On Vulnerability of Power and Total Graphs

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Abstract: In communication networks, greater degrees of stability or less vulnerability is required. The vulnerability of communication network measures the resistance of the network to the disruption of operation after the failure of certain stations or communication links. If the network begins losing communication links or processors, then there is a loss in its effectiveness. Thus, communication networks must be so designed that they do not easily get disrupted under external attack and, moreover, these are easily reconstructible if they do get disrupted. These desirable properties of networks can be measured by various graph parameters like toughness, integrity, scattering number, tenacity and rupture degree. Power graphs and total graphs constitute a large class of graphs and which are widely used in systems ranging from large supercomputers to small embedded systems-on-a-chip. In this paper, we firstly give the exact values for the integrity and toughness of powers of paths. After that, the vulnerability parameters such as integrity, toughness, rupture degree of total graphs of some special graphs are calculated. Finally, the relationships between some vulnerability parameters, namely the integrity, toughness, scattering number, tenacity and rupture degree are established.

Key–Words: Vulnerability, Integrity, Toughness, Scattering number, Tenacity, Rupture degree, Total graph, Power graph.

1 Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set V and edge set E. For $S \subseteq V(G)$, let $\omega(G - S)$ and $m(G - S)$, respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq V(G)$ is a cut set of G, if either $G - S$ is disconnected or $G - S$ has only one vertex. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x, and $\lfloor x \rfloor$ for the largest integer not larger than x. The distance $d_G(u, v)$ in a simple undirected graph G between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. We use Bondy and Murty [9] for terminology and notations not defined here.

A communication network is composed of processors and communication links. Links cuts, node interruptions, software errors or hardware failures, and transmission failures at various points can interrupt service for long periods of time, then there is a loss in its effectiveness. This event is called as the vulnerability of communication networks. In other words, the vulnerability of communication network measures the resistance of the network to the disruption of operation after the failure of certain processors or communication links. Network designers attach importance the vulnerability of a network, they want to design network with less vulnerability or more reliability. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

The communication network often has as considerable an impact on a network's performance as the processors themselves. Performance measures for communication networks are essential to guide the designers in choosing an appropriate topology.

In order to measure the performance, we are interested in the following performance metrics (there may be others):

(1) the number of elements that are not functioning,

(2) the number of remaining connected subnetworks,

(3) the size of a largest remaining group within which mutual communication can still occur.

Since the communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a node and a communication link between processors (stations) as an edge between corresponding nodes, there are many graph theoretical parameters can be used to describe the vulnerability of communication networks.

Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have been introduced that attempt to cope with this difficulty, including toughness and edge-toughness in $[6, 7, 12, 19, 28]$, integrity and edge-integrity in $[2, 3, 4, 5, 13, 18, 21]$, tenacity and edge-tenacity in $[1, 10, 11, 14, 15, 22, 24, 26, 29]$, scattering number in $[30, 31]$, and rupture degree in $[20, 22, 23, 25]$. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused.

For comparing, the following graph parameters are listed.

The connectivity is a parameter defined based on Quantity (1). The *connectivity* of an incomplete graph G is defined by

$$\kappa(G) = \min\{|S| : S \subset V(G), \omega(G - S) > 1\},$$

and that of the complete graph K_n is defined as $n-1$.

Both toughness and scattering number take into account Quantities (1) and (2). The *toughness* and *scattering number* of an incomplete connected graph G are defined by

$$\tau(G) = \min\{\frac{|S|}{\omega(G - S)} : S \subset V(G), \omega(G-S) > 1\}.$$

and

$$s(G) = \max\{\omega(G - S) - |S| : S \subset V(G),$$
$$\omega(G - S) > 1\},$$

respectively. For the complete graph K_n, we have $\tau(K_n) = \infty$.

The integrity is defined based on Quantities (1) and (3). The *integrity* of a graph G is defined by

$$I(G) = \min\{|S| + m(G - S) : S \subset V(G)\}.$$

Both the tenacity and rupture degree take into account all the three quantities. The *tenacity* and *rupture degree* of an incomplete connected graph G are defined by

$$T(G) = \min\{\frac{|S| + m(G - S)}{\omega(G - S)} :$$
$$S \subset V(G), \omega(G - S) > 1\},$$

and

$$r(G) = \max\{\omega(G - S) - |S| - m(G - S) :$$
$$S \subset V(G), \omega(G - S) > 1\},$$

respectively. And the tenacity and rupture degree of the complete graph K_n is defined as n and $n-1$, respectively.

The corresponding edge analogues of these concepts are defined similarly, see [5,19,28,29,32].

From the above definitions, we can see that the connectivity of a graph reflects the difficulty in breaking down a network into several pieces. This invariant is often too weak, since it does not take into account what remains after the corresponding graph is disconnected. Unlike the connectivity, each of the other vulnerability measures, i.e., toughness, scattering number, integrity, tenacity and rupture degree, reflects not only the difficulty in breaking down the network but also the damage that has been caused.

In [27], Moazzami *et al.* compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. In [11], Choudum *et al.* studied the tenacity of complete graph products and grids. In [22], Li *et al.* discussed the tenacity and rupture degree for permutation graphs of complete bipartite graphs. Cheng *et al.* [10] determined the maximum tenacity of trees and unicyclic graphs with given order and show the corresponding extremal graphs. These results are helpful in constructing stable networks with lower costs. Li [24] gave some results on the tenacity of gear graphs, powers of paths and the lexicographic product of some special graphs. Cozzens *et al.*[15] studied the tenacity of harary Graphs. In [23], Li *et al.* discussed the rupture degree of powers graphs of cycles. In [32], Zhang *et al.* studied some edge vulnerability parameters of split graphs. In [7, 13, 20, 26, 30], the authors proved that computing the vulnerability parameters such as integrity, scattering number, toughness, tenacity and rupture degree of a graph is NP-hard in general. So, it is an interesting problem to determine vulnerability parameters for some special graphs.

In this paper, we consider the problem of measuring the vulnerability of power and total graphs. In Section 2, we give some results on the vulnerability of power graphs. After that, we study the vulnerability of total graphs in sections 3. Finally, the relationships between some vulnerability parameters such as integrity, toughness, scattering number, tenacity and rupture degree are established in sections 4.

2 Vulnerability of Powers of Graphs

In this section, we consider the problem of computing the toughness and integrity of powers of paths. At first we give the concept of the *power graph* (or *k-th power*) G^k of a graph G.

Definition 1 *For an integer $k \geq 1$, the power graph G^k (the k-th power of a graph G) is defined as follows: $V(G) = V(G^k)$. Two distinct vertices u and v are adjacent in G^k if and only if the distance between the vertices u and v in G is at most k, i.e., $d_G(u, v) \leq k$.*

The second power of a graph is also called its square.

Remark 2 *We notice that G^1 is just G itself. So, we let $k \geq 2$ in the following.*

As a useful network, power of cycles and paths have arouse interests for many network designers. Barefoot et al. gave the exact values of integrity of powers of cycles in [2], and determined the connectivity, binding number and toughness of powers of cycles [3]. Vertex-neighbor-integrity of powers of cycles were studied in [16] by Cozzens and Wu. In [27] Moazzami gave the exact values for the tenacity of powers of cycles. Zhang and Yang [33] study the binding number of the Powers of Paths and cycles. In [24] Li gave the exact values for the tenacity of powers of paths.

It is easy to see that $P_n^k \cong K_n$ if $n \leq k+1$. So, in the following lemmas, we suppose that $2 \leq k \leq n-2$.

A vertex cut set S of a graph G is called a *τ-set* of G if it satisfies that $\tau(G) = \frac{|S|}{\omega(G-S)}$.

Lemma 3 *If S is a minimal τ-set for the graph P_n^k, $2 \leq k \leq n-2$, then S consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.*

Proof. We assume that the vertices of P_n^k are labelled by $v_1, v_2, \cdots, v_n$. Let S be a minimal τ-set of P_n^k and j be the smallest integer such that $T = \{v_j, v_{j+1}, \cdots, v_{j+t-1}\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. We distinguish two cases:

Case 1. If $T = S$, then S contains just t consecutive vertices $v_j, v_{j+1}, \cdots, v_{j+t-1}$. Since $T = S \neq V(P_n^k)$, and by the structure of P_n^k, S must leave exact two components of $P_n^k - S$, we have $j > 1$, $j + t - 1 < n$, $v_{j-1} \notin S$, $v_{j+t} \notin S$, and $v_{j+t} \neq v_{j-1}$. Therefore, $\{v_{j+t}, v_{j-1}\} \cap S = \emptyset$. By the structure of P_n^k we know that if S leave exact two components of $P_n^k - S$, S must contain at least k consecutive vertices. Thus we have $t \geq k$. Now suppose $t > k$. Delete v_{j+t-1} from the set S yielding a new set $S_1 = S - \{v_{j+t-1}\}$. Since $t > k$, the edge $v_{j+t-1}v_{j-1}$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_{j+t-1} in $P_n^k - S_1$. Then, $p \geq j + t$ and $p \leq j + t + k - 2$, and so v_p is also adjacent to v_{j+t} in $P_n^k - S_1$. Therefore, deleting v_{j+t-1} from S yields $\omega(P_n^k - S_1) = \omega(P_n^k - S)$. So,

$$\frac{|S'|}{\omega(P_n^k - S')} = \frac{|S| - 1}{\omega(P_n^k - S)} < \frac{|S|}{\omega(P_n^k - S)} = \tau(P_n^k).$$

which is contrary to our choice of S. Thus, $t = k$.

Case 2. If $T \subset S$. Since $S \neq V(P_n^k)$, $T \neq V(P_n^k)$, and S must leave at least two components of $P_n^k - S$, we have $j > 1$, $j+t-1 < n$, $v_{j-1} \notin S$, $v_{j+t} \notin S$, and $v_{j+t} \neq v_{j-1}$. Therefore, $\{v_{j+t}, v_{j-1}\} \cap S = \emptyset$. Now suppose $t < k$. Choose v_{j+i} such that $1 \leq i \leq t$, and delete v_{j+i} from S yielding a new set $S' = S - \{v_{j+i}\}$ with $|S'| = |S| - 1$. By the definition of P_n^k ($2 \leq k \leq n-2$) we know that the edges $v_{j+i}v_{j-1}$ and $v_{j+i}v_{j+t}$ are in $P_n^k - S'$. Consider a vertex v_p adjacent to v_{j+i} in $P_n^k - S'$. If $p \geq t + j + 1$, then $p < t + j + k$. So, v_p is also adjacent to v_{j+t} in $P_n^k - S'$. If $p < j - 1$, then $p \geq j - k$ and v_p is also adjacent to v_{j-1} in $P_n^k - S'$. Since $t < k$, then v_{j-1} and v_{j+t} are adjacent in $P_n^k - S'$. Therefore, we can conclude that deleting the vertex v_{j+i} from S does not change the number of components, and so $\omega(P_n^k - S') = \omega(P_n^k - S)$. Thus, we have

$$\frac{|S'|}{\omega(P_n^k - S')} = \frac{|S| - 1}{\omega(P_n^k - S)} < \frac{|S|)}{\omega(P_n^k - S)} = \tau(P_n^k).$$

This is contrary to our choice of S. Thus we must have $t \geq k$. Now suppose $t > k$. Delete v_{j+t-1} from the set S yielding a new set $S_1 = S - \{v_{j+t-1}\}$. Since $t > k$, the edge $v_{j+t-1}v_{j-1}$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_{j+t-1} in $P_n^k - S_1$. Then, $p \geq j + t$ and $p \leq j + t + k - 2$, and so v_p is also adjacent to v_{j+t} in $P_n^k - S_1$. Therefore, deleting v_{j+t-1} from S yields $\omega(P_n^k - S_1) = \omega(P_n^k - S)$. So,

$$\frac{|S_1|}{\omega(P_n^k - S_1)} = \frac{|S| - 1}{\omega(P_n^k - S)} < \frac{|S|}{\omega(P_n^k - S)} = T(P_n^k),$$

which is again contrary to our choice of S. Thus, $t = k$, and so S consists of the union of sets of exactly k consecutive vertices. □

Lemma 4 *There is a τ-set S for the graph P_n^k, such that all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$.*

Proof. Among all τ-sets of minimum order, consider those sets with maximum number of minimum order components, and we let s denote the order of a minimum component. Among these sets, let S be one with the fewest components of order s in P_n^k. Suppose $s \leq m(P_n^k - S) - 2$. Note that all of the components must be sets of consecutive vertices. Assume that C_p is a smallest component. Then $|V(C_p)| = s$, and without loss of generality, let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_e is a largest component, and so $|V(C_e)| = m(P_n^k - S) = m$ and let $C_e = \{v_j, v_{j+1}, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i| = p_i$ for $1 \leq i \leq a$, and let $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{p_i}}\}$. Now we construct the vertex set S' as

$$S' = S - \{v_{s+1}, v_{1_{p_1+1}}, v_{2_{p_2+1}}, \cdots, v_{a_{p_a+1}}\}$$
$$\cup \{v_{1_1}, v_{2_2}, \cdots, v_{a_1}, v_j\}.$$

Therefore,

$$|S'| = |S|,$$
$$m(P_n^k - S') \leq m(P_n^k - S)$$

and

$$\omega(P_n^k - S') = \omega(P_n^k - S).$$

So we have

$$\frac{|S'|}{\omega(P_n^k - S')} \leq \frac{|S|}{\omega(P_n^k - S)}.$$

Therefore,

$$\tau(P_n^k) = \frac{|S'|}{\omega(P_n^k - S')}.$$

But, $P_n^k - S'$ has one less components of order s than $P_n^k - S$, a contradiction. Thus, all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$. So,

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil.$$

This completes the proof. □

By the above two lemmas we give the exact values of toughness of the powers of paths.

Theorem 5 *Let P_n^k be a powers of a path P_n and $n = r(k+1) + s$ for $0 \leq s < k+1$. Then*

$$\tau(P_n^k) = \begin{cases} \infty, & if\ n \leq k+1 \\ \frac{k}{2}, & if\ n > k+1. \end{cases}$$

Proof. If $n \leq k+1$, then $P_n^k = K_n$, so, $\tau(P_n^k) = \infty$. If $n > k+1$, let S be a minimum τ-set of P_n^k. By Lemmas 3 and 4 we know that

$$|S| = k(\omega - 1)$$

and

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil.$$

Thus, by the definition of toughness we have

$$\tau(P_n^k) = \min\{\frac{k(\omega - 1)}{\omega}\}.$$

Now we consider the function

$$f(\omega) = \frac{k(\omega - 1)}{\omega}.$$

It is easy to see that

$$f'(\omega) = \frac{k}{\omega^2} > 0,$$

and so $f(\omega)$ is an increasing function and the minimum value occurs at the lower boundary. And we know that $2 \leq \omega \leq r+1$, so, we have $\omega = 2$. Then,

$$\tau(P_n^k) = \frac{k}{2}.$$

The proof is now completed. □

In the following section, we determine the integrity of powers of paths.

A vertex cut set S of a graph G is called an *I-set* of G if it satisfies that $I(G) = |S| + m(G - S)$.

Lemma 6 *If S is a minimal I-set for the graph P_n^k, $2 \leq k \leq n-2$, then S consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.*

Proof. We assume that the vertices of P_n^k are labelled by $v_1, v_2, \cdots, v_n$. Let S be a minimal I-set of P_n^k and j be the smallest integer such that $T = \{v_j, v_{j+1}, \cdots, v_{j+t-1}\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. We distinguish two cases:

Case 1. If $T = S$, then S contains just t consecutive vertices $v_j, v_{j+1}, \cdots, v_{j+t-1}$. Since $T = S \neq V(P_n^k)$, and by the structure of P_n^k, S must leave exact two components of $P_n^k - S$, we have $j > 1$, $j+t-1 < n$, $v_{j-1} \notin S$, $v_{j+t} \notin S$, and $v_{j+t} \neq v_{j-1}$. Therefore, $\{v_{j+t}, v_{j-1}\} \cap S = \emptyset$. By the structure of P_n^k we know that if S leave exact two components of $P_n^k - S$, S must contain at least k consecutive vertices. Thus we have $t \geq k$. Now suppose $t > k$. Delete v_{j+t-1} from the set S yielding a new set $S_1 = S - \{v_{j+t-1}\}$. Since $t > k$, the edge $v_{j+t-1}v_{j-1}$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_{j+t-1} in $P_n^k - S_1$. Then, $p \geq j+t$ and $p \leq j+t+k-2$, and so v_p is also adjacent to v_{j+t} in $P_n^k - S_1$. Therefore, deleting v_{j+t-1} from S yields $\omega(P_n^k - S_1) = \omega(P_n^k - S)$ and $m(P_n^k - S_1) \leq m(P_n^k - S) + 1$. So,

$$\begin{aligned}&|S_1| + m(P_n^k - S_1)\\ &\leq |S| - 1 + m(P_n^k - S) + 1\\ &= |S| + m(P_n^k - S)\\ &= I(P_n^k),\end{aligned}$$

which is contrary to our choice of S. Thus, $t = k$.

Case 2. If $T \subset S$. Since $S \neq V(P_n^k)$, $T \neq V(P_n^k)$, and S must leave at least two components of $P_n^k - S$, we have $j > 1$, $j+t-1 < n$, $v_{j-1} \notin S$, $v_{j+t} \notin S$, and $v_{j+t} \neq v_{j-1}$. Therefore, $\{v_{j+t}, v_{j-1}\} \cap S = \emptyset$. Now suppose $t < k$. Choose v_{j+i} such that $1 \leq i \leq t$, and delete v_{j+i} from S yielding a new set $S' = S - \{v_{j+i}\}$ with $|S'| = |S| - 1$. By the definition of P_n^k ($2 \leq k \leq n-2$) we know that the edges $v_{j+i}v_{j-1}$ and $v_{j+i}v_{j+t}$ are in $P_n^k - S'$. Consider a vertex v_p adjacent to v_{j+i} in $P_n^k - S'$. If $p \geq t+j+1$, then $p < t+j+k$. So, v_p is also adjacent to v_{j+t} in $P_n^k - S'$. If $p < j-1$, then $p \geq j-k$ and v_p is also adjacent to v_{j-1} in $P_n^k - S'$. Since $t < k$, then v_{j-1} and v_{j+t} are adjacent in $P_n^k - S'$. Therefore, we can conclude that deleting the vertex v_{j+i} from S does not change the number of components, and so $\omega(P_n^k - S') = \omega(P_n^k - S)$ and $m(P_n^k - S') \leq m(P_n^k - S) + 1$. Thus, we have

$$\begin{aligned}&|S'| + m(P_n^k - S')\\ &\leq |S| - 1 + m(P_n^k - S) + 1\\ &= |S| + m(P_n^k - S)\\ &= I(P_n^k).\end{aligned}$$

This is contrary to our choice of S. Thus we must have $t \geq k$. Now suppose $t > k$. Delete v_{j+t-1} from the set S yielding a new set $S_1 = S - \{v_{j+t-1}\}$. Since $t > k$, the edge $v_{j+t-1}v_{j-1}$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_{j+t-1} in $P_n^k - S_1$. Then, $p \geq j+t$ and $p \leq j+t+k-2$, and so v_p is also adjacent to v_{j+t} in $P_n^k - S_1$. Therefore, deleting v_{j+t-1} from S yields $\omega(P_n^k - S_1) = \omega(P_n^k - S)$ and $m(P_n^k - S_1)$=$m(P_n^k - S) + 1$. So,

$$\begin{aligned}&|S_1| + m(P_n^k - S_1)\\ &\leq |S| - 1 + m(P_n^k - S) + 1\\ &= |S| + m(P_n^k - S)\\ &= I(P_n^k),\end{aligned}$$

which is again contrary to our choice of S. Thus, $t = k$, and so S consists of the union of sets of exactly k consecutive vertices. □

Lemma 7 *There is an I-set S for the graph P_n^k, such that all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$.*

Proof. Among all I-sets of minimum order, consider those sets with maximum number of minimum order components, and we let s denote the order of a minimum component. Among these sets, let S be one with the fewest components of order s in P_n^k. Suppose $s \leq m(P_n^k - S) - 2$. Note that all of the components must be sets of consecutive vertices. Assume that C_p is a smallest component. Then $|V(C_p)| = s$, and without loss of generality, let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_e is a largest component, and so $|V(C_e)| = m(P_n^k - S) = m$ and let $C_e = \{v_j, v_{j+1}, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i| = p_i$ for $1 \leq i \leq a$, and let $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{p_i}}\}$. Now we construct the vertex set S' as $S' = S - \{v_{s+1}, v_{1_{p_1+1}}, v_{2_{p_2+1}}, \cdots, v_{a_{p_a+1}}\} \cup \{v_{1_1}, v_{2_2}, \cdots, v_{a_1}, v_j\}$. Therefore,

$$|S'| = |S|,$$

$$m(P_n^k - S') \leq m(P_n^k - S)$$

and

$$\omega(P_n^k - S') = \omega(P_n^k - S).$$

So we have

$$|S'| + m(P_n^k - S') \leq |S| + m(P_n^k - S).$$

Therefore, $I(P_n^k) = |S'| + m(P_n^k - S')$. But, $P_n^k - S'$ has one less components of order s than $P_n^k - S$, a contradiction. Thus, all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$. So,

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil.$$

The proof is now complete. □

By the above two lemmas we give the exact values of integrity of the powers of paths.

Theorem 8 *Let P_n^k be a powers of a path P_n and $n = r(k+1) + s$ for $0 \leq s < k+1$.*

$$I(P_n^k) = \begin{cases} n, \\ if\ n\ \leq k+1 \\ \min\{(\underline{\omega} - 1)k + \lceil \frac{n-k(\underline{\omega}-1)}{\underline{\omega}} \rceil, \\ \qquad (\overline{\omega} - 1)k + \lceil \frac{n-k(\overline{\omega}-1)}{\overline{\omega}} \rceil\}, \\ if\ n\ > k+1. \end{cases}$$

where $\underline{\omega} = \lfloor \sqrt{\frac{n+k}{k}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{n+k}{k}} \rceil$.

Proof. If $n \leq k+1$, then $P_n^k = K_n$, so, $I(P_n^k) = n$. If $n > k+1$, Let S be a minimum I-set of P_n^k. By Lemmas 6 and 7 we know that $|S| = k(\omega - 1)$ and $m(P_n^k - S) = \lceil \frac{n-k(\omega-1)}{\omega} \rceil$. Thus, from the definition of integrity we have

$$I(P_n^k) = \min\{k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil$$

$$|2 \leq \omega \leq r+1\}.$$

Now we consider the function

$$f(\omega) = k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil.$$

It is easy to see that

$$f'(\omega) = k + \lceil \frac{-n-k}{\omega^2} \rceil = \lceil \frac{k\omega^2 - (n+k)}{\omega^2} \rceil.$$

Since $\omega^2 > 0$, we have $f'(\omega) \geq 0$ if and only if

$$g(\omega) = k\omega^2 - (n+k) \geq 0.$$

Since the two roots of the equation

$$g(\omega) = k\omega^2 - (n+k) = 0$$

are

$$\omega_1 = -\sqrt{\frac{n+k}{k}}$$

and

$$\omega_2 = \sqrt{\frac{n+k}{k}}.$$

But $\omega_1 < 0$, and so it is deleted. It is easy to see that $\lfloor \omega_2 \rfloor \geq 2$, and we know that $2 \leq \omega \leq r+1$, so, we have the following cases:

Case 1. If $2 \leq \omega \leq \lfloor \omega_2 \rfloor$, we have $f'(\omega) \leq 0$, and so $f(\omega)$ is an decreasing function.

Case 2. If $\lceil \omega_2 \rceil \leq \omega \leq k$, then $f'(\omega) \geq 0$, and so $f(\omega)$ is a increasing function.

Thus the minimum value occurs when $\omega = \lfloor \omega_2 \rfloor$ or $\omega = \lceil \omega_2 \rceil$. Then,

$$I(P_n^k) = \min\{(\underline{\omega}-1)k + \lceil \frac{n-k(\underline{\omega}-1)}{\underline{\omega}} \rceil,$$

$$(\overline{\omega}-1)k + \lceil \frac{n-k(\overline{\omega}-1)}{\overline{\omega}} \rceil\},$$

where $\underline{\omega} = \lfloor \sqrt{\frac{n+k}{k}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{n+k}{k}} \rceil$.

The proof is now completed. □

3 Vulnerability of Total Graphs of Paths and Cycles

In this section, firstly we define total graph of a graph, then we obtain integrity, toughness, tenacity of total graphs of some basic graphs.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The concept of *total graph* $T(G)$ of graph G was introduced by Behzad [8] in 1966.

Definition 9 *The total graph of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.*

Definition 10 *([17]) The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each of its edge by a path of length 2, or equivalently, subdividing every edge of G once.*

The total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G. It is easy to see that $T(G)$ always contains both G and Line graph $L(G)$ as a induced subgraphs. Total graph is the largest graph that is formed by the adjacent relations of elements of a graph. It is is highly recommended for the design of interconnection networks.

Lemma 11 *([8]) For any graph G, $T(G) \cong (S(G))^2$.*

In [1], Aytaç, computing the tenacity of total graph of path and cycles, we will give formulas for computing exact values of tenacity of total graph of path and cycles, and our proof is simpler than that in [1].

Lemma 12 *([27]) Let C_n^k be a powers of a cycle C_n $(n \geq 2k+1)$ and $n = r(k+1)+s$ for $0 \leq s < k+1$. Then*

$$T(C_n^k) = k + \frac{1+\lceil \frac{s}{r} \rceil}{r}.$$

Remark 13 *The result in Lemma 12 only holds when $n \geq 2k+1$. If $n < 2k+1$, C_n^k is K_n, then $T(C_n^k) = n$.*

Theorem 14 *Let $T(C_n)$ be the total graph of C_n with order n $(n \geq 3)$, then the tenacity of $T(C_n)$ is*

$$T(T(C_n)) = 2 + \frac{1+\lceil \frac{s}{r} \rceil}{r}$$

where $2n = 3r + s$ for $0 \leq s < 3$.

Proof. By the definition of subdivision graph, we know that $S(C_n) = C_{2n}$. From Lemma 11, we have

$$T(C_n) \cong C_{2n}^2.$$

It is obvious that $2n > 5$, i.e., C_{2n}^2 is not a complete graph. So, by Lemma 12, we have

$$T(T(C_n)) = T(C_{2n}^2) = 2 + \frac{1 + \lceil \frac{s}{r} \rceil}{r}$$

where $2n = 3r + s$ for $0 \le s < 3$.

The proof is now complete. $\square$

Lemma 15 *([24]) Let P_n^k be a powers of a path P_n and $n = r(k+1) + s$ for $0 \le s < k+1$. Then*

$$T(P_n^k) = \begin{cases} n, \\ if\ n \le k+1 \\ \frac{k(r-1)+\lceil \frac{n-k(r-1)}{r} \rceil}{r}, \\ if\ n > k+1\ and\ s = 0 \\ \frac{kr+\lceil \frac{n-kr}{r+1} \rceil}{r+1}, \\ if\ n > k+1\ and\ s \neq 0. \end{cases}$$

Theorem 16 *Let $T(P_n)$ be the total graph of P_n with order n, then the tenacity of $T(P_n)$ is*

$$T(T(P_n)) = \begin{cases} 2n-1, \\ if\ n \le 2 \\ \frac{2(r-1)+\lceil \frac{2n-2r+1}{r} \rceil}{r}, \\ if\ n > 2\ and\ s = 0 \\ \frac{2r+\lceil \frac{2n-2r-1}{r+1} \rceil}{r+1}, \\ if\ n > 2\ and\ s \neq 0. \end{cases}$$

where $2n - 1 = 3r + s$ for $0 \le s < 3$.

Proof. By the definition of subdivision graph, we know that $S(P_n) = P_{2n-1}$. From Lemma 11, we have

$$T(P_n) \cong P_{2n-1}^2.$$

So by Lemma 15, we have

$$T(T(P_n)) = T(P_{2n-1}^2) = \begin{cases} 2n-1, \\ if\ n \le 2 \\ \frac{2(r-1)+\lceil \frac{2n-2r+1}{r} \rceil}{r}, \\ if\ n > 2\ and\ s = 0 \\ \frac{2r+\lceil \frac{2n-2r-1}{r+1} \rceil}{r+1}, \\ if\ n > 2\ and\ s \neq 0. \end{cases}$$

where $2n - 1 = 3r + s$ for $0 \le s < 3$.

The proof is now complete. $\square$

In [18], Dündar and Aytaç determined the integrity of total graphs via some parameters, but they did not give exact values for integrity of total graphs of path and cycles. We will do it in the next.

Lemma 17 *([2]) Let C_n^k be a powers of a cycle C_n, for $1 \le k \le \frac{n}{2}$, we have*

$$I(C_n^k) = k\lceil \sqrt{\frac{n}{k} + \frac{1}{4}} - \frac{3}{2} \rceil + \lceil \frac{n}{\lceil \sqrt{\frac{n}{k} + \frac{1}{4}} \rceil - \frac{1}{2}} \rceil.$$

Theorem 18 *Let $T(C_n)$ be the total graph of C_n with order n, then the integrity of $T(C_n)$ is*

$$I(T(C_n)) = 2\lceil \sqrt{n + \frac{1}{4}} - \frac{3}{2} \rceil + \lceil \frac{2n}{\lceil \sqrt{n + \frac{1}{4}} \rceil - \frac{1}{2}} \rceil.$$

Proof. By the definition of subdivision graph, we know that $S(C_n) = C_{2n}$. From Lemma 11, we have

$$T(C_n) \cong C_{2n}^2.$$

And so by Lemma 17, we have

$$\begin{aligned} I(T(C_n)) &= I(C_{2n}^2) \\ &= 2\lceil \sqrt{n + \frac{1}{4}} - \frac{3}{2} \rceil + \lceil \frac{2n}{\lceil \sqrt{n + \frac{1}{4}} \rceil - \frac{1}{2}} \rceil. \end{aligned}$$

The proof is now complete. $\square$

Theorem 19 *Let $T(P_n)$ be the total graph of P_n with order n, then the integrity of $T(P_n)$ is*

$$I(T(P_n)) = \begin{cases} 2n-1, \\ if\ n \le 2 \\ \min\{2(\underline{\omega}-1) + \lceil \frac{2n-1-2(\underline{\omega}-1)}{\underline{\omega}} \rceil, \\ 2(\overline{\omega}-1) + \lceil \frac{2n-1-2(\overline{\omega}-1)}{\overline{\omega}} \rceil\}, \\ if\ n > 2. \end{cases}$$

where $\underline{\omega} = \lfloor \sqrt{\frac{2n+1}{2}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{2n+1}{2}} \rceil$

Proof. By the definition of subdivision graph, we know that $S(P_n) = P_{2n-1}$. From Lemma 11, we have

$$T(P_n) \cong P_{2n-1}^2.$$

And so by Theorem 8, we have

$$I(T(P_n)) = I(P_{2n-1}^2)$$

$$= \begin{cases} 2n-1, \\ if\ n \leq 2 \\ \min\{2(\underline{\omega}-1)+\lceil \frac{2n-1-2(\underline{\omega}-1)}{\underline{\omega}} \rceil, \\ 2(\overline{\omega}-1)+\lceil \frac{2n-1-2(\overline{\omega}-1)}{\overline{\omega}} \rceil\}, \\ if\ n\ > 2. \end{cases}$$

where $\underline{\omega} = \lfloor \sqrt{\frac{2n+1}{2}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{2n+1}{2}} \rceil$. This complete the proof. □

Lemma 20 *([3]) Let C_n^k be the k-th power graph of C_n with order n, then the toughness of C_n^k is*

$$\tau(C_n^k) = k.$$

Theorem 21 *Let $T(C_n)$ be the total graph of C_n with order n, then the toughness of $T(C_n)$ is*

$$\tau(T(C_n)) = 2.$$

Proof. By the definition of subdivision graph, we know that $S(C_n) = C_{2n}$. From the Lemma 11, we have

$$T((C_n) \cong (S(C_n))^2.$$

And so by Lemma 20, we have

$$\tau(T(C_n)) = \tau(C_{2n}^2) = 2.$$

The proof is now complete. □

Theorem 22 *Let $T(P_n)$ be the total graph of P_n with order n, then the toughness of $T(P_n)$ is*

$$\tau(T(P_n)) = \begin{cases} \infty, & if\ n = 2 \\ 1, & if\ n\ > 2. \end{cases}$$

Proof. By the definition of subdivision graph, we know that $S(P_n) = P_{2n-1}$. From the Lemma 11, we have

$$T(P_n) \cong (S(P_n))^2.$$

Thus, by Theorem 5, we have

$$\tau(T(P_n)) = \tau(P_{2n-1}^2) = \begin{cases} \infty, & if\ n = 2 \\ 1, & if\ n\ > 2. \end{cases}$$

The proof is now complete. □

4 Relationships Between Some Vulnerability Parameters

In this section, the relationships between some vulnerability parameters such as integrity, toughness, tenacity, scattering number and rupture degree are established.

Theorem 23 *If G is an incomplete connected graph, $r(G)$, $\tau(G)$ and $\kappa(G)$ are the rupture degree, toughness and connectivity of G, respectively, then we have*

$$\tau(G) \geq \frac{2\kappa(G)}{r(G)+n+1}.$$

Proof. Suppose that S is a cut-set of G. Then, by the definition of rupture degree of G, we have

$$r(G) \geq \omega(G-S) - |S| - m(G-S).$$

It is obvious that

$$|S| + m(G-S) \leq n + 1 - \omega(G-S),$$

thus, by the above two inequalities we have

$$\omega(G-S) \leq \frac{r(G)+n+1}{2}.$$

And it is obvious that

$$|S| \geq \kappa(G).$$

So we have

$$\frac{|S|}{\omega(G-S)} \geq \frac{2}{r(G)+n+1}\kappa(G).$$

Then, by the definition of toughness and the choice of S, we have

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)}\} \geq \frac{2}{r(G)+n+1}\kappa(G).$$

The proof is thus completed. □

Remark 24 *The result in Theorem 23 is best possible, this can be shown by the graph $G = K_{1,n-1}$.*

Theorem 25 *If G is an incomplete connected graph of order n, $s(G)$, $T(G)$ and $I(G)$ are the scattering number, tenacity and integrity of G, respectively, then we have*

$$I(G) \leq \frac{s(G)+n}{2}T(G).$$

Proof. Let S be a cut-set of G. Then, by the definition of scattering number, we have

$$\omega(G-S)-|S|\leq s(G).$$

With the fact that

$$\omega(G-S)+|S|\leq n,$$

we have

$$\omega(G-S)\leq\frac{n+s(G)}{2}.$$

Thus,

$$\frac{|S|+m(G-S)}{\omega(G-S)}\geq\frac{2}{n+s(G)}(|S|+m(G-S)).$$

It is easily seen that

$$|S|+m(G-S)\geq I(G).$$

Then, by the definition of tenacity and the choice of S, we have

$$T(G)=max\{\frac{|S|+m(G-S)}{\omega(G-S)}\}\geq\frac{2}{n+s(G)}I(G).$$

Hence

$$I(G)\leq\frac{s(G)+n}{2}T(G).$$

The proof is thus completed. □

Remark 26 *The result in Theorem 25 is best possible, this can be shown by the graph* $G=K_{1,n-1}$.

Theorem 27 *If G is an incomplete connected graph, $T(G)$, $\tau(G)$ and $I(G)$ are the tenacity, toughness and integrity of G, respectively, then we have*

$$T(G)\geq\frac{\tau(G)+1}{n}I(G).$$

Proof. Suppose that S is a T-set of G. Then, by the definition of toughness, we have

$$\tau(G)\omega(G-S)\leq|S|.$$

With this fact and $\omega(G-S)+|S|\leq n$. We get that

$$2\leq\omega(G-S)\leq\frac{n}{\tau(G)+1},$$

It is easily seen that

$$|S|+m(G-S)\geq I(G).$$

So we have

$$\begin{aligned}T(G)&=\frac{|S|+m(G-S)}{\omega(G-S)}\\&\geq\frac{\tau(G)+1}{n}(|S|+m(G-S))\\&\geq\frac{\tau(G)+1}{n}I(G).\end{aligned}$$

The proof is thus completed. □

Remark 28 *The result in Theorem 27 is best possible, this can be shown by the graph* $G=K_{1,n-1}$.

Theorem 29 *If G is an incomplete connected graph of order n, $r(G)$, $T(G)$ and $I(G)$ are the rupture degree, tenacity and integrity of G, respectively, then we have*

$$r(G)\leq\frac{n+1}{T(G)+1}-I(G).$$

Proof. Let S be a cut-set of G. Then, by the definition of rupture degree, we have

$$r(G)\geq\omega(G-S)-|S|-m(G-S).$$

It is easy to see that

$$I(G)\leq|S|+m(G-S)\leq n+1-\omega(G-S).$$

Then, by the definition of tenacity, we have

$$T(G)\leq\frac{|S|+m(G-S)}{\omega(G-S)}\leq\frac{n+1-\omega(G-S)}{\omega(G-S)}.$$

So, we have

$$\omega(G-S)\leq\frac{n+1}{T(G)+1}.$$

On the other hand, we have that

$$\omega(G-S)-|S|-m(G-S)\leq\frac{n+1}{T(G)+1}-I(G).$$

By the definition of rupture and the choice of S, we know that

$$r(G)=\max\{\omega(G-S)-|S|-m(G-S)\}$$

$$\leq\frac{n+1}{T(G)+1}-I(G).$$

The proof is thus completed. □

Remark 30 *The result in Theorem 29 is best possible, this can be shown by the graph* $G=K_{1,n-1}$.

Theorem 31 *If G is an incomplete connected graph of order n, $r(G)$, $s(G)$ and $I(G)$ are the rupture degree, scattering number and integrity of G, respectively, then we have*

$$r(G)\leq(n+1-I(G))-\frac{2(2n-I(G))}{n+s(G)}.$$

Proof. Let S be a cut-set of G. Then, by the definition of scattering number, we have

$$\omega(G-S)-|S|\leq s(G).$$

With the fact that

$$\omega(G-S)+|S|\leq n,$$

we have

$$\omega(G-S)\leq \frac{n+s(G)}{2}.$$

It is easily seen that

$$m(G-S)\geq \frac{n-|S|}{\omega(G-S)}.$$

From

$$|S|+m(G-S)\leq n+1-\omega(G-S),$$

and

$$|S|\geq 1,$$

we have

$$|S|\geq \omega(G-S)-n+I(G),$$

So, we have that

$$|S|+m(G-S)\geq |S|+\frac{n-|S|}{\omega(G-S)}$$
$$=\frac{n+|S|(\omega(G-S)-1)}{\omega(G-S)}$$
$$\geq \frac{n+(\omega(G-S)-n+I(G))(\omega(G-S)-1)}{\omega(G-S)}$$
$$=\omega(G-S)+\frac{2n-I(G)}{\omega(G-S)}-(n+1-I(G)).$$

Therefore, by the fact that

$$\omega(G-S)\leq \frac{n+s(G)}{2}.$$

We have the following inequality

$$\omega(G-S)-|S|-m(G-S)$$
$$\leq (n+1-I(G))-\frac{2n-I(G)}{\omega(G-S)}$$
$$\leq (n+1-I(G))-(2n-I(G))\frac{2}{n+s(G)}$$

By the definition of rupture and the choice of S, we know that

$$r(G)=\max\{\omega(G-S)-|S|-m(G-S)\}$$
$$\leq (n+1-I(G))-(2n-I(G))\frac{2}{n+s(G)}.$$

The proof is thus completed. □

Remark 32 *The result in Theorem 31 is best possible, this can be shown by the graph* $G=K_{1,n-1}$.

5 Conclusion

If a system such as a communication network is modelled by a graph G, there are many graph theoretical parameters used to describe the vulnerability of communication networks including connectivity, integrity, toughness, binding number, tenacity and rupture degree. Two ways of measuring the vulnerability of a network is through the ease with which one can disrupt the network, and the cost of a disruption. Connectivity has the least cost as far as disrupting the network, but it does not take into account what remains after disruption. One can say that the disruption is less harmful if the disconnected network contains more components and much less harmful if the affected components are small. One can associate the cost with the number of the vertices destroyed to get small components and the reward with the number of the components remaining after destruction. In this paper, we have obtained the exact values for some graph theoretical parameters of powers of paths and of total graphs of some special graphs.

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