{k+1k} & \cdots & -P_{Nk} \\ P_{1k+1} & \cdots & P_{k-1k+1} & -P_{kk+1} & P_{k+1k+1} & \cdots & P_{Nk+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1N} & \cdots & P_{k-1N} & -P_{kN} & P_{k+1N} & \cdots & P_{NN} \end{bmatrix},$$

$$P(-k,-l;-k,-l)=\begin{bmatrix} P_{11} & P_{21} & \cdots & -P_{k1} & \cdots & -P_{l1} & \cdots & P_{N1} \\ P_{12} & P_{22} & \cdots & -P_{k2} & \cdots & -P_{l2} & \cdots & P_{N2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -P_{1k} & -P_{2k} & \cdots & P_{kk} & \cdots & -P_{lk} & \cdots & -P_{Nk} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -P_{1l} & -P_{2l} & \cdots & -P_{kl} & \cdots & P_{ll} & \cdots & -P_{Nl} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1N} & P_{2N} & \cdots & -P_{kN} & \cdots & -P_{lN} & \cdots & P_{NN} \end{bmatrix},$$

where $k<l$,

$$I=(I_1/a_1, I_2/a_2, \cdots, I_N/a_N)^T,$$

$$I(-k)=(I_1/a_1, \cdots, -I_k/a_k, \cdots, I_N/a_N)^T,$$

$$I(-k_1,-k_2)=(I_1/a_1, I_2/a_2, \cdots, - I_{k_1}/a_{k_1}, \cdots, -I_{k_2}/a_{k_2}, \cdots, I_N/a_N)^T,$$

and

$$E_N=(1,1,\cdots,1)^T_{1\times N}.$$

For vector $U=(u_1,u_2,\cdots,u_N)^T$ and $V=(v_1,v_2,\cdots,v_N)^T$, $U\geq V$ means that each pair of corresponding elements of U and V satisfies the inequality $u_i\geq v_i, i=1,2,\cdots,N$. Let

$$\begin{aligned} &X(t)=(x_1(t),x_2(t),\cdots,x_N(t))^T\in\mathbb{R}^N,\\ &S(t)=(s_1(t),s_2(t),\cdots,s_N(t))^T\in\mathbb{R}^N\\ &X^*=(x_1^*,x_2^*,\cdots,x_N^*)^T\in\mathbb{R}^N,\\ &S^*=(s_1^*,s_2^*,\cdots,s_N^*)^T\in\mathbb{R}^N,\\ &Z(t)=(X^T(t),S^T(t))^T,\\ &Z^*=(X^{*T},S^{*T})^T. \end{aligned}$$

Definition 1 *The point $Z^*=(X^{*T},S^{*T})^T\in R^{2N}$ is called an equilibrium point of system (2), if it satisfies the following equations*

$$\left\{\begin{array}{l} -a_ix_i^*+\sum\limits_{j=1}^{N}(b_{ij}+c_{ij})f_j(x_j^*)+B_is_i^*+I_i=0,\\ -s_i^*+\alpha f_i(x_i^*)=0, \quad i=1,2,\cdots,N. \end{array}\right. \tag{4}$$

From (4), we can obtain

$$\left\{\begin{array}{l} x_i^*=\frac{1}{a_i}[\sum\limits_{j=1}^{N}(b_{ij}+c_{ij})f_j(x_j^*)+\alpha B_if_i(x_i^*)+I_i],\\ s_i^*=\alpha f_i(x_i^*), \quad i=1,2,\cdots,N. \end{array}\right. \tag{5}$$

Definition 2 *Let*

$$\Omega=\{\prod_{l=1}^{N}(-\infty,-1)^{\delta^l}\times(1,+\infty)^{1-\delta^l}, \quad \delta^l=1 \text{ or } 0, l=1,2,\cdots,N\},$$

then $\Omega\times\Omega$ are said to be saturation regions of system (2)*, where*

$(-\infty,-1)=(-\infty,-1)^1\times(1,+\infty)^0$;

$(1,+\infty)=(-\infty,-1)^0\times(1,+\infty)^1$.

Hence, Ω is made up of 2^N elements.

In the following, denote

$$\Omega_{-1}=\{\prod_{l=1}^{N}(-\infty,-1)^{\delta^l}\times(1,+\infty)^{1-\delta^l}, \quad \delta^l=1, l=1,2,\cdots,N\}=(-\infty,-1)^N,$$

$$\Omega_{1}=\{\prod_{l=1}^{N}(-\infty,-1)^{\delta^l}\times(1,+\infty)^{1-\delta^l}, \quad \delta^l=0, l=1,2,\cdots,N\}=(1,+\infty)^N,$$

$$-(-\infty,-1)=(1,\infty), \ -(1,\infty)=(-\infty,-1),$$

$$(-\infty,-1)^{\delta^{(ik)}}=\left\{\begin{array}{ll}(-\infty,-1), & i\neq k,\\ (1,\infty), & i=k,\end{array}\right.$$

for $i,k=1,2,\cdots,N$.

$$(-\infty,-1)^{\delta^{(ikl)}}=\left\{\begin{array}{ll}(-\infty,-1), & i\neq k, i\neq l,\\ (1,\infty), & i=k \ \text{ or } \ i=l,\end{array}\right.$$

for $i,k,l=1,2,\cdots,N$.

Definition 3 *Define $\Re^N$ as the set of N-dimensional bipolar vectors, i.e.,*

$$\Re^N=\{U \mid U=(u_1,u_2,\cdots,u_N)^T\in\mathbb{R}^N, \quad u_i=1 \text{ or } -1, i=1,2,\cdots,N\}.$$

Hence, $\Re^N$ is made up of 2^N elements. For any $(d_1,d_2,\cdots,d_N)^T\in\Re^N$, let

$$D(d_l)=\left\{\begin{array}{ll}(1,\infty), & d_l=1\\ (-\infty,-1), & d_l=-1\end{array}\right., l\in\{1,2,\cdots,N\}.$$

Consequently, $(d_1,d_2,\cdots,d_N)^T$ and $\prod\limits_{i=1}^{N}D(d_i)=D(d_1)\times D(d_2)\times\cdots\times D(d_N)$ represent a one-to-one correspondence.

Definition 4 *The point Z^* is said to be an isolated equilibrium point of system* (2) *if there exists $\delta > 0$ such that Z^* is the only equilibrium point of system* (2) *in*

$$\{Z|\|Z - Z^*\| < \delta, Z \in \mathbb{R}^{2N}\}$$

where $Z = (x_1, x_2, \cdots, x_N, s_1, s_2, \cdots, s_N)^T$.

Obviously, the equilibrium point in the saturation region $\Omega \times \Omega$, then it is always an isolated equilibrium point.

Definition 5 *For vector function $U(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T$ and $n \times n$ order matrix $G = (g_{ij})_{n\times n}$, we define norm as following, respectively*

$$\|U(t)\| = (\sum_{i=1}^{n} |u_i(t)|^2)^{\frac{1}{2}}, \ \ \|G\| = (\sum_{i,j=1}^{n} |g_{ij}|^2)^{\frac{1}{2}}.$$

Definition 6 *The equilibrium point Z^* of system* (2) *is said to be locally exponentially stable in region D, if there exists constants $\delta > 0, \beta > 0$ such that*

$$\|Z(t; t_0, \phi) - Z^*\| \le \beta\|\phi\|_{t_0} e^{-\delta(t-t_0)}, t > t_0,$$

where $Z(t; t_0, \phi)$ is a solution of system (2) *with the initial condition $\phi(\vartheta) \in C((-\infty, t_0], D)$.*

3 Main results

In this section, we will derive some sufficient conditions to ensure the isolated and locally exponentially stable equilibrium points located in the saturation region. Without loss of generality, in the following discussion, we always assume $b_{ij} \ge 0, c_{ij} \ge 0, B_i > 0, I_i \ge 0$ and $\alpha > 1$.

Theorem 7 *For system* (2), *if there exists equilibrium point located in the saturation region, then the equilibrium point is locally exponentially stable.*

Proof. For $\forall(d_1, d_2, \cdots, d_N)^T \in \Re^N$, denote $\Omega_D = \prod_{i=1}^{N} D(d_i) \in \Omega$. If there exists equilibrium point located in the saturation region $\Omega_D \times \Omega_D$, let one of its equilibrium point is $Z^* = (X^{*T}, S^{*T})^T, X^* \in \Omega_D$, then we have $f(x_i^*) = d_i$ and

$$\begin{cases} -a_i x_i^* + \sum\limits_{j=1}^{N}(b_{ij} + c_{ij})d_j + B_i s_i^* + I_i = 0, \\ -s_i^* + \alpha d_i = 0, \ \ i = 1, 2, \cdots, N. \end{cases} \tag{6}$$

If $X(t), X(t - \tau(t)) \in \Omega_D$, then $f(x_i(t)) = d_i, f(x_i(t - \tau)) = d_i$, from (2),

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum\limits_{j=1}^{N}(b_{ij} + c_{ij})d_j + B_i s_i(t) + I_i, \\ \frac{ds_i(t)}{dt} = -s_i(t) + \alpha d_i, \ \ i = 1, 2, \cdots, N. \end{cases} \tag{7}$$

From (6) and (7),

$$\begin{cases} \frac{d(x_i(t)-x_i^*)}{dt} = -a_i(x_i(t) - x_i^*) + B_i(s_i(t) - s_i^*), \\ \frac{d(s_i(t)-s_i^*)}{dt} = -(s_i - s_i^*), \ \ i = 1, 2, \cdots, N. \end{cases} \tag{8}$$

Let

$$A_i = \begin{bmatrix} -a_i & B_i \\ 0 & -1 \end{bmatrix}, \quad Y_i(t) = \begin{bmatrix} x_i(t) - x_i^* \\ s_i(t) - s_i^* \end{bmatrix}.$$

It follows from (8) that

$$\frac{dY_i(t)}{dt} = A_i Y_i(t), \ \ i = 1, 2, \cdots, N. \tag{9}$$

From (9), we obtain

$$Y_i(t) = e^{A_i t} Y_i(0). \tag{10}$$

By calculation, we can obtain the eigenvalue of A_i

$$\lambda_1 = -a_i, \ \ \lambda_2 = -1.$$

Corresponding eigenvector of the λ_1 and λ_2, respectively

$$V_1 = (1, 0)^T, \quad V_2 = (1, (a_i - 1)/B_i)^T.$$

Thus, we obtain the fundamental solution matrix of system (9) is

$$\phi_i(t) = \begin{bmatrix} e^{-a_i t} & e^{-t} \\ 0 & \frac{a_i-1}{B_i} e^{-t} \end{bmatrix}.$$

By calculation, we can obtain

$$\phi_i^{-1}(0) = \frac{B_i}{a_i - 1} \begin{bmatrix} \frac{a_i-1}{B_i} & -1 \\ 0 & 1 \end{bmatrix}.$$

Since $e^{A_i t} = \phi(t)\phi^{-1}(0)$, we can obtain

$$e^{A_i t} = \left[\begin{array}{cc} e^{-a_i t} & \frac{B_i}{a_i - 1}(e^{-t} - e^{-a_i t}) \\ 0 & e^{-t} \end{array} \right].$$

Thus, we have

$$\begin{array}{l} \|e^{A_i t}\| = [e^{-2a_i t} + e^{-2t} + (\frac{B_i}{a_i-1})^2(e^{-t} - e^{-a_i t})^2]^{\frac{1}{2}} \\ \leq [\frac{(a_i-1)^2+B_i^2}{(a_i-1)^2}]^{\frac{1}{2}}[e^{-2a_i t} + e^{-2t}]^{\frac{1}{2}} \\ \leq \frac{\sqrt{2[(a_i-1)^2+B_i^2]}}{|a_i-1|}e^{-\delta t} \leq Me^{-\delta t}, \end{array} \quad (11)$$

where $\delta = \min_{1\leq i\leq N}\{1, a_i\}$, and

$$M = \max_{1\leq i\leq N}\{\frac{\sqrt{2[(a_i-1)^2+B_i^2]}}{|a_i-1|}\} > 1.$$

From (10) and (11), we obtain

$$\|Y_i(t)\| \leq M\|Y_i(0)\|e^{-\delta t}, t > 0.$$

Thus, we have

$$\begin{array}{l} |x_i(t) - x_i^*|^2 + |s_i(t) - s_i^*|^2 \\ \leq M^2(|\varphi_i(0) - x_i^*|^2 + |\psi_i(0) - s_i^*|^2)e^{-2\delta t}, \end{array}$$

for $t > 0, i = 1, 2, \cdots, N$.

Then we have

$$\|Z - Z^*\| \leq M\|\phi(0)\|e^{-\delta t}, t > 0, Z \in \mathbb{R}^{2N},$$

where

$$\|\phi(0)\| = (\sum_{j=1}^{N}[|\varphi_i(0) - x_i^*|^2 + |\psi_i(0) - s_i^*|^2])^{\frac{1}{2}}$$

from Definition 4 and Definition 6, the equilibrium point Z^* is isolated and locally exponentially stable located in the saturation region. □

Theorem 8 *For system (2), if*

1) $PE_N - I > E_N$*;*

2) there exists $l \in \{1, 2, \cdots, N\}$ *such that* $\forall k \in \{1, 2, \cdots, N\}, k \neq l$*,*

$$\sum_{j=1, j\neq k}^{N}(b_{lj} + c_{lj}) - (b_{lk} + c_{lk}) + \alpha B_l + I_l < a_l,$$

then system (2) has neither more nor less than 2 isolated and locally exponentially stable equilibrium points located in $\Omega_1 \times \Omega_1$*, and* $\Omega_{-1} \times \Omega_{-1}$*, respectively.*

Proof. We choose

$$Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T \in \mathbb{R}^{2N}$$

such that

$$\left\{ \begin{array}{l} -a_i x_i^* + \sum_{j=1}^{N}(b_{ij} + c_{ij}) + \alpha B_i + I_i = 0, \\ -s_i^* + \alpha = 0, \quad i = 1, 2, \cdots, N. \end{array} \right. \quad (12)$$

Since $a_i > 0, \alpha > 1, b_{ij} \geq 0, c_{ij} \geq 0, B_i > 0, I_i \geq 0 (i, j = 1, 2, \cdots, N)$, then from (12), we obtain

$$x_i^* = \frac{1}{a_i}[\sum_{j=1}^{N}(b_{ij} + c_{ij}) + \alpha B_i + I_i], \quad s_i^* = \alpha,$$

i.e.,

$$\left\{ \begin{array}{l} X^* = PE_N + I \geq PE_N - I > E_N, \\ S^* = (\alpha, \alpha, \cdots, \alpha)^T > e_N. \end{array} \right. \quad (13)$$

From (13), we have $x_i^* > 1, s_i^* = \alpha > 1, i = 1, 2, \cdots, N$, and

$$\left\{ \begin{array}{l} -a_i x_i^* + \sum_{j=1}^{N}(b_{ij} + c_{ij})f_j(x_j^*) + B_i s_i^* + I_i \\ \quad = -a_i x_i^* + \sum_{j=1}^{N}(b_{ij} + c_{ij}) + \alpha B_i + I_i = 0, \\ -s_i^* + \alpha f_i(x_i^*) = -s_i^* + \alpha = 0, \end{array} \right. \quad (14)$$

for $i = 1, 2, \cdots, N$.

Hence, $Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T$ is an equilibrium point of system (2) located in the saturation region

$\Omega_1 \times \Omega_1 = (1, +\infty)^N \times (1, +\infty)^N$.

We choose

$$Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T \in \mathbb{R}^{2N}$$

such that

$$\left\{ \begin{array}{l} -a_i x_i^* - \sum_{j=1}^{N}(b_{ij} + b_{ij}) - \alpha B_i + I_i = 0, \\ -s_i^* - \alpha = 0, \quad i = 1, 2, \cdots, N. \end{array} \right. \quad (15)$$

From (15), we obtain

$$\left\{ \begin{array}{l} X^* = -PE_N + I < -E_N, \\ S^* = (-\alpha, -\alpha, \cdots, -\alpha)^T < -E_N. \end{array} \right. \quad (16)$$

From (16), we have $x_i^* < -1, s_i^* = -\alpha < -1, \quad i = 1, 2, \cdots, N$, and

$$\begin{cases} -a_i x_i^* + \sum\limits_{j=1}^{N}(a_{ij}+b_{ij})f_j(x_j^*) + B_i s_i^* + I_i \\ \quad = -a_i x_i^* - \sum\limits_{j=1}^{n}(a_{ij}+b_{ij}) - \alpha B_i + I_i = 0, \\ -s_i^* + \alpha f_i(x_i^*) = -s_i^* - \alpha = 0, \end{cases} \tag{17}$$

for $i = 1, 2, \cdots, N$.
Hence, $Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T$ is an equilibrium point of system (2) located in the saturation region $\Omega_{-1} \times \Omega_{-1} = (-\infty, -1)^N \times (-\infty, -1)^N$.

On the other hand, according to condition 2) of Theorem 8, there exists $l \in \{1, 2, \cdots, N\}$, such that $\forall k \in \{1, 2, \cdots, N\}, k \neq l$,

$$\sum_{j=1, j\neq k}^{N}(b_{lj}+c_{lj}) - (b_{lk}+c_{lk}) + \alpha B_l + I_l < a_l.$$

Assume that there exists another equilibrium point $\bar{X} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N) \in \Omega$, $\bar{S} = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_N) \in \Omega$, and $\bar{X} \notin \Omega_1$, $\bar{X} \notin \Omega_{-1}$, without loss of generality, assume
$\bar{X} \in \Omega_{(l)} = D(d_1) \times D(d_2) \times \cdots \times D(d_{l-1}) \times D(1) \times D(d_{l+1}) \times \cdots D(d_N)$,
thus we have $\bar{x}_l > 1, \bar{s}_l = \alpha (l \leq N)$. Then there exists $k \in \{1, 2, \cdots, N\}, k \neq l$, such that $d_k = -1$(otherwise, we have $\bar{X} \in \Omega_1$) and

$$\begin{aligned} \bar{x}_l &= \tfrac{1}{a_l}[\sum_{j=1}^{N}(b_{lj}+b_{lj})f_j(\bar{x}_j) + B_l \bar{s}_l + I_l] \\ &= \tfrac{1}{a_l}[\sum_{j=1}^{N}(b_{lj}+c_{lj})d_j + B_l \bar{s}_l + I_l] \end{aligned}$$

$$\leq \frac{1}{a_l}[\sum_{j=1, j\neq k}^{N}(b_{lj}+c_{lj}) - (b_{lk}+c_{lk}) + \alpha B_l + I_l] < 1, \tag{18}$$

for $i = 1, 2, \cdots, N$, i.e., $\bar{x}_l < 1$, this is in contradiction with $\bar{x}_l > 1$.

Then the system (2) has no another equilibrium point. Hence, Theorem 8 holds. □

Theorem 9 *For system (2), if there exists $q, k \in \{1, 2, 3, \cdots, n\}, k \neq q$, such that*
1) $P(-k; -k)E_N - I > E_N$;
2) $\forall l \in \{1, 2, 3, \cdots, N\}, l \neq k, l \neq q$

$$\sum_{j=1, j\neq l}^{N}(b_{qj}+c_{qj}) - (b_{ql}+c_{ql}) + \alpha B_q + I_q < a_q,$$

then system (2) has neither more nor less than 4 isolated and locally exponentially stable equilibrium points located in the saturation region.

Proof. If the conditions of Theorem 3. hold, then

$$PE_N - I \geq P(-k; -k)E_N - I > E_N.$$

From Theorem 8, we know system (2) has 2 isolated and locally exponentially stable equilibrium points located in the saturation region $\Omega_1 \times \Omega_1$, and $\Omega_{-1} \times \Omega_{-1}$, respectively.

For $k \in \{1, 2, 3, \cdots, n\}, k \neq q$, we choose $Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T \in R^{2N}$, such that

$$\begin{cases} -a_i x_i^* + \sum\limits_{j=1, j\neq k}^{N}(b_{ij}+c_{ij}) \\ \quad -(b_{ik}+c_{ik}) + B_i s_i^* + I_i = 0, \\ s_i^* = \alpha, i = 1, 2, \cdots, N, i \neq k, s_k^* = -\alpha, \end{cases} \tag{19}$$

we have

$$(x_1^*, x_2^*, \cdots, x_N^*)^T = P(-k)e_N + I. \tag{20}$$

From (20), we can obtain

$$\begin{aligned} &(x_1^*, x_2^*, \cdots, -x_k^*, \cdots, x_N^*)^T \\ &= P(-k; -k)E_N + I(-k) \\ &\geq P(-k; -k)E_N - I > E_N. \end{aligned} \tag{21}$$

From (19) and (21), we obtain

$$x_i^* = \begin{cases} > 1, & i \neq k \\ < -1, & i = k \end{cases}, \quad i = 1, 2, \cdots, N.$$

$$s_i^* = \begin{cases} \alpha > 1, & i \neq k \\ -\alpha < -1, & i = k \end{cases}, \quad i = 1, 2, \cdots, N.$$

Thus, we have

$$\begin{cases} -a_i x_i^* + \sum\limits_{j=1}^{N}(b_{ij}+c_{ij})f_j(x_j^*) + B_i s_i^* + I_i \\ \quad = -a_i x_i^* + \sum\limits_{j=1, j\neq k}^{N}(b_{ij}+c_{ij}) \\ \quad -(b_{ik}+c_{ik}) + B_i s_i^* + I_i = 0, \\ -s_i^* + \alpha f_i(x_i^*) = 0, \quad i = 1, 2, \cdots, N. \end{cases} \tag{22}$$

Then,

$$Z^* = (x_1^*, x_2^*, \cdots, x_N^*, s_1^*, s_2^*, \cdots, s_N^*)^T \in \mathbb{R}^{2N}$$

is an equilibrium point of systems (2), and

$$Z^* \in \Omega_{(-k)} = \prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ik)}}] \times \prod_{j=1}^{N}[-(-\infty,-1)^{\delta^{(jk)}}] \in \Omega \times \Omega.$$

It is similar to prove that $-Z^*$ is also an equilibrium point of systems (2), and

$$-Z^* \in \Omega_{(k)} = \prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(ik)}}] \times \prod_{j=1}^{N}[(-\infty,-1)^{\delta^{(jk)}}] \in \Omega \times \Omega.$$

If the conditions 2) of Theorem 9 hold, i.e., for $\forall l \in \{1,2,3,\cdots,N\}, l \neq k, l \neq q$ such that

$$\sum_{j=1,j\neq l}^{N}(b_{qj}+c_{qj}) - (b_{ql}+c_{ql}) + \alpha B_q + I_q < a_q,$$

assume that there exists another equilibrium point $\bar{X} = (\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_N) \in \Omega$ and $\bar{S} = (\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_N) \in \Omega$, without loss of generality, assume $\bar{X} \in \Omega_{(q)} = D(d_1) \times D(d_2) \times \cdots \times D(d_{q-1}) \times D(1) \times D(d_{q+1}) \times \cdots D(d_N)$, i.e., $\bar{x}_q > 1, \bar{s}_q = \alpha$. Then, there exists $l \in \{1,2,,3,\cdots,N\}, l \neq k, l \neq q$, such that $d_l = -1$ (otherwise, $\bar{x} \in \Omega_1$ or $\bar{x} \in \Omega_{(-k)}$). Since

$$\begin{aligned}\bar{x}_q &= \tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})f_j(\bar{x}_j) + B_q\bar{s}_q + I_q] \\ &= \tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})d_j + B_q\bar{s}_q + I_q] \\ &\leq \tfrac{1}{a_q}[\sum_{j=1,j\neq l}^{N}(b_{qj}+c_{qj}) - (b_{ql}+c_{ql}) + \alpha B_q + I_q] \\ &< 1.\end{aligned} \tag{23}$$

From (23), we obtain $\bar{x}_q < 1$, this is in contradiction with $\bar{x}_q > 1$. Then the system (2) has no another equilibrium point.

Hence, systems (2) has neither more nor less than 4 isolated and locally exponentially stable equilibrium points located in the saturation region

$(-\infty,-1)^N \times (-\infty,-1)^N$,
$(1,+\infty)^N \times (1,+\infty)^N$,
$\prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ik)}}] \times \prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ik)}}]$,
and
$\prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(ik)}}] \times \prod_{i=1}^{n}[(-\infty,-1)^{\delta^{(ik)}}]$,
respectively.

Theorem 10 *For system (2), if there exists* $q \in \{1,2,\cdots,N\}$, $\forall h \in T = \{T_1,T_2,\cdots,T_M\} \subset \{1,2,3,\cdots,N\}, T_k \neq q (k=1,2,\cdots,M)$, *and* $T_1 < T_2 < \cdots < T_M, M \leq N-1$, *such that*

1) $P(-h;-h)E_N - I > E_N$;

2) $\forall l \in \{1,2,3,\cdots,N\} - T - \{q\}$,

$$\sum_{j=1,j\neq l}^{N}(b_{qj}+c_{qj}) - (b_{ql}+c_{ql}) + \alpha B_q + I_q < a_q,$$

or $\forall m,r \in T, m \neq r$,

$$\sum_{j=1,j\neq m,r}^{N}(b_{qj}+c_{qj}) - (b_{qm}+c_{qm}) - (b_{qr}+c_{qr}) + \alpha B_q + I_q < a_q,$$

then system (2) has neither more nor less than $2+2M$ *isolated and locally exponentially stable equilibrium points located in the saturation region.*

Proof. If the conditions of Theorem 10 hold, then for $\forall h \in T = \{T_1,T_2,\cdots,T_M\} \subset \{1,2,3,\cdots,N\}, T_k \neq q (k=1,2,\cdots,M)$, we have

$$P(-h;-h)E_N - I > E_N.$$

From Theorem 9, for $\forall h \in T$, we know system (2) has 4 isolated and locally exponentially stable equilibrium points located in the saturation region. Since $T = \{T_1,T_2,\cdots,T_M\}$ is made up of M elements, thus, system (2) has $2+2M$ isolated and locally exponentially stable equilibrium points located in the saturation region

$(1,+\infty)^N \times (1,+\infty)^N$,
$(-\infty,-1)^N \times (-\infty,-1)^N$,

$\prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ih)}}] \times \prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ih)}}]$,
$\prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(ih)}}] \times \prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(ih)}}]$,
for $h = T_1,T_2,\cdots,T_M$.

If the conditions 2) of Theorem 10 hold, i.e., for $\forall l \in \{1,2,3,\cdots,N\} - T - \{q\}$,

$$\sum_{j=1,j\neq l}^{N}(b_{qj}+c_{qj}) - (b_{ql}+c_{ql}) + \alpha B_q + I_q < a_q,$$

or $\forall m,r \in T, m \neq r$,

$$\sum_{j=1,j\neq m,r}^{N}(b_{qj}+c_{qj}) - (b_{qm}+c_{qm})$$

$-(b_{qr}+c_{qr})+\alpha B_q+I_q<a_q.$

Assume that there exists another equilibrium point $\bar{X}=(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_N)\in\Omega$, and $\bar{S}=(\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_N)\in\Omega$, without loss of generality, assume $\bar{X}\in\Omega_{(q)}=D(d_1)\times D(d_2)\times\cdots\times D(d_{q-1})\times D(1)\times D(d_{q+1})\times\cdots D(d_N)$, i.e., $\bar{x}_q>1,\bar{s}_q=\alpha$, then, there exists $l\in\{1,2,3,\cdots,m\}-T-\{q\}$, such that $d_l=-1$, or there exist $m,r\in T,m\neq r$, such that $d_m=-1,d_r=-1$ (otherwise, $\bar{X}$ is only one among $2+2M$ equilibrium points above). Since

$$\begin{aligned}\bar{x}_q&=\tfrac{1}{a_q}[\sum_{j=1}^{n}(b_{qj}+c_{qj})f_j(\bar{x}_j)+B_q\bar{s}_q+I_q]\\&=\tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})d_j+B_q\bar{s}_q+I_q]\\&\leq\tfrac{1}{a_q}[\sum_{j=1,j\neq l}^{N}(b_{qj}+c_{qj})-(b_{ql}+c_{ql})+\alpha B_q+I_q]\\&<1.\end{aligned}\tag{24}$$

or

$$\begin{aligned}\bar{x}_q&=\tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})f_j(\bar{x}_j)+B_q\bar{s}_q+I_q]\\&=\tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})d_j+B_q\bar{s}_q+I_q]\\&=\tfrac{1}{a_q}[\sum_{j=1,j\neq m,r}^{N}(b_{qj}+c_{qj})-(b_{qm}+c_{qm})\\&-(b_{qr}+c_{qr})+\alpha B_q+I_q]<1.\end{aligned}\tag{25}$$

From (24) or (25), we obtain $\bar{x}_q<1$, this is in contradiction with $\bar{x}_q>1$.

Hence, systems (2) has neither more nor less than $2+2M$ isolated and locally exponentially stable equilibrium points located in the saturation region.

Theorem 11 *For systems (2), if there exists $q\in\{1,2,\cdots,n\}$, such that*

1) $\forall k,l\in\{1,2,3,\cdots,N\},k\neq l$, and $k\neq q,l\neq q$,

$$P(-k,-l;-k,-l)E_N-I>E_N,\tag{26}$$

2) $\forall g,h,m\in\{1,2,3,\cdots,N\},g,h,m\neq q$, and g,h,m are not equal one another

$$\begin{aligned}&\sum_{j=1,j\neq g,h,m}^{N}(a_{qj}+b_{qj})-\sum_{j=g,h,m}(a_{qj}+b_{qj})\\&+\alpha B_q+I_q<a_q,\end{aligned}\tag{27}$$

then system (2) has neither more nor less than $2+2(N-1)+(N-1)(N-2)$ isolated and locally exponentially stable equilibrium points located in the saturation region.

Proof. If the conditions of (26) hold, then for $\forall T,\bar{T}\in\{1,2,3,\cdots,N\},T\neq\bar{T}$, and $T,\bar{T}\neq q$, we have

$$\begin{aligned}&P(-T;-T)E_N-I\\&\quad\geq P(-T,-\bar{T};-T,-\bar{T})E_N-I>E_N.\end{aligned}$$

Thus, the conditions $1)$ of Theorem 10 hold. From Theorem 10, we know system (2) has $2+2(N-1)$ isolated and locally exponentially stable equilibrium points located in the saturation region

$(1,+\infty)^N\times(1,+\infty)^N$,

$(-\infty,-1)^N\times(-\infty,-1)^N$,

$\prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(iT)}}]\times\prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(iT)}}]$,

and

$\prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(iT)}}]\times\prod_{i=1}^{N}[(-\infty,-1)^{\delta^{(iT)}}]$,

$T\in\{1,2,\cdots,N\},T\neq q$, respectively.

For $\forall k,l\in\{1,2,3,\cdots,N\},k\neq l$, and $k,l\neq q$, we choose

$Z^*=(x_1^*,x_2^*,\cdots,x_N^*,s_1^*,s_2^*,\cdots,s_N^*)^T$,

such that

$$\begin{cases}-a_ix_i^*+\sum_{j=1,j\neq k,l}^{N}(b_{ij}+c_{ij})-(b_{ik}+c_{ik})\\\quad-(b_{il}+c_{il})+B_is_i^*+I_i=0,\\s_i^*=\alpha,i=1,2,\cdots,N,i\neq k,l,\\\quad s_k^*=-\alpha,s_l^*=-\alpha.\end{cases}\tag{28}$$

Without loss of generality, we assume $k<l$, from (28), we can obtain

$$\begin{aligned}&(x_1^*,x_2^*,\cdots,-x_k^*,\cdots,-x_l^*,\cdots,z_N^*)^T\\&=P(-k,-l;-k,-l)E_N+I(-k,-l)\\&\geq P(-k,-l;-k,-l)E_N-I>E_N.\end{aligned}\tag{29}$$

From (28) and (29), we have

$$x_i^*=\begin{cases}>1, & i\neq k,l\\<-1, & i=k\ or\ i=l\end{cases},$$

$$s_i^*=\begin{cases}\alpha>1, & i\neq k,l\\-\alpha<-1, & i=k\ or\ i=l,\end{cases}$$

for $i=1,2,\cdots,N$.

Thus, $Z^*\in\Omega_{(-k,-l)}=\prod_{i=1}^{N}[-(-\infty,-1)^{\delta^{(ikl)}}]\times$

$\prod_{j=1}^{N}[-(-\infty,-1)^{\delta(jkl)}]$, and

$$\begin{cases} -a_ix_i^*+\sum_{j=1}^{N}(b_{ij}+c_{ij})f_j(x_j^*)+B_is_i^*+I_i \\ \quad =-a_ix_i^*+\sum_{j=1,j\neq k,l}^{N}(b_{ij}+c_{ij})-(b_{ik}+c_{ik}) \\ \quad -(a_{il}+b_{il})+B_is_i^*+I_i=0, \\ -s_i^*+\alpha f_i(x_i^*)=0, \quad i=1,2,\cdots,N, \end{cases} \tag{30}$$

i.e., Z^* is an equilibrium points of systems (2) located in the saturation region of $\Omega_{(-k,-l)}\times\Omega_{(-k,-l)}$.

It is similar to prove that $-Z^*$ is also an equilibrium point of systems (2), and

$$-Z^*\in\Omega_{(k,l)} = \prod_{i=1}^{N}[(-\infty,-1)^{\delta(ikl)}]\times\prod_{j=1}^{N}[(-\infty,-1)^{\delta(jkl)}].$$

On the other hand, since $\{(k,l)|k,l\in\{1,2,3,\cdots,N\},k\neq l$, and $k,l\neq q\}$ is made up of $(N-1)\times(N-2)$ elements, and take note $(-\infty,-1)^{\delta(ilk)}=(-\infty,-1)^{\delta(ikl)}$, thus, system (2) has $(N-1)\times(N-2)$ isolated and locally exponentially stable equilibrium points located in the saturation region

$$\bigcup_{k=1,k\neq q}^{N}\bigcup_{l=k+1,l\neq q}^{N}\{\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(ikl)}] \times\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(ikl)}]\bigcup\prod_{i=1}^{N}[(-\infty,-1)^{\delta(ikl)}] \times\prod_{i=1}^{N}[(-\infty,-1)^{\delta(ikl)}]\},$$

respectively.

If the conditions 2) of Theorem 11 hold, i.e., for $\forall g,h,m\in\{1,2,3,\cdots,N\},g,h,m\neq q$ which are not equal one another, such that (27) holds. Assume that there exists another equilibrium point $\bar{X}=(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_N)\in\Omega$, and $\bar{S}=(\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_N)\in\Omega$, without loss of generality, we assume $\bar{X}\in\Omega_{(q)}=D(d_1)\times D(d_2)\times\cdots\times D(d_{q-1})\times D(1)\times D(d_{q+1})\times\cdots D(d_N)$, i.e., $\bar{x}_q>1,\bar{s}_q=\alpha$. Then, there exist $g,h,m\in\{1,2,3,\cdots,N\},g,h,m\neq q$, such that $d_g=-1,d_h=-1$, and $d_m=-1$ (otherwise, $\bar{Z}$ is only one among $2+2(N-1)+(N-1)(N-2)$ equilibrium points above). Since

$$\begin{aligned} \bar{x}_q &= \tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})f_j(\bar{x}_j)+B_q\bar{s}_q+I_q] \\ &= \tfrac{1}{a_q}[\sum_{j=1}^{N}(b_{qj}+c_{qj})d_j+B_q\bar{s}_q+I_q] \\ &\leq \tfrac{1}{a_q}[\sum_{j=1,j\neq g,h,m}^{N}(b_{qj}+c_{qj}) \\ &\quad -\sum_{j=g,h,m}(b_{qj}+c_{qj})+\alpha B_q+I_q] \\ &< 1, \end{aligned}$$

we obtain $\bar{x}_q<1$, this is in contradiction with $\bar{x}_q>1$.

Hence, systems (2) has neither more nor less than $2+2(N-1)+(N-1)(N-2)$ isolated and locally exponentially stable equilibrium points located in the saturation region

$$(1,+\infty)^N\times(1,+\infty)^N,$$
$$(-\infty,-1)^N\times(-\infty,-1)^N,$$
$$\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(iT)}]\times\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(iT)}],$$
$$\prod_{i=1}^{N}[(-\infty,-1)^{\delta(iT)}]\times\prod_{i=1}^{N}[(-\infty,-1)^{\delta(iT)}],$$

for $T=1,2,3,\cdots,N,T\neq q$, and

$$\bigcup_{k=1,k\neq q}^{N}\bigcup_{l=k+1,l\neq q}^{N}\{\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(ikl)}] \times\prod_{i=1}^{N}[-(-\infty,-1)^{\delta(ikl)}]\bigcup\prod_{i=1}^{N}[(-\infty,-1)^{\delta(ikl)}] \times\prod_{i=1}^{N}[(-\infty,-1)^{\delta(ikl)}]\},$$ respectively.

4 Numerical Simulation

In this section, we give four examples to show our results validity. Consider the following competitive neural networks with time delays($N=4$)

$$\begin{cases} \frac{dx_1(t)}{dt}=-a_1x_1(t)+\sum_{j=1}^{4}b_{1j}f_j(x_j(t)) \\ \qquad +\sum_{j=1}^{4}c_{1j}f_j(x_j(t-\tau_{1j}(t)))+B_1s_1(t)+I_1, \\ \frac{dx_2(t)}{dt}=-a_2x_2(t)+\sum_{j=1}^{4}b_{2j}f_j(x_j(t)) \\ \qquad +\sum_{j=1}^{4}c_{2j}f_j(x_j(t-\tau_{2j}(t)))+B_2s_2(t)+I_2, \end{cases}$$

$$\begin{cases} \frac{dx_3(t)}{dt} = -a_3x_3(t) + \sum\limits_{j=1}^{4} b_{3j}f_j(x_j(t)) \\ \qquad + \sum\limits_{j=1}^{4} c_{3j}f_j(x_j(t-\tau_{3j}(t))) + B_3s_3(t) + I_3, \\ \frac{dx_4(t)}{dt} = -a_4x_4(t) + \sum\limits_{j=1}^{4} b_{4j}f_j(x_j(t)) \\ \qquad + \sum\limits_{j=1}^{4} c_{4j}f_j(x_j(t-\tau_{4j}(t))) + B_4s_4(t) + I_4, \\ \frac{ds_1(t)}{dt} = -s_1(t) + \alpha f_1(x_1(t)), \\ \frac{ds_2(t)}{dt} = -s_2(t) + \alpha f_2(x_2(t)), \\ \frac{ds_3(t)}{dt} = -s_3(t) + \alpha f_3(x_3(t)), \\ \frac{ds_4(t)}{dt} = -s_4(t) + \alpha f_4(x_4(t)), \end{cases} \tag{31}$$

where $f_j(x) = \frac{1}{2}[|x+1| - |x-1|], j = 1, 2, 3, 4.$

Example 1. If we set

$$\begin{array}{llll} \alpha = 2, & a_1 = 8, & a_2 = 2, & a_3 = 3, \\ a_4 = 5, & b_{11} = 1, & b_{12} = \frac{1}{2}, & b_{13} = \frac{3}{2}, \\ b_{14} = \frac{1}{4}, & b_{21} = \frac{1}{2}, & b_{22} = \frac{1}{4}, & b_{23} = \frac{3}{4}, \\ b_{24} = \frac{1}{8}, & b_{31} = \frac{1}{4}, & b_{32} = \frac{1}{2}, & b_{33} = \frac{1}{8}, \\ b_{34} = \frac{3}{4}, & b_{41} = 1, & b_{42} = \frac{1}{2}, & b_{43} = \frac{1}{4}, \\ b_{44} = \frac{1}{2}, & c_{11} = 1, & c_{12} = \frac{3}{2}, & c_{13} = \frac{1}{2}, \\ c_{14} = \frac{7}{4}, & c_{21} = \frac{1}{2}, & c_{22} = \frac{3}{4}, & c_{23} = \frac{1}{4}, \\ c_{24} = \frac{7}{8}, & c_{31} = \frac{3}{4}, & c_{32} = \frac{1}{2}, & c_{33} = \frac{7}{8}, \\ c_{34} = \frac{1}{4}, & c_{41} = \frac{1}{2}, & c_{42} = 1, & c_{43} = \frac{5}{4}, \\ c_{44} = 1, & B_1 = 1, & B_2 = \frac{1}{2}, & B_3 = 2, \\ B_4 = 3, & I_1 = 1, & I_2 = 2, & I_3 = 3, \\ I_4 = 4, & \tau_{1j} = 0.3, & \tau_{2j} = 0.2, & \tau_{3j} = 0.4, \\ \tau_{4j} = 0.1, & j = 1, 2, 3, 4. & & \end{array}$$

We have

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} & \frac{3}{2} \end{bmatrix}, \quad I = \begin{bmatrix} \frac{1}{8} \\ 1 \\ 1 \\ \frac{4}{5} \end{bmatrix}.$$

Then the equilibrium point of systems (31) satisfies the following equation

$$\begin{cases} -8x_1 + 4f_1(x_1) + 2f_2(x_2) \\ +2f_3(x_3) + 2f_4(x_4) + 1 = 0, \\ \quad -2x_2 + f_1(x_1) + 2f_2(x_2) \\ +f_3(x_3) + f_4(x_4) + 2 = 0, \\ \quad -3x_3 + f_1(x_1) + f_2(x_2) \\ +5f_3(x_3) + f_4(x_4) + 3 = 0, \\ \quad -5x_4 + \frac{3}{2}f_1(x_1) + \frac{3}{2}f_2(x_2) \\ \quad +\frac{3}{2}f_3(x_3) + \frac{15}{2}f_4(x_4) + 4 = 0, \end{cases}$$

$$\begin{cases} -s_1 + 2f_1(x_1) = 0, \\ -s_2 + 2f_2(x_2) = 0, \\ -s_3 + 2f_3(x_3) = 0, \\ -s_4 + 2f_4(x_4) = 0. \end{cases} \tag{32}$$

Let $l = 1$, we have the following results by simple calculation

$$Pe_4 - I > e_4,$$
$$\sum_{j=1, j\neq k}^{4} (b_{1j} + c_{1j}) - (b_{1k} + c_{1k}) + \alpha B_1 + I_1 < a_1,$$

for $k \in \{2, 3, 4\}$. Then, the conditions of Theorem 8 hold. And we can obtain only two equilibrium points of systems (32), i.e.,

$$(\tfrac{11}{8}, \tfrac{7}{2}, \tfrac{11}{3}, \tfrac{16}{5}, 2, 2, 2, 2),$$
$$(-\tfrac{9}{8}, -\tfrac{3}{2}, -\tfrac{5}{3}, -\tfrac{8}{5}, -2, -2, -2, -2).$$

Evidently, this consequence is coincident with the results of Theorem 8. Figs. 1-2 depict the time responses of state variables of $x_1(t), x_2(t), x_3(t), x_4(t), s_1(t), s_2(t), s_3(t), s_4(t)$ of system in example 1, respectively.

Example 2. If we set $\alpha = 2, a_1 = 6,\ a_2 = 1,\ a_3 = \frac{1}{2},\ a_4 = 3$, and

$$\begin{array}{llll} b_{11} = \frac{1}{2}, & b_{12} = \frac{1}{4}, & b_{13} = 1, & b_{14} = 2, \\ b_{21} = \frac{1}{2}, & b_{22} = 1, & b_{23} = \frac{1}{4}, & b_{24} = \frac{3}{4}, \\ b_{31} = \frac{1}{2}, & b_{32} = \frac{1}{4}, & b_{33} = \frac{1}{2}, & b_{34} = \frac{3}{4}, \\ b_{41} = \frac{1}{2}, & b_{42} = \frac{1}{4}, & b_{43} = \frac{1}{2}, & b_{44} = 1, \\ c_{11} = \frac{1}{2}, & c_{12} = \frac{3}{4}, & c_{13} = 2, & c_{14} = 1, \\ c_{21} = \frac{1}{2}, & c_{22} = 1, & c_{23} = \frac{3}{4}, & c_{24} = \frac{1}{4}, \\ c_{31} = \frac{1}{2}, & c_{32} = \frac{3}{4}, & c_{33} = \frac{1}{2}, & c_{34} = \frac{1}{4}, \\ c_{41} = \frac{1}{2}, & c_{42} = \frac{3}{4}, & c_{43} = \frac{3}{2}, & c_{44} = 1, \\ B_1 = 1, & B_2 = 2, & B_3 = \frac{1}{2}, & B_4 = 2, \\ I_1 = 1, & I_2 = 1, & I_3 = 2, & I_4 = 4, \\ \tau_{1j} = 0.3, & \tau_{2j} = 0.2, & \tau_{3j} = 0.4, & \tau_{4j} = 0.1, \\ j = 1, 2, 3, 4. & & & \end{array}$$

We have

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} \\ 1 & 6 & 1 & 1 \\ 2 & 2 & 4 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 2 \end{bmatrix}, \quad I = \begin{bmatrix} \frac{1}{6} \\ 1 \\ 4 \\ \frac{4}{3} \end{bmatrix}.$$

Then the equilibrium point of systems (31) satis-

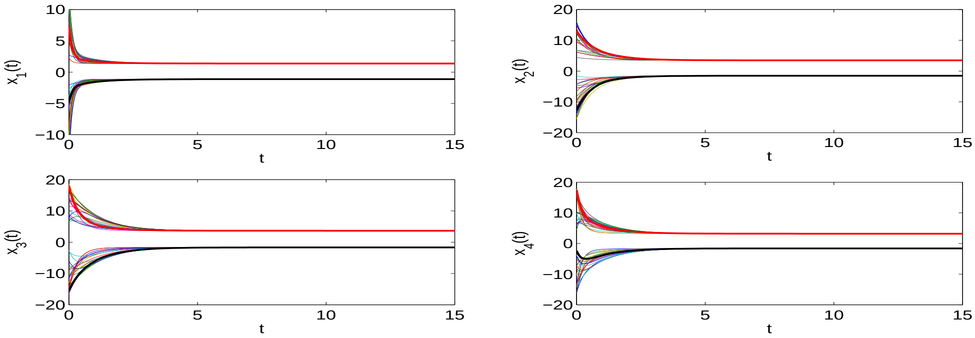

Fig.1. Transient response of state variables $x_1(t),x_2(t),x_3(t),x_4(t)$ of Example 4.1

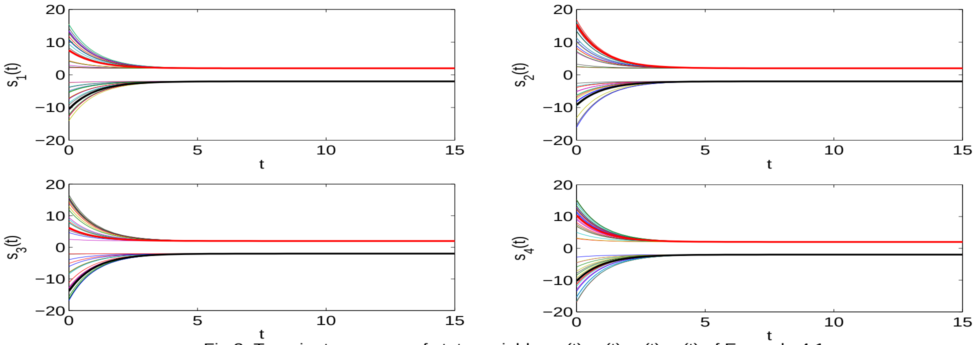

Fig.2. Transient response of state variables $s_1(t),s_2(t),s_3(t),s_4(t)$ of Example 4.1

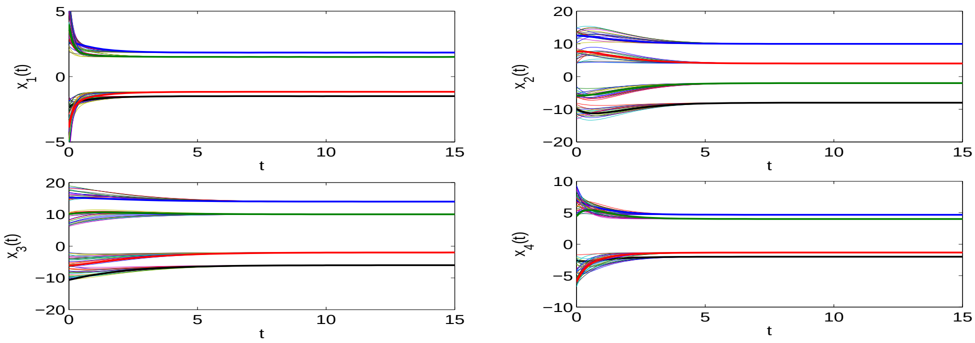

Fig.3. Transient response of state variables $x_1(t),x_2(t),x_3(t),x_4(t)$ of Example 4.2

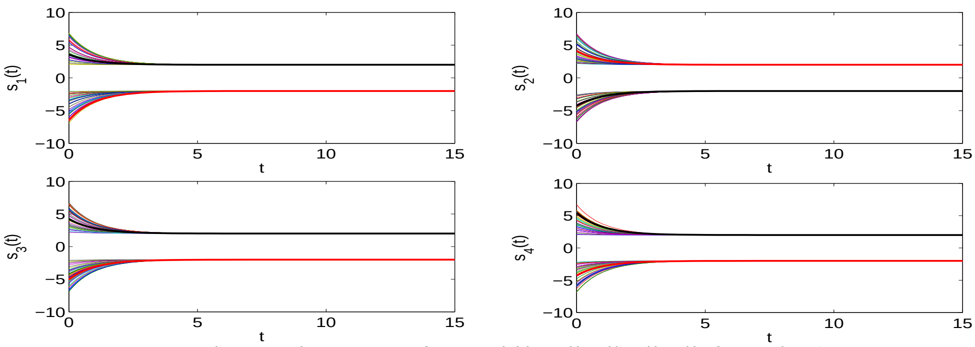

Fig.4. Transient response of state variables $s_1(t),s_2(t),s_3(t),s_4(t)$ of Example 4.2

fies the following equation

$$\left\{\begin{array}{l} -6x_1+3f_1(x_1)+f_2(x_2) \\ \quad +3f_3(x_3)+3f_4(x_4)+1=0, \\ -x_2+f_1(x_1)+6f_2(x_2) \\ \quad +f_3(x_3)+f_4(x_4)+1=0, \\ -\frac{1}{2}x_3+f_1(x_1)+f_2(x_2) \\ \quad +2f_3(x_3)+f_4(x_4)+2=0, \\ -3x_4+f_1(x_1)+f_2(x_2)+2f_3(x_3) \\ \quad +6f_4(x_4)+4=0, \\ -s_1+2f_1(x_1)=0, \\ -s_2+2f_2(x_2)=0, \\ -s_3+2f_3(x_3)=0, \\ -s_4+2f_4(x_4)=0. \end{array}\right. \tag{33}$$

Let $q=1, k=2$, we have the following results by simple calculation

$$P(-2;-2)e_4-I>e_4,$$
$$\sum_{j=1,j\neq l}^{4}(b_{1j}+c_{1j})-(b_{1l}+c_{1l})+\alpha B_1+I_1<a_1,$$

for $l\in\{3,4\}$. Then, the conditions of Theorem 9 hold. And we can obtain only four equilibrium points of systems (33), i.e.,

$$(\tfrac{11}{6},10,14,\tfrac{14}{3},2,2,2,2),$$
$$(-\tfrac{3}{2},-8,-6,-2,-2,-2,-2,-2),$$
$$(\tfrac{3}{2},-2,10,4,2,-2,2,2)$$
$$(-\tfrac{7}{6},4,-2,-\tfrac{4}{3},-2,2,-2,-2).$$

Evidently, this consequence is coincident with the results of Theorem 9. Figs. 3-4 depict the time responses of state variables of $x_1(t),x_2(t),x_3(t),x_4(t),s_1(t),s_2(t),s_3(t),s_4(t)$ of system in example 2, respectively.

Example 3. If we set $\alpha=2,\ a_1=4,\ a_2=1,\ a_3=2,\ a_4=3,$
$b_{11}=\frac{1}{4},\ b_{12}=\frac{1}{2},\ b_{13}=\frac{1}{4},\ b_{14}=2,$
$b_{21}=\frac{1}{4},\ b_{22}=1,\ b_{23}=\frac{1}{2},\ b_{24}=\frac{1}{8},$
$b_{31}=\frac{1}{8},\ b_{32}=\frac{1}{2},\ b_{33}=\frac{3}{4},\ b_{34}=\frac{1}{4},$
$b_{41}=\frac{1}{8},\ b_{42}=\frac{1}{2},\ b_{43}=\frac{1}{4},\ b_{44}=\frac{1}{4},$
$c_{11}=\frac{1}{4},\ c_{12}=\frac{1}{2},\ c_{13}=\frac{3}{4},\ c_{14}=1,$
$c_{21}=\frac{1}{4},\ c_{22}=1,\ c_{23}=\frac{3}{2},\ c_{24}=\frac{1}{8},$
$c_{31}=\frac{1}{8},\ c_{32}=\frac{1}{2},\ c_{33}=\frac{1}{4},\ c_{34}=\frac{1}{4},$
$c_{41}=\frac{3}{8},\ c_{42}=\frac{1}{2}, B_1=1,\ B_2=2,$
$B_3=3,\ B_4=4,\ I_1=1,\ I_2=2,$
$I_3=3,\ I_4=5,\ \tau_{1j}=0.3,\ \tau_{2j}=0.2,$
$\tau_{3j}=0.4,\ \tau_{4j}=0.1, j=1,2,3,4.$

We have

$$P=\begin{bmatrix} \frac{5}{8} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 6 & 2 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{7}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{17}{6} \end{bmatrix},\quad I=\begin{bmatrix} \frac{1}{4} \\ 2 \\ \frac{3}{2} \\ \frac{5}{3} \end{bmatrix}.$$

Then the equilibrium point of systems (31) satisfies the following equation

$$\left\{\begin{array}{l} -4x_1+\frac{5}{2}f_1(x_1)+f_2(x_2) \\ \quad +f_3(x_3)+3f_4(x_4)+1=0, \\ -x_2+\frac{1}{2}f_1(x_1)+6f_2(x_2)+ \\ \quad 2f_3(x_3)+\frac{1}{4}f_4(x_4)+2=0, \\ -2x_3+\frac{1}{4}f_1(x_1)+f_2(x_2)+ \\ \quad 7f_3(x_3)+\frac{1}{2}f_4(x_4)+3=0, \\ -3x_4+\frac{1}{2}f_1(x_1)+f_2(x_2)+ \\ \quad f_3(x_3)+\frac{17}{2}f_4(x_4)+5=0, \\ -s_1+2f_1(x_1)=0, \\ -s_2+2f_2(x_2)=0, \\ -s_3+2f_3(x_3)=0, \\ -s_4+2f_4(x_4)=0. \end{array}\right. \tag{34}$$

Let $q=1, h=2,3$, we have the following results by simple calculation

$$P(-2;-2)e_4-I>e_4,\ P(-3;-3)e_4-I>e_4,$$
$$\sum_{j=1,j\neq l}^{4}(b_{1j}+c_{1j})-(b_{1l}+c_{1l})+\alpha B_1+I_1<a_1,$$

for $l=4$. Then, the conditions of Theorem 10 hold. And we can obtain only six equilibrium points of systems (34), i.e.,

$$(\tfrac{17}{8},\tfrac{43}{4},\tfrac{47}{8},\tfrac{16}{3},2,2,2,2),$$
$$(-\tfrac{13}{8},-\tfrac{27}{4},-\tfrac{23}{8},-2,-2,-2,-2,-2),$$
$$(\tfrac{13}{8},-\tfrac{5}{4},\tfrac{39}{8},\tfrac{14}{3},2,-2,2,2),$$
$$(-\tfrac{9}{8},\tfrac{21}{4},-\tfrac{15}{8},-\tfrac{4}{3},-2,2,-2,-2),$$
$$(\tfrac{13}{8},\tfrac{27}{4},-\tfrac{9}{8},\tfrac{14}{3},2,2,-2,2),$$
$$(-\tfrac{9}{8},-\tfrac{11}{4},\tfrac{33}{8},-\tfrac{4}{3},-2,-2,2,-2).$$

Evidently, this consequence is coincident with the results of Theorem 10. Figs. 5-6 depict the time responses of state variables of $x_1(t),x_2(t),x_3(t),x_4(t),s_1(t),s_2(t),s_3(t),s_4(t)$ of system in example 3, respectively.

Example 4. If we set
$\alpha=2,\ a_1=1,\ a_2=2,\ a_3=3,\ a_4=1,$
$b_{11}=2,\ b_{12}=1,\ b_{13}=\frac{1}{2},\ b_{14}=\frac{3}{2},$

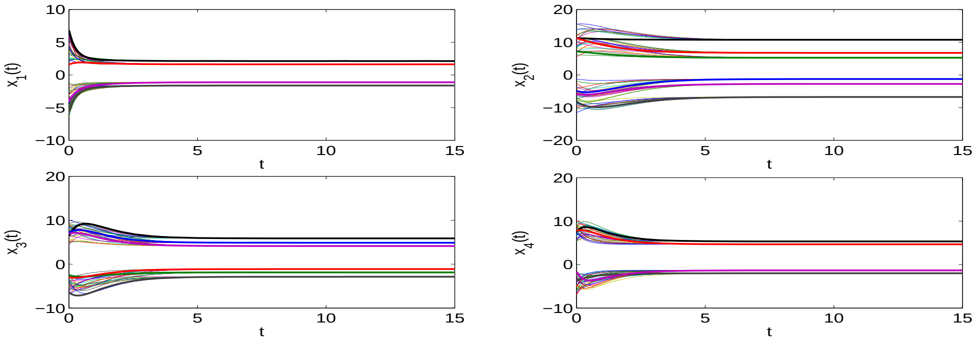

Fig.5. Transient response of state variables $x_1(t),x_2(t),x_3(t),x_4(t)$ of Example 4.3

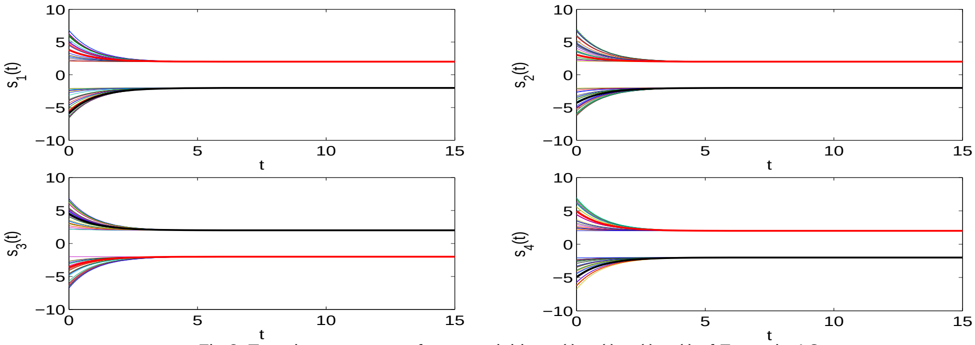

Fig.6. Transient response of state variables $s_1(t),s_2(t),s_3(t),s_4(t)$ of Example 4.3

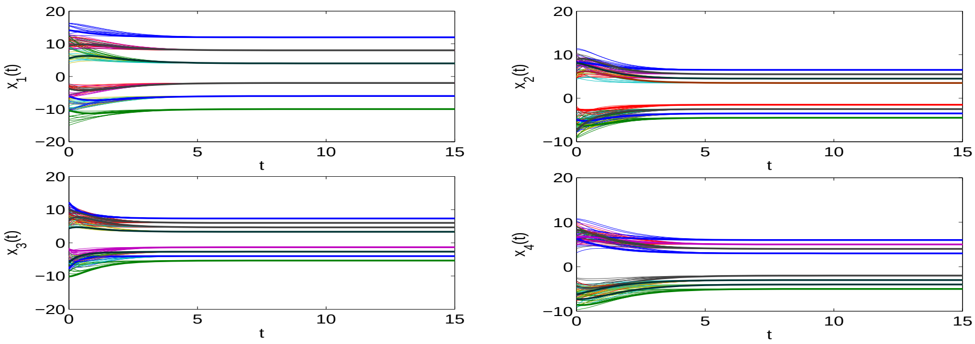

Fig.7. Transient response of state variables $x_1(t),x_2(t),x_3(t),x_4(t)$ of Example 4.4

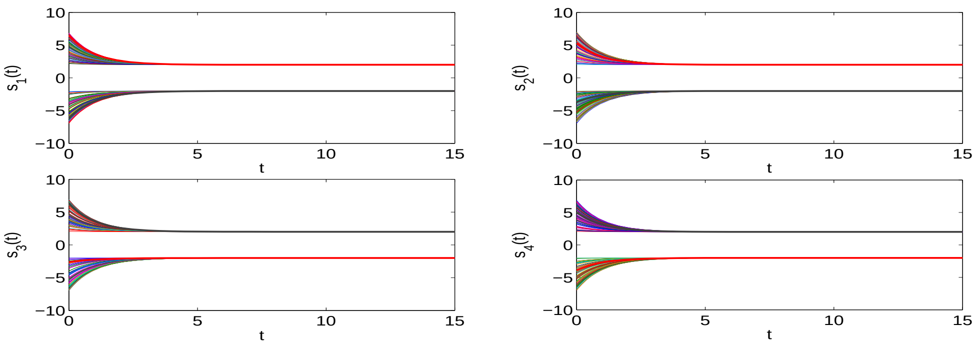

Fig.8. Transient response of state variables $s_1(t),s_2(t),s_3(t),s_4(t)$ of Example 4.4

$b_{21} = \frac{1}{2},\ b_{22} = 2,\ b_{23} = \frac{1}{4},\ b_{24} = \frac{3}{4},$
$b_{31} = 1,\ b_{32} = \frac{1}{2},\ b_{33} = 4,\ b_{34} = \frac{3}{2},$
$b_{41} = \frac{1}{4},\ b_{42} = \frac{1}{8},\ b_{43} = \frac{3}{8},\ b_{44} = 1,$
$c_{11} = 1,\ c_{12} = 1,\ c_{13} = \frac{3}{2},\ c_{14} = \frac{1}{2},$
$c_{21} = \frac{1}{2},\ c_{22} = 2,\ c_{23} = \frac{3}{4},\ c_{24} = \frac{1}{4},$
$c_{31} = 1,\ c_{32} = \frac{3}{2},\ c_{33} = 3,\ c_{34} = \frac{1}{2},$
$c_{41} = \frac{1}{4},\ c_{42} = \frac{3}{8},\ c_{43} = \frac{1}{8},\ c_{44} = 1,$
$B_1 = 1,\ B_2 = 2,\ B_3 = 3,\ B_4 = 1,$
$I_1 = 1,\ I_2 = 2,\ I_3 = 3,\ I_4 = \frac{1}{2},$
$\tau_{1j} = 0.3,\ \tau_{2j} = 0.2,\ \tau_{3j} = 0.4,\ \tau_{4j} = 0.1,$
$j = 1, 2, 3, 4.$

We have

$$P = \begin{bmatrix} 5 & 2 & 2 & 2 \\ \frac{1}{2} & 4 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{2}{3} & \frac{13}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

Then the equilibrium point of systems (31) satisfies the following equation

$$\begin{cases} -x_1 + 5f_1(x_1) + 2f_2(x_2) \\ \quad +2f_3(x_3) + 2f_4(x_4) + 1 = 0, \\ -2x_2 + f_1(x_1) + 8f_2(x_2) \\ \quad +f_3(x_3) + f_4(x_4) + 2 = 0, \\ -3x_3 + 2f_1(x_1) + 2f_2(x_2) \\ \quad +13f_3(x_3) + 2f_4(x_4) + 3 = 0, \\ -x_4 + \frac{1}{2}f_1(x_1) + \frac{1}{2}f_2(x_2) \\ \quad +\frac{1}{2}f_3(x_3) + 4f_4(x_4) + \frac{1}{2} = 0, \\ -s_1 + 2f_1(x_1) = 0, \\ -s_2 + 2f_2(x_2) = 0, \\ -s_3 + 2f_3(x_3) = 0, \\ -s_4 + 2f_4(x_4) = 0. \end{cases} \tag{35}$$

Let $q = 1, k, l \in \{2, 3, 4\}, k \neq l$, we have the following results by simple calculation

$$\begin{aligned} &P(-2, -3; -2, -3)e_4 - I > e_4, \\ &P(-2, -4; -2, -4)e_4 - I > e_4, \\ &P(-3, -4; -3, -4)e_4 - I > e_4, \\ &\sum_{j=1, j\neq g,h,m}^{4} (a_{1j} + b_{1j}) - \sum_{j=g,h,m} (a_{1j} + b_{1j}) \\ &+\alpha B_1 + I_1 < a_1, \end{aligned}$$

$\forall g, h, m \in \{2, 3, 4\}$ and g, h, m are not equal one another.

Then, the conditions of Theorem 11 hold. And we can obtain only fourteen equilibrium points of systems (35), i.e.,

$$\begin{aligned} &(12, \tfrac{13}{2}, \tfrac{22}{3}, 6, 2, 2, 2, 2), \\ &(-10, -\tfrac{9}{2}, -\tfrac{16}{3}, -5, -2, -2, -2, -2), \\ &(8, -\tfrac{3}{2}, 6, 5, 2, -2, 2, 2), \\ &(-6, \tfrac{7}{2}, -4, -4, -2, 2, -2, -2), \\ &(8, \tfrac{11}{2}, -\tfrac{4}{3}, 5, 2, 2, -2, 2), \\ &(-6, -\tfrac{7}{2}, \tfrac{10}{3}, -4, -2, -2, 2, -2), \\ &(8, \tfrac{11}{2}, 6, -2, 2, 2, 2, -2), \\ &(-6, -\tfrac{7}{2}, -4, 3, -2, -2, -2, 2) \\ &(8, -\tfrac{5}{2}, -\tfrac{8}{3}, 5, 2, -2, -2, 2), \\ &(-2, \tfrac{9}{2}, \tfrac{14}{3}, -3, -2, 2, 2, -2), \\ &(4, -\tfrac{5}{2}, \tfrac{14}{3}, -3, 2, -2, 2, -2), \\ &(-2, \tfrac{9}{2}, -\tfrac{8}{3}, 4, -2, 2, -2, 2), \\ &(4, \tfrac{9}{2}, -\tfrac{8}{3}, -3, 2, 2, -2, -2), \\ &(-2, -\tfrac{5}{2}, \tfrac{14}{3}, 4, -2, -2, 2, 2). \end{aligned}$$

Evidently, this consequence is coincident with the results of Theorem 11. Figs. 7-8 depict the time responses of state variables of $x_1(t), x_2(t), x_3(t), x_4(t), s_1(t), s_2(t), s_3(t), s_4(t)$ of system in example 4, respectively.

5 Conclusions

In this paper, based on the stability theory, we investigate the Multistability of a class of competitive neural networks with time delays, and obtain some sufficient conditions to ensure the existence and locally exponential stability of the equilibrium points of the systems in the saturation region. And according to the peculiarity of the saturation regions, these sufficient conditions, which only depend on the synaptic weights matrices P and the external input vector I, are very easy to be verified. Moreover, four examples show our results are effective.

Acknowledgement

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