New Asymptotic and Algorithmic Results in Calculation of Random Graphs Connectivity

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Abstract: This paper is devoted to calculations of connectivity probabilities and distributions of connectivity components numbers in planar and recursively defined and parallel-sequential random graphs. These calculations are based on asymptotic formulas for connectivity probabilities and on recurrent relations for different deterministic and stochastic characteristics of random graphs and on limit theorems for Markov chains with finite state set. Main aim of these paper is to decrease calculational complexity of considered problems solutions.

Key–Words: Planar graph, cross section, connectivity probability, recursive definition, calculational complexity, central limit theorem, law of large numbers.

1 Introduction

This paper contains authors‘ results in actual recent years problem of reliability analysis of complicated random networks. Specialists from different sciences physics, biology, technical sciences, nanotechnology, sociology) interest in these models [1] – [8]. This fundamental problem is considered as in reliability theory [9] – [13] so probability theory papers [14] – [24].

One of the most difficult problems in an investigation of complicated random networks is large complexity of necessary calculations and even NP problem. In this paper the authors concentrated their attention on an investigation of a specifics of random networks and on a derivation of asymptotic formulas and on a construction of recurrent algorithms. Such approach allowed to decrease a complexity of calculations in considered problems sometimes even to linear.

Large interest of specialists in reliability theory is called a concept and a calculation of connectivity probability in random networks. In [25] – [27] upper and low estimates of the connectivity probability are constructed for general type networks on a base of maximal systems of disjoint frames. For small numbers of arcs in [28] accelerated algorithms of a calculation of reliability polynomial coefficients are constructed. In [29] this problem is solved using the Monte-Carlo method with some combinatory formulas. To calculate the connectivity probability in rectangle lattices the transfer matrix method is used [30]. But an increasing of arcs number leads to large complexity and so it is worthy to develop asymptotic methods.

In this paper an analog of Burtin-Pittel asymptotic formula [31] for disconnection probability of random graph with high reliable arcs is constructed in terms of cross sections with minimal volume. Minimal volume equals to maximal flow in random ports [32] and its calculation has cubic complexity. But an enumeration of cross sections with minimal volume has geometric complexity. So we consider widely used planar graphs for which we prove that a definition of these characteristics has no more than cubic complexity by a number of faces. And there is a lot of graphs [33, Ch. IV] for which this complexity is linear and smaller. These results are based on a consideration of dual graphs [34], [35] in which cross sections generate cycles [36], [37]. Numerical experiment confirms an accuracy and a performance of suggested method.

Accuracy calculations of connectivity probability and another reliability characteristics usually are NP problems. So there is a necessity to describe practically interesting classes of random networks with fast algorithms of their analysis. For example such calculations in parallel-sequential networks defined by recursive formulas [8], [9] have linear complexity by a number of arcs. Such an approach to fast algorithms construction for recursively defined networks has still not received adequate attention. In this paper we consider a class of random networks recursively defined by a gluing in single node and construct algorithms of their connectivity probability calculation and a calculation of a probability that there is closed way passing through all network nodes. Last characteristic immediately leads to a salesman problem which is N-P complicated. Nevertheless a solution of this problem has linear complexity because of recursive definition of considered networks.

Another important characteristic of random networks is a number of connectivity components. In [38], [39] limit distributions of numbers of connectivity components in random image of n - element discrete set are obtained. Analogous problem for structurally complicated systems with many states is solved by means of probability - algebraic methods [40], [41] with geometric complexity. But such problem was not considered for parallel-sequential connections widely used in reliability theory. In this paper for special sequence of parallel-sequential connections recursive formulas characterizing random numbers of connectivity components are obtained. Variants of the law of large numbers and the central limit theorem for the random numbers of connectivity components are proved and their parameters are calculated using recurrent formulas for auxiliary conditional moments of these numbers.

2 Asymptotic analysis of connectivity probability in random planar graphs

In this section, complete asymptotic formulas for an disconnection probability in random graphs with high reliable arcs are obtained. A definition of coefficients in these formulas have geometric complexity by a number of arcs. But a consideration of planar graphs and dual graphs allow to solve this problem with no more than cubic complexity by a number of graph faces.

2.1 Asymptotic formulas

Consider non oriented connected graph G with finite sets of nodes U and arcs W. Suppose that each pair of nodes in G may be connected no more than single arc and there are not loops. Denote $\mathcal{L}(u,v)$ the set of all cross sections in G which divide nodes $u,v \in U,\ u \neq v$, and define the set $\mathcal{L} = \bigcup\limits_{u\neq v} \mathcal{L}(u,v)$ of all cross sections in G. Graph cross section is such set of arcs which deletion makes it non connected. Put $d(L)$ a number of arcs in the cross section L and

$$D(u,v) = \min(d(L):\ L \in \mathcal{L}(u,v)),$$

$$D = \min_{u\neq v} D(u,v),\ \mathcal{L}_* = \{L \in \mathcal{L}:\ d(L) = D\},$$

C is a number of cross sections in the set $\mathcal{L}_*$. Suppose that graph arcs work independently with probabilities $p(w),\ w \in W$.

Theorem 1 *If* $\overline{p}(w) = 1 - p(w) = h,\ w \in W$, *then graph disconnection probability*

$$\overline{P} \sim Ch^D,\ h \to 0. \tag{1}$$

Theorem 2 *If* $\overline{p}(w) \sim c_w h,\ h \to 0,\ w \in W$, *then*

$$\overline{P} \sim \sum_{L\in\mathcal{L}_*} h^D \prod_{w\in L} c_w,\ h \to 0.$$

Theorems 1, 2 are generalizations of the Burtin-Pittel asymptotic formula [31].

2.2 Calculation of constants C, D

Suppose that the graph G has m arcs and r nodes.

Theorem 3 *If the number c_1 of arcs in G, which do not belong to any cycle is positive then $D = 1,\ C = c_1$. The complexity of c_1 calculation is $O(mr^3)$.*

Assume that the graph G is planar and its each arc belongs to some cycle. Arcs of planar graph divide a plane into faces [34, Ch. 1]. Denote n the number of faces in G. Confront the graph G its dual graph G^*. Each face z in G accords the node z^* in G^*, each arc w in G belonging faces z_1, z_2, accords an arc w^* connecting nodes $z_1^*,\ z_2^*$ in G^*.

A set of arcs $\{w_1, \dots, w_d\}$ in G accords some subgraph R^* in G^*. For its definition each arc $w_i,\ 1 \leq i \leq d$, accords a pair of faces which contain this arc. Then this pair of faces accords a pair of nodes in R^* connected by the arc w^*. Say that the graph R^* is generated by the set of arcs $\{w_1, \dots, w_d\}$.

Theorem 4 *The set of cross sections $\mathcal{L}_*$ consists of all sets of arcs $\{w_1, \dots, w_d\}$, which generate cycles with minimal length D^* in the dual graph G^* and*

$$D = D^* \leq 5.$$

This statement is a corollary of the Whithney theorem and the Euler formula [34, Theorem 1.5, Corollary of Theorem 1.6], [35].

Suppose that elements a_{ij} of the matrix A define a number of arcs which belong to $z_i \cap z_j,\ i \neq j,\ a_{ii} = 0$, in the planar graph G.

Corollary 1 *If* $\max_{1\le i,j\le n} a_{ij} > 1$ *then*

$$D = 2,\ C = \frac{1}{4}\sum_{1\le i,j\le n} a_{ij}(a_{ij}-1) \qquad (2)$$

and a complexity of constants D, C calculation by the formula (2) is squared by n. If for $i < j$ $a_{ij} > 1$ only for $j = n$ then this complexity is linear.

Define c_i the number of cycles with length i, $i = 3, 4, 5$, in G^*. Assume that all cycles $u_1 \to u_2 \to \ldots \to u_k \to u_1$ which consist of same set of nodes $\{u_1, \ldots, u_k\}$ and differ by an initial node u_1 and by a direction of a bypass coincide. Elements of power A^l, $l > 1$, of a matrix A denote by $a_{ij}^{(l)}$.

Corollary 2 *If* $\max_{1\le i,j\le n} a_{ij} = 1$ *then*

$$D = \min(i:\ c_i > 0,\ i = 3, 4, 5),\ C = c_D, \qquad (3)$$

$$c_3 = \frac{1}{6}trA^3,\ c_4 = \frac{1}{8}\left(trA^4 - 2m - 2\sum_{1\le i\ne j\le n} a_{ij}^{(2)}\right),$$

$$c_5 = \frac{1}{8}\left(trA^5 - 5trA^3 - 5\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} - 2\right)a_{ii}^{(3)}\right).$$

Complexity of the constants D, C calculation using the formula (3) is cubic by n.

The formulas of c_3, c_4, c_5 calculation are obtained in [36], see also [37, Formulas (16), (17)].

Corollary 3 *In general case the following formulas are true:*

$$D = \min(i:\ c_i > 0,\ 1 \le i \le 5),\ C = c_D, \qquad (4)$$

Consider a connected graph G' which consists of plane faces in three dimensional space. Suppose that each pair of faces has not joint points or has joint node or has joint arc and each arc belongs at least to two faces. Take a set of arcs $\{w_i,\ 1 \le i \le d\}$ from G' and confront each arc w_i a pair of faces z_i, z^i which contain this arc. Then the set of arcs $\{w_1, \ldots, w_d\}$ accords some (non unique) graph Γ_d with the nodes z_i, z^i, $1 \le i \le d$, and arcs $\{w_1, \ldots, w_d\}$ which connect these nodes.

Theorem 5 *If the graph Γ_d is acyclic then the set of arcs $\{w_1, \ldots, w_d\}$ which generates it is not cross section in G'.*

Corollary 4 *Suppose that the set $\mathcal{L}'$ of arcs sets which generate cycles with minimal length D^* and which are cross sections in G' is not empty. Then $D = D^*$, $\mathcal{L}_* = \mathcal{L}'$.*

Example 1 *In fig. 1 there are examples of planar graphs with representatives of cross sections from the set $\mathcal{L}_*$:*

1) an integer rectangle with a length M and a width N ($\mathcal{L}_*$ consists of arcs pairs connected with angle nodes),

2) a honeycomb structure ($\mathcal{L}_*$ consists of all possible pairs of arcs which belong to internal and external faces simultaneously),

3) a tube which is constructed by a gluing of opposite sides (with a length M) of integer rectangle ($\mathcal{L}_*$ consists of arcs triplets which have common butt node) if $N > 3$.

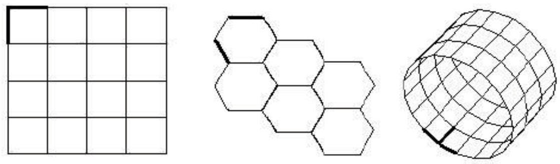

Fig. 1. Planar graphs with cross sections dedicated by bold type.

Example 2 *In fig. 2 there are graphs with examples of their cross sections from the set $\mathcal{L}_*$:*

1) a graph constructed from integer rectangle by a gluing of pairs of its opposite sides ($\mathcal{L}_*$ consists of arcs quads which have common node),

2) a graph constructed from unit cubes with integer coordinates of their nodes ($\mathcal{L}_*$ consists of arcs triplets which contain a cube node, in this node the cube does not intersect or has only common node with another cube).

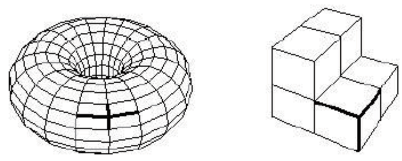

Fig. 2. Graph G' with dedicated cross sections.

2.3 Numerical experiment

Calculate the disconnection probability of honeycomb structure (fig. 1 in center) using theorem 1 and Corollary 1 and by the Monte-Carlo method with 10^6 realizations. Failure probability of each arc is 0.005. Results of calculations are represented in the table. Time of calculations by asymptotic method is few seconds and by the Monte-Carlo method is some hours.

Size structure	Asymptotic method	Monte-Carlo method	Relative error
2×2	0.00045	0.000439	2.4 %
3×3	0.00055	0.000526	4.3 %
3×4	0.00060	0.000579	3.5 %
3×5	0.00065	0.000621	4.5 %
4×4	0.00065	0.000619	4.8 %
5×5	0.00075	0.000732	2.4 %

The authors thanks A.S. Losev for a realization of a numerical experiment.

3 Stochastic and deterministic characteristics of recursively defined networks

In this section, we led to the idea of constructing recursive definitions of networks, calculating different characteristics by recursive formulas and deriving linear bounds for the calculated number of arithmetic operations of our algorithms. We attempt such an approach for calculations of reliability and the solution of a salesman problem. This suggested approach may also lead to recursive formulas for the calculation of other characteristics of networks connected with physical applications, rather than with reliability theory or the theory of transportation networks.

3.1 Definitions

Suppose that $\Gamma = \{U, W\}$ is a non oriented graph with a finite set of nodes U, a finite set of arcs W and dedicated initial and final nodes $u, v \in U$. Each arc $w \in W$ is characterized by a set of positive numbers: a probability of work $p_w,\ 0 < p_w < 1$, a length d_w and a weight (ability to handle) s_w. Let us introduce the following characteristics of the graph Γ:
1) its number of nodes $l(\Gamma)$ and number of arcs $m(\Gamma)$;
2) a probability $P_\Gamma = P_\Gamma(p_w,\ w \in W)$ that there is a path from u to v;
3) a probability $P'_\Gamma = P'_\Gamma(p_w,\ w \in W)$ that there is a path from u to v through all nodes of the graph;
4) a probability $P''_\Gamma = P''_\Gamma(p_w,\ w \in W)$ of a closed path through all nodes of the graph Γ;
5) the length of a shortest path $D_\Gamma{=}D_\Gamma(d_w, w \in W)$ from u to v of the graph Γ;
6) the length of a shortest path $D'_\Gamma{=}D'_\Gamma(d_w, w \in W)$ from u to v through all nodes of the graph Γ;
7) the length $D''_\Gamma = D''_\Gamma(d_w, w \in W)$ of a shortest closed path through all nodes of the graph Γ;
8) the minimal weight for cross sections $S_\Gamma = S_\Gamma(s_w,\ w \in W)$ from u to v of the graph Γ;
9) the number of arithmetic operations $n(P_\Gamma), n(P'_\Gamma)$, $n(P''_\Gamma)$, $n(D_\Gamma)$, $n(D'_\Gamma)$, $n(D''_\Gamma)$, $n(S_\Gamma)$ necessary to calculate P_Γ, P'_Γ, P''_Γ, D_Γ, D'_Γ, D''_Γ, S_Γ appropriately. If in the graph Γ, there is not a path connecting u, v (through all graph nodes) then

$$P_\Gamma = 0,\ D_\Gamma = \infty,\ S_\Gamma = 0\ (P'_\Gamma = 0,\ D'_\Gamma = \infty).$$

3.2 Ports constructed by a replacement of arcs

Suppose that $\mathcal{B}_*$ is a family of ports Γ with nonintersecting sets of nodes. Define a class $\mathcal{B}$ of ports with a set of generators $\mathcal{B}_*,\ \mathcal{B}_* \subset \mathcal{B}$ by the following condition. If a port $\Gamma = \{U, W\} \in \mathcal{B}_*$ with $W = \{w_1, \dots, w_m\}$, and ports $\Gamma_1 = \{U_1, W_1\}, \dots, \Gamma_m = \{U_m, W_m\} \in \mathcal{B}$ with $U_1 \cap \dots \cap U_m = \emptyset$, then a port $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m)$ constructed from Γ by replacing arcs $w_1, \dots, w_m$ by ports $\Gamma_1, \dots, \Gamma_m$, also belongs to the class $\mathcal{B}$.

Algorithms for calculating the reliability P, the length of a shortest path D, the minimal weight S for ports from the class $\mathcal{B}$ are based on the recursive formulas

$$P_{\Gamma'} = P_\Gamma(P_{\Gamma_1}, \dots, P_{\Gamma_m}),\ D_{\Gamma'} = D_\Gamma(D_{\Gamma_1}, ..., D_{\Gamma_m}),$$

$$S_{\Gamma'} = S_\Gamma(S_{\Gamma_1}, ..., S_{\Gamma_m}). \quad (5)$$

A calculation of the complexity of these algorithms is defined by the following statement.

Theorem 6 *Suppose that* $\inf_{\Gamma \in \mathcal{B}_*} m(\Gamma) > 1$. *If arcs of* Γ *work independently and* $\sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma) < \infty$ *then*

$$n(P_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma),\ \Gamma \in \mathcal{B}. \quad (6)$$

Analogous statements are true for $n(P'_\Gamma)$, $n(D_\Gamma)$, $n(D'_\Gamma)$, $n(S_\Gamma)$.

From this theorem we see that to calculate the probabilities P_Γ, P'_Γ linear numbers of arithmetic operations by $n(P_\Gamma),\ n(P'_\Gamma)$ are necessary. Note that for ports of general type, $n(D_\Gamma)$ increases as a square of $m(\Gamma)$ while $n(S_\Gamma)$ increases as a cube of $m(\Gamma)$, $n(P_\Gamma),\ n(P'_\Gamma)$ and $n(D'_\Gamma)$ increase as geometric progressions of $m(\Gamma)$ [9, 10, 42].

3.3 Networks constructed by clustering of nodes

A salesman problem. Suppose that $\mathcal{D}_*$ is a family of networks Γ with nonintersecting sets of arcs. Define recursively a class of networks $\mathcal{D},\ \mathcal{D}_* \subset \mathcal{D}$ by the following condition. If for a pair of networks $\Gamma_1 = \{U_1, W_1\} \in \mathcal{D},\ \Gamma_2 = \{U_2, W_2\} \in \mathcal{D}_*, W_1 \cap W_2 =$

$\emptyset,\ U_1 \cap U_2 = \{z\}$ (with a single node z) then $\Gamma_1 \cup \Gamma_2 \in \mathcal{D}$.

Define recursively a number $k(\Gamma),\ \Gamma \in \mathcal{D}$:

$$k(\Gamma) = \begin{cases} 1,\ \Gamma \in \mathcal{D}_*, \\ k(\Gamma_1) + k(\Gamma_2),\ \Gamma = \Gamma_1 \cup \Gamma_2, \\ \Gamma_1 \in \mathcal{D},\ \Gamma_2 \in \mathcal{D}_*. \end{cases}$$

It is clear that $k(\Gamma) \le l(\Gamma)$.

Algorithms of the reliability P'' and calculations of the length of a shortest path D'' for networks from the class $\mathcal{D}$ are based on the recursive formulas

$$P''_{\Gamma_1\cup\Gamma_2} = P''_{\Gamma_1}P''_{\Gamma_2},\ D''_{\Gamma_1\cup\Gamma_2} = D''_{\Gamma_1} + D''_{\Gamma_2}. \quad (7)$$

A calculation of the complexity of these algorithms is defined by the following statement.

Theorem 7 *Suppose that* $\Gamma_1, ..., \Gamma_l$ *is a finite family of networks with nonintersecting sets of arcs. If* $\mathcal{D}_*$ *consists of* $\Gamma_1, ..., \Gamma_l$ *independent copies, then for any* $\Gamma \in \mathcal{D}$:

$$n(P''_\Gamma) \le k(\Gamma) + \sum_{i=1}^{l} n(P''_{\Gamma_i}),$$

$$n(D''_\Gamma) \le k(\Gamma) + \sum_{i=1}^{l} n(D''_{\Gamma_i}). \quad (8)$$

This analog of the salesman problem has a linear solution complexity depending on the number of nodes $l(\Gamma)$.

A problem of Floid and Steinberg. In the problem considered in [43] complete families of

$$\{D_\Gamma,\ u,v \in U,\ u \ne v\},\ \{S_\Gamma,\ u,v \in U,\ u \ne v\}$$

but not their elements are calculated. In this paper families

$$\{D_\Gamma,\ u,v \in U,\ u \ne v\},\ \{S_\Gamma,\ u,v \in U,\ u \ne v\},$$

$$\{P_\Gamma,\ u,v \in U,\ u \ne v\}$$

are calculated on the basis of recursive formulas: suppose that $\Gamma' \in \mathcal{D},\ \Gamma'' \in \mathcal{D}_*,\ U' \cap U'' = \{z\}$, then

$$D_{\Gamma'\cup\Gamma''} = \begin{cases} D_{\Gamma'},\ u,v \in U', \\ D_{\Gamma''},\ u,v \in U'', \\ D_{\Gamma'} + D_{\Gamma''},\ u \in U',\ v \in U'', \end{cases}$$

$$S_{\Gamma'\cup\Gamma''} = \begin{cases} S_{\Gamma'},\ u,v \in U', \\ S_{\Gamma''},\ u,v \in U'', \\ \min(S_{\Gamma'}, S_{\Gamma''}),\ u \in U',\ v \in U'', \end{cases}$$

$$P_{\Gamma'\cup\Gamma''} = \begin{cases} P_{\Gamma'},\ u,v \in U', \\ P_{\Gamma''},\ u,v \in U'', \\ P_{\Gamma'}P_{\Gamma''},\ u \in U',\ v \in U''. \end{cases} \quad (9)$$

In the last equalities which recursively define D, S, P, the quantities $D_{\Gamma'},\ S_{\Gamma'},\ P_{\Gamma'}$ characterize connections between nodes u, z, while the quantities $D_{\Gamma''},\ S_{\Gamma''},\ P_{\Gamma''}$ characterize connections between nodes z, v. A calculation of the complexity of algorithms based on recursive formulas (9) is defined by the following statement.

Theorem 8 *Under the conditions of theorem 7 for any* $\Gamma \in \mathcal{D}$:

$$\begin{aligned} \tfrac{l(\Gamma)(l(\Gamma)-1)}{2} &\le \sum_{u,v\in U,\ u\ne v} n(D_\Gamma) \\ &\le \tfrac{l(\Gamma)(l(\Gamma)-1)}{2} + \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\ne v} n(D_{\Gamma_i}), \end{aligned} \quad (10)$$

Analogous inequalities are true for $n(S_{\Gamma_i}),\ n(P_{\Gamma_i})$.

From this theorem we obtain

$$\begin{aligned} &\lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\ne v} n(D_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} \\ &= \lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\ne v} n(S_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} \\ &= \lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\ne v} n(P_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} = 1. \end{aligned} \quad (11)$$

Hence, asymptotically for $l(\Gamma) \to \infty$, to calculate the length of the shortest path D or a minimal weight S, or a reliability P for a single pair of initial and final nodes, only a single arithmetic operation is necessary.

4 Probability characteristics of connectivity components numbers in parallel-sequential connections

In this section recursive formulas for generating functions, first and second moments of numbers of connectivity components in parallel-sequential connections of arcs with different failure probabilities are proved. These proofs are based on an introduction of two dimensional random vector which consists of an indicator of connection between initial and final nodes and a number of connectivity components in parallel-sequential connection.

Limit theorems for numbers of connectivity components in parallel-sequential connections with identical arcs are obtained. These theorems are based on recursive formulas for numbers of connectivity components and the law of large numbers and the central limit theorem for discrete Markov chain.

4.1 Connections with non identical arcs

Define recursively the class $\mathcal{A}$ of parallel-sequential connections of ports. Suppose that there is enumerable set $\mathcal{A}_1$ is of arcs w (generating set) and $\mathcal{A}_1 \subset \mathcal{A}$. If the ports $g_1, g_2 \in \mathcal{A}$ with arcs sets W_1, W_2, accordingly and $W_1 \cap W_2 = \emptyset$ then their sequential $g_1 \to g_2$ and parallel $g_1 \parallel g_2$ connections belong to the class $\mathcal{A}$, and arcs sets in these connections coincide with $W_1 \cup W_2$. Initial node in the port $g_1 \to g_2$ (in the port $g_1 \parallel g_2$) coincides with initial node in the port g_1 (with a gluing of initial nodes of the ports g_1, g_2), and final node coincides with final node in g_2 (with a gluing of final nodes ing_1, g_2).

Denote by α_g random variable which equals one if in random realization of the port g initial and final nodes are connected and zero if these nodes are not connected. Consider random variable η_g which characterizes a number of connectivity components in its random realization (if an arc of a graph fails its nodes conserves in this graph). Then for $g_1,\ g_2 \in \mathcal{A}$, $W_1 \cap W_2 = \emptyset$, the following formulas are true:

$$\alpha_{g_1 \to g_2} = \alpha_{g_1} \wedge \alpha_{g_2},\ \alpha_{g_1 \| g_2} = \alpha_{g_1} \vee \alpha_{g_2},$$

$$\eta_{g_1 \to g_2} = \eta_{g_1} + \eta_{g_2} - 1, \quad (12)$$

$$\eta_{g_1 \| g_2} = \eta_{g_1} + \eta_{g_2} - 2 + \alpha_{g_1} \wedge \alpha_{g_2}, \quad (13)$$

$$P(\alpha_{g_1 \to g_2} = 1) = P(\alpha_{g_1} = 1) P(\alpha_{g_2} = 1),$$

$$P(\alpha_{g_1 \| g_2} = 0) = P(\alpha_{g_1} = 0) P(\alpha_{g_2} = 0). \quad (14)$$

Assume that

$$p_w = P(\alpha_w = 1),\ q_w = 1 - p_w,\ w \in \mathcal{A}_1,$$

consequently for $w \in \mathcal{A}_1$

$$P((\alpha_w, \eta_w) = (1,1)) = p_w,$$

$$P((\alpha_w, \eta_w) = (0,2)) = q_w. \quad (15)$$

Theorem 9 *If $g_1, g_2 \in \mathcal{A},\ W_1 \bigcap W_2 = \emptyset$, and random vectors $(\alpha_{g_1}, \eta_{g_1})$ and $(\alpha_{g_2}, \eta_{g_2})$ are independent then*

$$M\left(z^{\eta_w} | \alpha_w = a\right) = z^{2-a},\ w \in \mathcal{A}_1,$$

$$M\left(z^{\eta_{g_1 \to g_2}} | \alpha_{g_1 \to g_2} = a\right) =$$

$$= \sum_{a_1, a_2:\ a_1 \wedge a_2 = a} \frac{P(\alpha_{g_1} = a_1) P(\alpha_{g_2} = a_2)}{P(\alpha_{g_1 \to g_2} = a)} \times$$

$$\times M\left(z^{\eta_{g_1}} | \alpha_{g_1} = a_1\right) M\left(z^{\eta_{g_2}} | \alpha_{g_2} = a_2\right) z^{-1}, \quad (16)$$

$$M\left(z^{\eta_{g_1 \| g_2}} | \alpha_{g_1 \| g_2} = a\right) =$$

$$= \sum_{a_1, a_2:\ a_1 \vee a_2 = a} \frac{P(\alpha_{g_1} = a_1) P(\alpha_{g_2} = a_2)}{z^{2 - a_1 \wedge a_2} P(\alpha_{g_1 \| g_2} = a)} \times$$

$$\times M\left(z^{\eta_{g_1}} | \alpha_{g_1} = a_1\right) M\left(z^{\eta_{g_2}} | \alpha_{g_2} = a_2\right), \quad (17)$$

and

$$M z^{\eta_g} = \sum_{a=0,1} P(\alpha_g = a) M\left(z^{\eta_g} | \alpha_g = a\right),\ g \in \mathcal{A}.$$

Using the equalities (16), (17) and the formulas

$$P(\eta_g = 0 | \alpha_g = a) \equiv 0,\ M\left(z^{\eta_g} | \alpha_g = a\right) =$$

$$= \sum_{k \geq 1} z^k P(\eta_g = k | \alpha_g = a),$$

obtain the following statement.

Theorem 10 *If $g_1, g_2 \in \mathcal{A},\ W_1 \bigcap W_2 = \emptyset$, then*

$$P(\eta_w{=}1 | \alpha_w{=}1) = P(\eta_w{=}2 | \alpha_w{=}0) = 1,\ w \in \mathcal{A}_1,$$

$$P(\eta_{g_1 \to g_2} = k | \alpha_{g_1 \to g_2} = a)$$

$$= \sum_{a_1, a_2:\ a_1 \wedge a_2 = a} \sum_{j=1}^{k} \frac{P(\alpha_{g_1}{=}a_1) P(\alpha_{g_2}{=}a_2)}{P(\alpha_{g_1 \to g_2}{=}a)} \times$$

$$P(\eta_{g_1} = j | \alpha_{g_1} = a_1) P(\eta_{g_2} = k - j + 1 | \alpha_{g_2} = a_2),$$

$$P(\eta_{g_1 \| g_2} = k | \alpha_{g_1 \| g_2} = a)$$

$$= \sum_{a_1, a_2:\ a_1 \vee a_2 = a} \sum_{j=1}^{k+1-a_1 \wedge a_2} \frac{P(\alpha_{g_1}{=}a_1) P(\alpha_{g_2}{=}a_2)}{P(\alpha_{g_1 \| g_2}{=}a)} \times$$

$$\times P(\eta_{g_1}{=}j | \alpha_{g_1}{=}a_1) P(\eta_{g_2}{=}k{-}j{+}2{-}a_1 | \alpha_{g_2}{=}a_2 \wedge a_2).$$

Corollary 5 *Suppose that $Q(g) = P((\eta_g, \alpha_g) = (1,1)) = P(\eta_g = 1)$ is connectivity probability of the graph g and $S(g) = P((\eta_g, \alpha_g) = (2,0))$. From the theorem 10, the equalities (14) and the complete probability formula obtain:*

$$Q(w) = p_w,\ S(w) = 1 - p_w,\ w \in \mathcal{A}_1,$$

and for $g_1, g_2 \in \mathcal{A},\ W_1 \bigcap W_2 = \emptyset$,

$$Q(g_1 \to g_2) {=} Q(g_1) Q(g_2),\ S(g_1 \parallel g_2) {=} S(g_1) S(g_2),$$

$$S(g_1 \to g_2) = Q(g_1) S(g_2) + S(g_1) Q(g_2),$$

$$Q(g_1 \parallel g_2) = Q(g_1) S(g_2) {+} S(g_1) Q(g_2) {+} Q(g_1) Q(g_2).$$

From the theorem 10 obtain the following statement.

Theorem 11 *If $g_1, g_2 \in \mathcal{A},\ W_1 \bigcap W_2 = \emptyset$, then*

$$M\left(\eta_{g_1 \to g_2} | \alpha_{g_1 \to g_2} = a\right) =$$

$$= \sum_{a_1, a_2:\ a_1 \wedge a_2 = a} \frac{P(\alpha_{g_1} = a_1) P(\alpha_{g_2} = a_2)}{P(\alpha_{g_1 \to g_2} = a)} \times$$

$$\times(A_1 + A_2 - 1),$$

$$M\left(\eta_{g_1\|g_2}|\alpha_{g_1\|g_2} = a\right) =$$

$$= \sum_{a_1,a_2:\ a_1\vee a_2=a} \frac{P(\alpha_{g_1} = a_1)P(\alpha_{g_2} = a_2)}{P(\alpha_{g_1\|g_2} = a)}\times$$

$$\times(A_1 + A_2 - 2 + a_1 \wedge a_2),$$

$$M\left(\eta^2_{g_1\to g_2}|\alpha_{g_1\to g_2} = a\right) =$$

$$= \sum_{a_1,a_2:\ a_1\wedge a_2=a} \frac{P(\alpha_{g_1} = a_1)P(\alpha_{g_2} = a_2)}{P(\alpha_{g_1\to g_2} = a)}\times$$

$$\times(B_1 + B_2 + 1 + 2A_1A_2 - 2A_1 - 2A_2),$$

$$M\left(\eta^2_{g_1\|g_2}|\alpha_{g_1\|g_2} = a\right) =$$

$$= \sum_{a_1,a_2:\ a_1\vee a_2=a} \frac{P(\alpha_{g_1} = a_1)P(\alpha_{g_2} = a_2)}{P(\alpha_{g_1\|g_2} = a)}\times$$

$$\times[B_1 + B_2 + 4 + 2A_1A_2 - 4A_1 - 4A_2+$$

$$+(2A_1 + 2A_2 - 3)a_1 \wedge a_2],$$

where

$$A_i = M(\eta_{g_i}|\alpha_{g_i} = a_i),\ B_i = M(\eta^2_{g_i}|\alpha_{g_i} = a_i),$$

and for $g \in \mathcal{A}$

$$M\eta^k_g = \sum_{a=0,1} P(\alpha_g = a)M(\eta^k_g|\alpha_g = a),\ k = 1, 2.$$

Remark 1 *Using obtained recurrent formulas computer program for a calculation of generating functions and distributions of connectivity components numbers in parallel-sequential connections was made. A complexity of these calculations is linear by numbers of arcs in parallel-sequential connections.*

4.2 Connections with identical arcs

Consider the sequence $\mathcal{A}_n,\ n \geq 1$, of ports defined recursively by a sequential or parallel connection of new arc b_n to the port $\mathcal{A}_n$. Denote a type of connection by $||$ or $\to$, accordingly. Suppose that random variable ω_n characterizes a type of the arc b_n connection to the port $\mathcal{A}_n$ and put

$$\pi_\to = P(\omega_n =\to),\ \pi_{||} = P(\omega_n = ||) = 1 - \pi_\to,$$

$$0 < \pi_\to < 1.$$

Here random variable β_n characterizes a state of the arc b_n :

$$P(\beta_n = 1) = P(b_n \text{ working state} = p,$$

$$P(\beta_n = 0) = 1 - p = q,\ 0 < p < 1.$$

The sequences of random variables $\{\omega_n,\ n \geq 1\}$, $\{\beta_n,\ n \geq 1\}$ are independent and each of them consists of independent and identically distributed random variables.

The port $\mathcal{A}_n$ with randomly working arcs is characterized by random vector (α_n, η_n), there α_n is an indicator of a connectivity between initial and final nodes of parallel-sequential connection $\mathcal{A}_n$ and η_n is a number of connectivity components in $\mathcal{A}_n$. Introduce auxiliary random variables

$$\overrightarrow{\alpha}_{n+1} = \alpha_n \wedge \beta_n,\ \overrightarrow{\eta}_{n+1} = \eta_n + 1 - \beta_n, \quad (18)$$

$$\overline{\overline{\alpha}}_{n+1} = \alpha_n \vee \beta_n,\ \overline{\overline{\eta}}_{n+1} = \eta_n - \beta_n + \alpha_n\beta_n, \quad (19)$$

then

$$(\alpha_{n+1}, \eta_{n+1}) = I(\omega_n =\to)(\overrightarrow{\alpha}_{n+1}, \overrightarrow{\eta}_{n+1})+$$

$$+I(\omega_n =\|)(\overline{\overline{\alpha}}_{n+1}, \overline{\overline{\eta}}_{n+1}), \quad (20)$$

where $I(C)$ is an indicator of an event C.

Denote $\Delta_{n+1} = \eta_{n+1} - \eta_n$, then the sequence $X_k = (\alpha_k, \Delta_k),\ k \geq 1$, is Markov chain with the states set $\mathcal{X} = \{(i, j),\ i = 0, 1,\ j = -1, 0, 1\}$ (which consists of six states) as follows

$$(\alpha_{n+1}, \Delta_{n+1}) = I(\omega_n =\to)(\alpha_n\beta_n, 1 - \beta_n)+$$

$$+I(\omega_n =\|)(\alpha_n \vee \beta_n, -\beta_n + \alpha_n\beta_n).$$

From the equalities (18) - (20) and the conditions $0 < p < 1,\ 0 < \pi_\to < 1$, we see that Markov chain $X_k,\ k \geq 1$, states are interconnected. Consequently from the central limit theorem for discrete Markov chains with finite states set [44, chapters V, VI] there are normally distributed random vector $N(0, \mathcal{B})$ with the dimension six and with zero mean and with covariance matrix $\mathcal{B}$ and real numbers $A(x),\ x \in \mathcal{X}$, which do not depend on initial state X_1 so that for any real $t(x),\ x \in \mathcal{X}$,

$$P\left(\frac{N_n(x) - nA(x)}{\sqrt{n}} > t(x),\ x \in \mathcal{X}\right) \to$$

$$\to P(N(0, \mathcal{B}) > (t(x),\ x \in \mathcal{X})),\ n \to \infty. \quad (21)$$

Here $N_n(x) = \sum_{k=1}^{n} I(X_k = x)$ and the inequality $N(0, \mathcal{B}) > (t(x),\ x \in \mathcal{X})$ is defined componentwise.

Introduce auxiliary numbers $a(x),\ x \in \mathcal{X}$, by the equalities:

$$a(i, 0) = 0,\ a(i, 1) = 1,\ a(i, -1) = -1,\ i = 0, 1.$$

From the formula (21) it is simple to obtain that there is normally distributed random variable $N(0, B)$ with zero mean and with the covariance B so that for any real t

$$P\left(\frac{\sum_{x\in\mathcal{X}} a(x)(N_n(x)-nA(x))}{\sqrt{n}} > t\right) \to$$

$$\to P(N(0,B) > t),\ n\to\infty. \tag{22}$$

Using obvious equality

$$\sum_{x\in\mathcal{X}} a(x)N_n(x) = \sum_{k=1}^{n}\Delta_k = \eta_n,\ n\ge 1,$$

rewrite the formula (22) as follows

$$P\left(\frac{\eta_n - nA}{\sqrt{n}} > t\right) \to P(N(0,B) > t),\ n\to\infty,$$

$$A = \sum_{x\in\mathcal{X}} a(x)A(x). \tag{23}$$

Remark 2 *A calculation of the vector $(A(x),\ x\in\mathcal{X}))$ and especially of covariance matrix $\mathcal{B}$ in the formula (21) is sufficiently complicated procedure [44, chapters V, VI]. So to define the mean A and the covariance B we use following limit formulas*

$$A = \lim_{n\to\infty}\frac{M\eta_n}{n},\ B = \lim_{n\to\infty}\frac{D\eta_n}{n}, \tag{24}$$

which are corollaries of the formula (23) with special initial distribution of X_1.

Choose random vector $(\alpha_1,\Delta_1) = (\alpha_1,\eta_1)$, which does not depend on random sequences $\{\omega_n,\ n\ge 1\}, \{\beta_n,\ n\ge 1\}$ and satisfies the equalities

$$P((\alpha_1,\eta_1)=(1,1)) = P = \frac{\pi_{||}p}{\pi_{||}p+\pi_{\to}q},$$

$$P((\alpha_1,\eta_1)=(0,2)) = Q = 1-P, \tag{25}$$

with $P(\alpha_n=1)\equiv P,\ P(\alpha_n=0)\equiv Q$. Random sequence $\alpha_n,\ n\ge 1$, is stationary Markov chain.

Theorem 12 *The equalities*

$$A = Q\pi_{\to}q, \tag{26}$$

$$B = \pi_{\to}qQ(1-\pi_{\to}qQ+2PQ) > 0. \tag{27}$$

are true.

Remark 3 *From Remark 2 is is possible to replace the condition (25) by more natural suggestion*

$$P((\alpha_1,\eta_1)=(1,1)) = p,\ P((\alpha_1,\eta_1)=(0,2)) = q$$

so that the equalities (23), (26), (27) are true also.

5 Proofs of main statements

The proof of theorem 1. Suppose that V_L is a random event that all arcs in cross section L fail. Then

$$\overline{P} = P\left(\left(\bigcup_{L\in\mathcal{L}_*} V_L\right)\bigcup\left(\bigcup_{L\in\mathcal{L}\setminus\mathcal{L}_*} V_L\right)\right)\sim$$

$$P\left(\bigcup_{L\in\mathcal{L}_*} V_L\right),\ h\to 0.$$

As $P(V_L)=o(h^D),\ L\in\mathcal{L}\setminus\mathcal{L}_*,\ h\to 0$, so

$$P\left(\bigcup_{L\in\mathcal{L}_*} V_L\right)\sim Ch^D,\ h\to 0.$$

The proof of theorem 5. Suppose that arcs set $\{w_1,\dots,w_d\}$ from the graph G generates acyclic graph R^*. Prove that each arc $w_i,\ 1\le i\le d$, may be bypassed in G by a way which does not contain arcs of this set.

The subgraph R^* consists of trees $S_1^*,\dots,S_m^*$ which do not connect with each other. Arrange each tree $S_i^*,\ 1\le i\le m$, on a plane so that in each node z^* arcs connected with this node follow each other as their preimages on the face z if we bypass this face in some direction. Confront each tree S_i^* closed way which bypasses once all its arcs from both sides, $1\le i\le m$ (fig. 3).

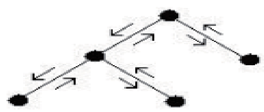

Fig. 3. Bypass of arcs in tree.

Accord ways Γ_i^* bypassing tree S_i^* arcs a closed way Γ_i, which passes in the graph G through all nodes of arcs $\{w_1,\dots,w_d\}$ which generate the tree S_i^* (fig. 4). The way Γ_i has not arcs from the set $\{w_1,\dots,w_d\}$. Consequently each arc from $\{w_1,\dots,w_d\}$ may be bypassed in G by a way which does not contain arcs from this set. So the set $\{w_1,\dots,w_d\}$ from the graph $G,\ d\le D^*$, which does not generate a cycle in G^* does not belong to the set of cross sections $\mathcal{L}_*$.

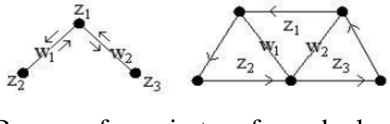

Fig. 4. Bypass of arcs in tree from dual graph and in initial graph.

The proof of theorem 6. We prove the inequality

$$n(P_\Gamma) \leq (m(\Gamma)-1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma),$$

from which all other statements of this theorem may be established analogously. From the conditions of the theorem for $\Gamma \in \mathcal{B}_*$ the inequality (6) is true. Suppose that the inequality (6) is true for $\Gamma_1, \ldots, \Gamma_m \in \mathcal{B}$ and $\Gamma' = \Gamma(\Gamma_1, \ldots, \Gamma_m)$. Then from the equations (5) and the equality $m(\Gamma') = m(\Gamma_1) + \ldots + m(\Gamma_m)$, we obtain that

$$n(P_{\Gamma'}) = n(P_{\Gamma_1}) + \ldots + n(P_{\Gamma_m}) + n(P_\Gamma),$$

$$n(P_{\Gamma'}) \leq \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma)(m(\Gamma_1)+\ldots+m(\Gamma_m)-m+1) \leq$$

$$\leq (m(\Gamma')-1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma).$$

The proof of theorem 7. We prove the first inequality in (8), the second inequality may be proved analogously. It is clear that this inequality is true for all $\Gamma \in \mathcal{D}_*$.

Suppose that $\Gamma_1 \in \mathcal{D},\ \Gamma_2 \in \mathcal{D}_*,\ R$ is a closed path through all nodes of the network $\Gamma_1 \cup \Gamma_2$ and z is its initial node. Divide R into closed paths with the initial node z, which belong entirely to Γ_1 or to Γ_2. Connect all closed paths which belong to Γ_1 and construct a closed path which passes through all nodes of Γ_1. Analogously construct a closed path which passes through all nodes of Γ_2. Then $P''_{\Gamma_1 \cup \Gamma_2} = P''_{\Gamma_1} P''_{\Gamma_2}$.

Suppose that the first inequality from (8) is true for Γ_1, then from $P''_{\Gamma_1 \cup \Gamma_2} = P''_{\Gamma_1} P''_{\Gamma_2}$

$$n(P''_{\Gamma_1 \cup \Gamma_2}) \leq \sum_{i=1}^{l} n(P''_{\Gamma_i}) + k(\Gamma_1) + 1 =$$

$$= \sum_{i=1}^{l} n(P''_{\Gamma_i}) + k(\Gamma_1 \cup \Gamma_2).$$

The proof of theorem 8. Suppose that the inequality (10) is true for Γ', then from the recursive formulas (9) and from the equality $l(\Gamma' \cup \Gamma'') = l(\Gamma') + l(\Gamma'') - 1$ we obtain

$$\sum_{u,v \in U' \cup U'',\ u \neq v} n(D_{\Gamma' \cup \Gamma''}) \leq \sum_{i=1}^{l} \sum_{u,v \in U_i,\ u \neq v} n(D_{\Gamma_i}) +$$

$$+ \frac{l(\Gamma_1)(l(\Gamma_1)-1)}{2} + \frac{l(\Gamma_2)(l(\Gamma_2)-1)}{2} +$$

$$+ (l(\Gamma_1 - 1))(l(\Gamma_2)-1) = \sum_{i=1}^{l} \sum_{u,v \in U_i,\ u \neq v} n(D_{\Gamma_i}) +$$

$$+ \frac{l(\Gamma_1 \cup \Gamma_2)(l(\Gamma_1 \cup \Gamma_2)-1)}{2}.$$

Analogous inequalities may be obtained for S, P.

The proof of theorem 9. Prove the formula (16). At first remark that for independent two dimensional discrete random vectors (defined on common probability space) (X_1, Y_1) and (X_2, Y_2) with meanings (x_{1i}, y_{1i}) and $(x_{2i}, y_{2i}),\ i = 1, 2, \ldots, I$, accordingly the formula

$$P\left(Y_1 = y_{1i},\ Y_2 = y_{2k} | X_1 = x_{1i},\ X_2 = x_{2k}\right) =$$

$$= \prod_{j=1}^{2} P\left(Y_j = y_{ji} | X_j = x_{ji}\right). \tag{28}$$

is true. Introduce the event $A = \{f(X_1, X_2) = b\}$, where f is a function of two real variable and b is some number. Then from the formula (28) and the complete probability formula we obtain

$$P\left(Y_1 = y_1,\ Y_2 = y_2,\ A\right) =$$

$$= \sum_{x_{1j}, x_{2m}:\ f(x_{1j}, x_{2m}) = b} P\left(X_1 = x_{1j}\right) P\left(Y_1 = y_1 | X_1 = x_{1j}\right) \times$$

$$\times P\left(X_2 = x_{1m}\right) P\left(Y_2 = y_2 | X_2 = x_{2m}\right).$$

Consequently we have

$$M\left(Y_1 Y_2 | A\right) = \frac{\sum\limits_{y_{1i}, y_{2k}} y_{1i} y_{2k} P\left(Y_1 = y_{1i}, Y_2 = y_{2k}, A\right)}{P(A)}$$

$$= \frac{1}{P(A)} \sum_{x_{1j}, x_{2m}:\ f(x_{1j}, x_{2m}) = b} P\left(X_1 = x_{1j}\right) P\left(X_2 = x_{1m}\right) \times$$

$$\times M\left(Y_1 | X_1 = x_{1j}\right) M\left(Y_2 | X_2 = x_{2m}\right).$$

From this equality and the formula (12) we obtain the formula (16). Analogously from the formula (13) it is possible to prove the equality (17).

The proof of theorem 12. To define the constants A, B from (24) we construct recurrent algorithm. Denote

$$A_n = M\left(\eta_n | a_n = 1\right),\ B_n = M\left(\eta_n | a_n = 0\right),$$

$$M_n = M\eta_n = A_n P + B_n Q, \tag{29}$$

$$A'_n = M\left(\eta_n^2 | a_n = 1\right),\ B'_n = M\left(\eta_n^2 | a_n = 0\right),$$

$$M'_n = M\eta'_n = A'_n P + B'_n Q, \tag{30}$$

where

$$A_1 = 1,\ B_1 = 2,\ A'_1 = 1,\ B'_1 = 4. \tag{31}$$

Using the formulas (18) - (20), (29) and theorem 11 obtain for $n \geq 1$:

$$A_{n+1} = \frac{1}{P}\Big(A_n P \pi_{\to} p + A_n P \pi_{\|} p +$$
$$+(B_n - 1)Q\pi_{\|} p + A_n P \pi_{\|} q\Big),$$
$$B_{n+1} = \frac{1}{Q}\left(B_n Q \pi_{\to} p + (A_n + 1)P\pi_{\to} q +\right.$$
$$\left.+(B_n + 1)Q\pi_{\to} q + B_n Q \pi_{\|} q\right),$$
$$M_{n+1} = M_n - Q\pi_{\|} p + P\pi_{\to} q + Q\pi_{\to} q =$$
$$= M_n + Q\pi_{\to} q,$$

consequently

$$M_{n+1} = M_1 + nQ\pi_{\to} q,\ n \geq 1,\ M_1 = 1 + Q. \quad (32)$$

Then from (24) we obtain the equality (26).

And

$$A_{n+1} - B_{n+1} = (A_n - B_n)\lambda - (2\pi_{\to} q + \pi_{\|} p),\ n \geq 1,$$
$$\lambda = \pi_{\|} q + \pi_{\to} p < 1, \quad (33)$$

so

$$A_{n+1} - B_{n+1} = -\left[\lambda^n + (2\pi_{\to} q + \pi_{\|} p)\frac{1-\lambda^n}{1-\lambda}\right] =$$
$$= \lambda^n Q - 1 - Q,$$
$$A_{n+1}P + B_{n+1}Q = M_{n+1},$$

consequently

$$A_{n+1} = M_{n+1} + Q[\lambda^n Q - 1 - Q],$$
$$B_{n+1} = M_{n+1} - P[\lambda^n Q - 1 - Q],\ n \geq 1. \quad (34)$$

Begin now a calculation of M'_{n+1}. Using the formulas (18) - (20), (30) and Theorem 11 obtain for $n \geq 1$:

$$A'_{n+1} = \frac{1}{P}\Big(A'_n P \pi_{\to} p + A'_n P \pi_{\|} p +$$
$$+(B'_n - 2B_n + 1)Q\pi_{\|} p + A'_n P \pi_{\|} q\Big),$$
$$B'_{n+1} = \frac{1}{Q}\left(B'_n Q \pi_{\to} p + (A'_n + 2A_n + 1)P\pi_{\to} q +\right.$$
$$\left.+(B'_n + 2B_n + 1)Q\pi_{\to} q + B'_n Q \pi_{\|} q\right),$$
$$M'_{n+1} = M'_n + 2A_n Q\pi_{\|} p + 2B_n Q(\pi_{\to} q - \pi_{\|} p) +$$
$$+\pi_{\to} q(1 + P).$$

So from (34) we obtain

$$M'_{n+1} = M'_1 + 2Q\pi_{\|} p \sum_{k=0}^{n-1} A_{k+1} +$$
$$+2Q(\pi_{\to} q - \pi_{\|} p)\sum_{k=0}^{n-1} B_{k+1} + n\pi_{\to} q(1 + P) =$$
$$= M'_1 + 2Q\pi_{\to} q \sum_{k=0}^{n-1} M_{k+1} - 2nQP(1 + Q)\pi_{\|} p +$$
$$+n\pi_{\to} q(1 + P) + 2\pi_{\to} q P^2 Q \frac{1-\lambda^n}{1-\lambda} =$$
$$= M'_1 + 2Q\pi_{\to} q\left(n(1 + Q) + \frac{\pi_{\to} q Q n(n-1)}{2}\right) -$$
$$-2nQP(1 + Q)\pi_{\|} p +$$
$$+n\pi_{\to} q(1 + P) + 2P^2 Q^2(1 - \lambda^n),\ M'_1 = 1 + 3Q.$$

Consequently

$$D\eta_{n+1} = M'_{n+1} - M^2_{n+1} = 2P^2 Q^2(1 - \lambda^n) + QP +$$
$$+n\pi_{\to} q[1 + P - Q^2\pi_{\to} q - 2P^2(1 + Q)].$$

Then from (24), (33) we have

$$B = \pi_{\to} q(1 + P - Q^2\pi_{\to} q - 2P^2(1 + Q)) =$$
$$= \pi_{\to} qQ(1 - \pi_{\to} qQ + 2PQ) > 0.$$

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