Finite group with c-normal or s-quasinormally embedded subgroups

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Abstract: If P is a p-group for some prime p we shall write $\mathscr{M}(P)$ to denote the set of all maximal subgroups of P and $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible. In this paper, the structure of a finite group G under some assumptions on the c-normal or s-quasinormally embedded subgroups in $\mathscr{M}_d(P)$, for each prime p, and Sylow p-subgroups P of G is researched. Some known results are generalized.

Key–Words: c-normal subgroup; s-quasinormally embedded subgroup; supersolvable groups; Sylow p-subgroup; Finite groups.

1 Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathscr{M}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G. A interesting topic in group theory is to study the influence of the elements of $\mathscr{M}(G)$ on the structure of G. A typical result in this direction is due to Srinivasan [1]. He proved that G is supersolvable provided that every member of $\mathscr{M}(G)$ is normal in G. This result has been widely generalized. One direction of generalization is to replace the normality condition of maximal subgroups of Sylow subgroups by a weaker condition; and the other direction of generalization is to minimize the number of maximal subgroups of Sylow subgroups. As a result, many interesting results have been subsequently obtained by many authors (for example, see [7, 8, 10, 12, 24-42]). It has been particularly observed that the property of normality for some maximal subgroups of Sylow subgroups gave a lot of useful information on the structure of groups.

A subgroup H of G is called s-quasinormal in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [2] and has been studied extensively by Deskins [3] and Schmidt [4]. More recently, Ballester-Bolinches and Pedraza-Aquilera [5] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. A subgroup H of G is said to be s-quasinormally embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-quasinormal subgroup of G. In [5], Ballester-Bolinches and Pedraza-Aquilera showed that, if every subgroup in $\mathscr{M}(G)$ is s-quasinormally embedded in G, then G is supersolvable. Assad and Heliel [6] showed that G is p-nilpotent for the smallest prime p dividing $|G|$ if and only if all members of $\mathscr{M}(P)$ are s-quasinormally embedded in G, where P is a Sylow p-subgroup of G. In the same paper, they showed that a group G belongs to $\mathscr{F}$, a saturated formation containing all supersolvable groups, if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(H)$ is s-quasinormally embedded in G. In the paper [7], the research in this direction has been continued further by considering a subset $\mathscr{M}_d(G)$ of $\mathscr{M}(G)$. In [8], Li and Wang have proved that $G \in \mathscr{F}$, a saturated formation containing all supersolvable groups, if and only if there is a normal subgroup H such that $G/H \in F$ and every member of $\mathscr{M}(F^*(H))$, where $F^*(H)$ is the generalized Fitting subgroup of H, is s-quasinormally embedded in G.

As another generalization of the normality, Wang [9] introduced the following concept: A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G. In [9], Wang showed that G is supersolvable if every member of $\mathscr{M}(G)$ is c-normal. Wang$'$s result has been generalized by some authors(see [10-14], etc). For example, Guo and Shum showed in [12] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is c-normal, then G is p-nilpotent. In [14], Wei, Wang and Li showed that $G \in \mathscr{F}$ if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and if every member of $\mathscr{M}(F^*(H))$ is c-normal in G. The research on c-normal subgroups has formed a series, which is similar to the series of s-quasinormal subgroups. However, the two series are independent of each other. The aim of this article is to unify and improve the results of [1], [5], [9] and some of [10].

If P is a p-group for some prime p we shall write $\mathscr{M}(P)$ to denote the set of all maximal subgroups of P and $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible.

Such subset $\mathscr{M}_d(P)$ is not unique for a fixed P in general. We know that

$$|\mathscr{M}(P)| = \frac{p^d-1}{p-1}, |\mathscr{M}_d(P)| = d, \lim_{d\to\infty} \frac{p^d-1}{(p-1)d} = \infty,$$

so $|\mathscr{M}(P)| \gg |\mathscr{M}_d(P)|$.

In this paper, we study the influence of the members of some fixed $\mathscr{M}_d(P)$ on the structure of group G. Our results are more general.

A class $\mathscr{F}$ of finite groups is called a formation if $G \in \mathscr{F}$ and $N \trianglelefteq G$ then $G/N \in \mathscr{F}$; and if $G/N_i(i = 1, 2) \in \mathscr{F}$ then $G/(N_1 \cap N_2) \in \mathscr{F}$. If, in addition, $G/\Phi(G) \in \mathscr{F}$ implies $G \in \mathscr{F}$, we call $\mathscr{F}$ saturated. An interesting example is the class of all supersolvable groups, which is denoted by $\mathscr{U}$.

The following notation is used in the paper. If H is a subgroup of the group G, then by H_G we denote the normal core of H in G, the largest normal subgroup of G which is contained in H. Also, G_p always denotes a Sylow p-subgroup of G, $\Phi(G)$ is the Frattini subgroup of G. The rest of our notation and terminology are standard. The reader may refer to ref.[23].

2 Basic definitions and preliminary results

In this section, we give some results that are needed in this paper.

Definition 1 *[2] A subgroup H of G is called s-quasinormal in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G.*

Definition 2 *[5] A subgroup H of G is said to be s-quasinormally embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-quasinormal subgroup of G.*

Definition 3 *[9] A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G.*

Remark 4 *we will show that there are groups with s-quasinormally embedded subgroups which are not c-normal. Conversely, there are also groups with c-normal subgroups which are not s-quasinormally embedded. This means that there is no obvious general relationship between these two notions.*

Example 5 *Every Sylow subgroup of any simple non-abelian group is s-quasinormally embedded but not c-normal*

Example 6 *Consider $G = S_4$, the symmetric group of degree 4. Take $\alpha = (34)$ and $\beta = (123)$. Then $G = \langle\alpha\rangle A_4$ and $\langle\alpha\rangle \cap A_4 = 1$, and hence $\langle\alpha\rangle$ is c-normal in G. However $\langle\alpha\rangle$ is not s-quasinormally embedded in G. In fact, if $\langle\alpha\rangle$ is a Sylow 2-subgroup of some s-quasinormal subgroup K of G, then $K\langle\beta\rangle$ is a group. Since $|K\langle\beta\rangle : \langle\beta\rangle| = 2$, we have $\langle\beta\rangle \lhd K\langle\beta\rangle$ and so $\langle\alpha\rangle\langle\beta\rangle = \langle\beta\rangle\langle\alpha\rangle$, which is a contradiction.*

Lemma 7 *Suppose that U is an s-quasinormally embedded subgroup of G and that K is a normal subgroup of G. Then:*

(a) U is s-quasinormally embedded in H whenever $U \leq H \leq G$;

(b) UK is s-quasinormally embedded in G and UK/K is s-quasinormally embedded in G/K.

Lemma 8 *[9] Let $X \leq H \leq G$ and $N \trianglelefteq G$. Then:*

(a) If X is c-normal in G, then X is also c-normal in H;

(b) If X is c-normal in G, then XN/N is c-normal in G/N.

In order to prove our main theorem, we need the following important lemma.

Lemma 9 *[2] If H is an s-quasinormal subgroup of the group G, then H/H_G is nilpotent.*

Lemma 10 *[6] For a nilpotent subgroup H of G, the following two statements are equivalent:*

(a) H is s-quasinormal in G;

(b) The Sylow subgroups of H are s-quasinormal in G.

Lemma 11 *[6] Let G be a group and let P_0 be a maximal subgroup of P. Then the following two statements are equivalent:*

(a) P_0 is normal in G;

(b) P_0 is s-quasinormal in G.

The following Tate′s theorem will be used in the proof of our Theorem 14

Lemma 12 *[15] If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \bigcap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 13 *[16] Let N be a normal subgroup of a group $G(N \neq 1)$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G that are contained in $F(N)$.*

3 Main results

Theorem 14 *Let p be a prime dividing the order of a group G, $(|G|, p-1) = 1$, and let P be a Sylow p-subgroup of G. Then the following statements are equivalent:*

(a) G is p-nilpotent;

(b) every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G.

Proof: Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$. By hypothesis, each P_i is either c-normal or s-quasinormally embedded in G. Without loss of generality, let I_1 be the subset of $\{1, ..., d\}$ such that every $P_i (i \in I_1)$ is c-normal in G and I_2 is the subset such that every $P_i (i \in I_2)$ is s-quasinormally embedded in G. We prove the theorem by the following claims:

(1) $O_{p'}(G) = 1$.

Set $N = O_{p'}(G)$. Consider the quotient group G/N. We know that PN/N is a Sylow p-subgroup of G/N, $N_{G/N}(PN/N) = N_G(P)N/N$ and $\mathscr{M}(PN/N) = \{P_1N/N, ..., P_mN/N\}$. Now, by Lemma 7 and Lemma 8, we see easily that G/N satisfies the condition. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent and hence G itself is p-nilpotent, a contradiction. Thus claim (1) holds.

(2) G/P_{iG} is p-nilpotent for all $i \in I_1$, where P_{iG} is the core of P_i in G.

In this case, P_i is a c-normal subgroup of G. We know that there exists a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i = P_{iG}$. Hence,

$$G/P_{iG} = P_i/P_{iG} \cdot K_i/P_{iG}, P_i \cap K_i = P_{iG}.$$

Therefore,

$$|K_i/P_{iG}|_p = |G : P_i|_p = |P : P_i| = p.$$

As p is the smallest prime dividing $|G|$, we know that K_i/P_{iG} is p-nilpotent by Burnside′s theorem. Therefore, K_i/P_{iG} has a normal Hall p'-subgroup H/P_{iG}. We see that H/P_{iG} is also a normal Hall p'-subgroup of G/P_{iG} because K_i/P_{iG} is normal in G/P_{iG}. It follows that G/P_{iG} is p-nilpotent for all $i \in I_1$.

For every $P_i (i \in I_2)$, there exists an s-quasinormal subgroup H_i of G such that P_i is a Sylow p-subgroup of H_i.

(3) G/H_{iG} is p-nilpotent for all $i \in I_2$, where H_{iG} is the core of H_i in G.

In fact, As H_i is an s-quasinormal subgroup of G and P_i is a Sylow p-subgroup of H_i, it follows that H_i/H_{iG} is s-quasinormal in G/H_{iG}, and the Lemma 9 asserts that H_i/H_{iG} is nilpotent. Hence, H_i/H_{iG} is an s-quasinormal nilpotent subgroup of G/H_{iG}. By Lemma 10, every Sylow subgroup of H_i/H_{iG} is s-quasinormal in G/H_{iG}. Since P_iH_i/H_{iG} is a Sylow p-subgroup of H_i/H_{iG}, it follows that P_iH_i/H_{iG} is s-quasinormal in G/H_{iG}. Thus, Lemma 11 indicates that P_iH_i/H_{iG} is normal in G/H_{iG}. Therefore, $P_iH_{iG} \trianglelefteq G$. Noting that P_i is a Sylow p-subgroup of H_i, we have $P_i \leq H_{iG}$. Therefore, $|G/H_{iG}|_p = p$. Now, as p is the smallest prime dividing $|G|$, by Burnside′s theorem, we see that G/H_{iG} is p-nilpotent for each $i \in I_2$, which proves (3).

Let

$$N = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_{iG}).$$

(4) N is p-nilpotent.

First, as all P_{iG} and H_{iG} are normal in G, we get $N \trianglelefteq G$. Second, we consider the subgroup $P \cap N$. Recall that P_i is a Sylow p-subgroup of H_{iG} and $P_i \leq P$, so $P \cap H_{iG} \leq P_i$. Moreover, $P_i \leq P \cap H_{iG}$. We have $P \cap H_{iG} = P_i$. Therefore,

$$\begin{aligned} P \cap N &= (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_{iG} \cap P) \\ &= (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} P_i) \\ &= \Phi(P). \end{aligned}$$

Applying Lemma 12, we know that N is p-nilpotent.

(5) Final contradiction.

Now, N possesses a Hall p'-normal subgroup $N_{p'}$ such that $N = N_pN_{p'}$, where N_p is a Sylow p-subgroup of N. Then, $N_{p'}$ char $N \trianglelefteq G$, so $N_{p'}$ is normal in G, and hence, $N_{p'} \leq O_{p'}(G)$. It follows by $O_{p'}(G) = 1$ that $N_{p'} = 1$. Consequently, N is a normal p-subgroup of G, and so, $N = P \cap N =$

$\Phi(P)$. Also, note that the class of p-nilpotent groups is a formation, by steps (2) and (3), we have G/N must be p-nilpotent. It follows that $G/\Phi(P)$ is p-nilpotent. Moreover, by III, 3.3 Hilfs-Satz in [20], $\Phi(P) \leq \Phi(G)$, so $G/\Phi(G)$ is p-nilpotent. It follows that G would be p-nilpotent, contrary to the choice of G. □

Corollary 15 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Proof: Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By hypothesis, every member of $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G. In particular, G satisfies the condition of Theorem 3.1, so G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 7 and 8, U satisfies the hypothesis. It follows by induction that U, and hence G possess the Sylow town property of supersolvable type. □

The following corollaries are immediate from Theorem 14 and Corollary 15

Corollary 16 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is c-normal in G, then G is p-nilpotent.*

Corollary 17 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is normal in G, then G is p-nilpotent.*

Corollary 18 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 19 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is s-quasinormal in G, then G is p-nilpotent.*

Corollary 20 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is quasinormal in G, then G is p-nilpotent.*

Corollary 21 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 22 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormal in G, then G is p-nilpotent.*

Corollary 23 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is either c-normal or quasinormal in G, then G is p-nilpotent.*

Corollary 24 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is c-normal in G, then G is p-nilpotent*

Corollary 25 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is normal in G, then G is p-nilpotent.*

Corollary 26 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 27 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is s-quasinormal in G, then G is p-nilpotent*

Corollary 28 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is quasinormal in G, then G is p-nilpotent.*

Corollary 29 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is either c-normal or s-quasinormal in G, then G is p-nilpotent.*

Corollary 30 *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}_d(P)$ is either c-normal or quasinormal in G, then G is p-nilpotent.*

Corollary 31 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is c-normal in G, then G has a Sylow tower of supersolvable type.*

Corollary 32 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is normal in G, then G has a Sylow tower of supersolvable type.*

Corollary 33 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is s-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Corollary 34 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is s-quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 35 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 36 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Corollary 37 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 38 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}(P)$ is either c-normal or quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 39 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is c-normal in G, then G has a Sylow tower of supersolvable type.*

Corollary 40 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is normal in G, then G has a Sylow tower of supersolvable type.*

Corollary 41 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is s-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Corollary 42 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is s-quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 43 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 44 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is either c-normal or s-quasinormal in G, then G has a Sylow tower of supersolvable type.*

Corollary 45 *Suppose that G is a group. For each Sylow subgroup P of G, if every member of $\mathscr{M}_d(P)$ is either c-normal or quasinormal in G, then G has a Sylow tower of supersolvable type.*

Theorem 46 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G. Then G is p-nilpotent.*

Proof: It is easy to see that the theorem holds when $p = 2$ by Theorem 14, so it suffices to prove the theorem for the case when p is odd. Suppose that the theorem is not true, and let G be a counterexample of the smallest order. We have the following claims:

(1) $O_{p'}(G) = 1$.

In fact, $O_{p'}(G) \neq 1$. Consider the quotient group $G/O_{p'}(G)$. By Lemma 7 and Lemma 8, we see easily that $G/O_{p'}(G)$ satisfies the condition of the theorem. It follows that $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Hence G itself is p-nilpotent, which is a contradiction. Thus claim (1) holds.

(2) If $P \leq H \leq G$, then H is p-nilpotent.

Noting that $N_H(P) \leq N_G(P)$, we have $N_H(P)$ is p-nilpotent. By Lemma 7 and Lemma **??**, H satisfies the hypotheses of the theorem. By the choice of G, H is p-nilpotent, as desired.

(3) $G = PQ$, where Q is a Sylow q-subgroup of G, $q \neq p$.

By the choice of G, G is not p-nilpotent. In the light of a result of Thompson (Corollary in Ref. [21]), there exists a nontrivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Now, T char $P \trianglelefteq N_G(P)$, which gives $T \trianglelefteq N_G(P)$. Therefore, $N_G(P) \leq N_G(T)$. By (2), we get $N_G(T) = G$, and hence, $T = O_p(G)$. Applying the result of Thompson again, we have $G/O_p(G)$ is p-nilpotent; therefore, G is p-solvable. Thus, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Consequently, $PQ = G$, as desired.

(4) Final contradiction.

We now make use of the above claims to finish our proof. As $O_{p'}(G) = 1$, we have $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leq \Phi(P)$, then $N \leq \Phi(G)$ by III, 3.3 in Ref. [20], and the quotient group G/N satisfies the hypotheses of the theorem, thus G/N is p-nilpotent by the choice of G. It follows that $G/\Phi(G)$ is p-nilpotent, and hence, G is p-nilpotent, which is a contradiction. Thus, $N \leq \Phi(G)$ cannot happen, so $N \nleq \Phi(G)$. Because $\Phi(P) = \bigcap_{i=1}^{d} P_i$, where $P_i \in \mathscr{M}_d(P)$, without loss of generality, we may assume that $N \nleq P_1$. Put $N_1 = N \cap P_1$. Then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. Also, by hypotheses, P_1 is c-normal or S-quasinormally emdedded in G.

We claim that N is a cyclic subgroup of order p.

Case 1. If P_1 is c-normal in G, then there exists a normal subgroup K of G such that $G = P_1K$ and $P_1 \cap K = 1$. Since N is a minimal normal subgroup of G with $N \nleq P_1$, we have $N \leq K$. Thus $N_1 = 1$ and N is a cyclic subgroup of order p.

Case 2. If P_1 is s-quasinormally embedded in G, then there exists an S-quasinormal subgroup H of G, such that $P_1 \in \mathrm{Syl}_p(H)$. Thus, HQ is a subgroup of G. As $N \trianglelefteq G$, we have

$$N_1 = N \cap HQ \trianglelefteq HQ,$$

and it follows that

$$N_1 \trianglelefteq \langle HQ, N\rangle = G.$$

Moreover, since N is a minimal normal subgroup of G, we have $N_1 = 1$, and N is a cyclic subgroup of order p.

Now, $NP_1 = P$ and $N \cap P_1 = 1$. By W. Gaschutz′s Theorem (I, 17.4 in Ref. [20]), there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Of course, $N \nleq \Phi(G)$. By Lemma 2.13, we have $O_p(G) = R_1 \times \cdots \times R_r$, where R_i, $i = 1, ..., r$, are minimal normal subgroups of G of order p. Therefore, we get

$$P \leq \bigcap_{i=1}^{r} C_G(R_i) = C_G(O_p(G)).$$

Moreover, by Theorem 9.3.1 in Ref. [22] and (3), $C_G(O_p(G)) \leq O_p(G)$, it follows that $P = O_p(G)$, and so, $G = N_G(P)$. Now, we apply the hypotheses that $N_G(P)$ is p-nilpotent to conclude that G is p-nilpotent. This is a contradiction, which completes the proof. □

The following corollaries are immediate from Theorem 46

Corollary 47 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is c-normal in G. Then G is p-nilpotent.*

Corollary 48 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is normal in G. Then G is p-nilpotent.*

Corollary 49 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 50 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is s-quasinormal in G. Then G is p-nilpotent.*

Corollary 51 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is quasinormal in G. Then G is p-nilpotent.*

Corollary 52 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 53 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormal in G. Then G is p-nilpotent.*

Corollary 54 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of $\mathscr{M}(P)$ is either c-normal or quasinormal in G. Then G is p-nilpotent.*

Corollary 55 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is c-normal in G. Then G is p-nilpotent.*

Corollary 56 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is normal in G. Then G is p-nilpotent.*

Corollary 57 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 58 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is s-quasinormal in G. Then G is p-nilpotent.*

Corollary 59 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is quasinormal in G. Then G is p-nilpotent.*

Corollary 60 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormal in G. Then G is p-nilpotent.*

Corollary 61 *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent and every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or quasinormal in G. Then G is p-nilpotent.*

Remark 62 *In proving our Theorem 3.33, Corollary 3.34, $\cdots$, Corollary 3.48, the assumption that $N_G(P)$ is p-nilpotent is essential. To illustrate the situation, we consider $G = A_5$ and $p = 5$. In this case, since every maximal subgroup of Sylow 5-subgroup of G is 1, we see that every maximal subgroup of Sylow 5-subgroup of G is c-normal and s-quasinormally embedded in G, but G is not 5-nilpotent.*

Theorem 63 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G. Then G is p-supersolvable.*

Proof: Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$ with $\Phi(P) = \bigcap_{i=1}^{d} P_i$. We shall finish the proof by the following claims:

(1) $O_{p'}(G) = 1$ and $\Phi(O_p(G)) = 1$.

This follows from the choice of G, as in the proof of Theorem **??**

(2) Every G-chief factor contained in $O_p(G)$ is cyclic.

As G is p-solvable and $O_{p'}(G) = 1$, we have $O_p(G) > 1$. Thus, we can find a minimal normal subgroup N of G contained in $O_p(G)$. If $N \leq \Phi(P)$, then $N \leq \Phi(G)$, and the quotient group G/N satisfies the hypotheses of the theorem, by the minimality of G, G/N is p-supersolvable. As the class of p-supersolvable groups is a saturated formation, we have G is p-supersolvable, which is a contradiction. Thus, $N \nleq \Phi(P)$. We may assume that $N \nleq P_1$. Let $N_1 = N \cap P_1$. Then, $|N : N_1| = p$. By hypotheses, P_1 is c-normal or S-quasinormally emdedded in G.

We claim that N is a cyclic subgroup of order p.

Case 1. If P_1 is c-normal in G, then there exists a normal subgroup K of G such that $G = P_1K$ and $P_1 \cap K = 1$. Since N is a minimal normal subgroup of G with $N \nleq P_1$, we have $N \leq K$. Thus $N_1 = 1$ and N is a cyclic subgroup of order p.

Case 2. If P_1 is s-quasinormally embedded in G, then there exists an s-quasinormal subgroup H of G, such that $P_1 \in \mathrm{Syl}_p(H)$. Thus, HQ is a subgroup of G. As $N \trianglelefteq G$, we have

$$N_1 = N \cap HQ \trianglelefteq HQ,$$

and it follows that

$$N_1 \trianglelefteq \langle HQ, N\rangle = G.$$

Moreover, since N is a minimal normal subgroup of G, we have $N_1 = 1$, and N is a cyclic subgroup of order p.

Consequently, $N \cap P_1 = 1$. By W. Gaschutz$'$s Theorem (I, 17.4 in Ref. [20]), there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Of course, $N \nleq \Phi(G)$. Now, we can apply Lemma 13 to conclude that $O_p(G)$ is a direct product of normal subgroups of G of order p. Thus, (2) follows.

(3) Final contradiction.

Since $G/C_G(R_i)$ is a cyclic group of order $p-1$, of course,

$$G/\bigcap_{i=1}^{r} C_G(R_i) = G/C_G(O_p(G))$$

is p-supersolvable. On the other hand, as G is p-solvable and $O_{p'}(G) = 1$, by Theorem 9.3.1 in Ref. [22], $C_G(O_p(G)) \leqslant O_p(G)$. Thus, $G/O_p(G)$ is p-supersolvable. Now, claim (2) implies that G is p-supersolvable, the proof is then finished. $\square$

The following corollaries are immediate from Theorem 63

Corollary 64 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is c-normal in G. Then G is p-supersolvable.*

Corollary 65 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is normal in G. Then G is p-supersolvable.*

Corollary 66 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is s-quasinormally embedded in G. Then G is p-supersolvable.*

Corollary 67 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is s-quasinormal in G. Then G is p-supersolvable.*

Corollary 68 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is quasinormal in G. Then G is p-supersolvable.*

Corollary 69 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormally embedded in G. Then G is p-supersolvable.*

Corollary 70 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is either c-normal or s-quasinormal in G. Then G is p-supersolvable.*

Corollary 71 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}(P)$ is either c-normal or quasinormal in G. Then G is p-supersolvable.*

Corollary 72 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is c-normal in G. Then G is p-supersolvable.*

Corollary 73 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is normal in G. Then G is p-supersolvable.*

Corollary 74 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is s-quasinormally embedded in G. Then G is p-supersolvable.*

Corollary 75 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is s-quasinormal in G. Then G is p-supersolvable.*

Corollary 76 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is quasinormal in G. Then G is p-supersolvable.*

Corollary 77 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is either c-normal or s-quasinormal in G. Then G is p-supersolvable.*

Corollary 78 *Let G be a p-solvable group for a prime p, and let P be a Sylow p-subgroup of G. Assume that every member of $\mathscr{M}_{\mathrm{d}}(P)$ is either c-normal or quasinormal in G. Then G is p-supersolvable.*

Remark 79 *The hypothesis that G is p-solvable in Theorem 63, Corollary 64, $\cdots$, Corollary 78 cannot be removed. To illustrate the situation, we consider for example the group $G = A_5$, the alternating group of degree 5. Clearly 1 is the maximal subgroup of any Sylow 5-subgroup of G and $F_5(G) = 1$. However, G is not 5-supersolvable.*

4 Conclusion

The results explained in the previous sections show that the method that we replace conditions for all maximal subgroups of Sylow subgroups of G by conditions referring to only some of maximal subgroups of Sylow subgroups of G in order to investigate the structure of a finite group is very useful. Results of this type are interesting since they can be used to simplify the proofs of new or known properties related to maximal subgroups of Sylow subgroups of G. In addition, there are many other generalizations of the normality, for example, c^*-normality in [35]; π-quasinormally embedded subgroups in [8]; SS-quasinormal subgroups in [19]; X-semipermutable subgroups in [25]. As an application, we may consider using the above special subgroups to characterize the structure of finite groups.

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