# Chaos Control and Synchronization of a Fractional-order Autonomous System

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*Abstract:* Liu, Liu, and Liu, in the paper "A novel three-dimensional autonomous chaos system, Chaos, Solitons and Fractals. 39 (2009) 1950-1958", introduce a novel three-dimensional autonomous chaotic system. In this paper, the fractional-order case is considered. The lowest order for the system to remain chaotic is found via numerical simulation. Stability analysis of the fractional-order system is studied using the fractional Routh-Hurwitz criteria. Furthermore, the fractional Routh-Hurwitz conditions are used to control chaos in the proposed fractional-order system to its equilibria. Based on the fractional Routh-Hurwitz conditions and using specific choice of linear controllers, it is shown that the fractional-order autonomous system can be controlled to its equilibrium points. In addition, the synchronization of the fractional-order system and the fractional-order Liu system is studied using active control technique. Numerical results show the effectiveness of the theoretical analysis.

*Key–Words:* Fractional-order; Chaos; Predictor-correctors scheme; The fractional Routh-Hurwitz criteria; Feedback control; Synchronization; Active control

## 1 Introduction

As a 300-year-old mathematical topic, fractional calculus has always attracted the interest of many famous ancient mathematicians, including L'Hospital, Leibniz, Liouville, Riemann, Grunwald, and Letnikov [1]. Although it has a long history, it was not applied in our real life because it seems to be more difficult and fewer theories have been established than for classical differential equations. In recent decades, fractional-order differential equations have been found to be a powerful tool in more and more fields, such as physics, chemistry, biology, economics, and other complex systems [2, 3]. The interest in the study of fractional-order nonlinear systems lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties, which are not taken into account in the classical integer-order models. Studying dynamics in fractional-order nonlinear systems has become an interesting topic and the fractional calculus is playing a more and more important role for analysis of the nonlinear dynamical systems. Some work has been done in the field, and the chaos and control in fractional-order systems have been studied, including Lorenz system [4], Chua system [5], Chen system [6], Rossler system [7], Newton-Leipnic system [8], Liu system [9] and so on.

In 2009, Liu et al. gave a novel three-dimensional autonomous chaotic system [10], which is not different from Liu system in Ref. [11]. In this paper, the dynamical behaviors and feedback control to steady state of the fractional-order autonomous system are studied. The lowest order for the system to remain chaotic is found via numerical simulation. Furthermore, we are going to use the fractional Routh-Hurwitz conditions given in [12, 13, 14] to study the stability conditions for the fractional-order system, and the conditions for linear feedback control are obtained as well. Moreover, the synchronization of the fractional-order autonomous system and the fractional-order Liu system is also studied using the active control method in Ref. [15]. This paper is organized as follows: Section 2 gives the integer-order autonomous system. Section 3 presents the fractional derivatives and the lowest order for the fractional-order autonomous system to remain chaotic. Section 4 gives stability analysis of the fractional-order chaotic autonomous system. Section 5 presents chaos control of the fractional-order autonomous system, and the numerical simulation is also given. In Section 6, the active control technique is applied to synchronize the novel fractional-order autonomous system and the fractional-order Liu system [9]. Finally, in Section 7, concluding comments are given.

## 2 The integer-order chaotic autonomous system

Ref. [10] reported a three-dimensional autonomous system which relies on two multipliers and one quadratic term to introduce the nonlinearity necessary for folding trajectories. The chaotic attractor obtained from the new system according to the detailed numerical as well as theoretical analysis is also the butterfly shaped attractor, exhibiting the abundant and complex chaotic dynamics. This chaotic system is a new attractor which is similar to Lorenz chaotic attractor. The chaotic system is described by the following nonlinear integer-order differential equations, called system (1):

$$\begin{cases} \frac{dx}{dt} = -ax - ey^2 \\ \frac{dy}{dt} = by + kxz \\ \frac{dz}{dt} = -cz - mxy \end{cases} \tag{1}$$

Liu et al. had studied and analyzed its forming mechanism. The compound structure of the butterfly attractor obtained by merging together two simple attractors after performing one mirror operation was explored.

When $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$, $m = 2$, the system (1) has three real equilibria $E_1(0, 0, 0)$, $E_2(-1.250, -1.118, -0.559)$ and $E_3(-1.250, 1.118, 0.559)$. The chaotic attractor and the equilibria are show in Fig.1, and the initial value of the system is selected as $(0.2, 0, 0.5)$,as in Ref. [10].

At equilibrium $E_1(0, 0, 0)$, the Jacobian matrix of system (1) is given by

$$J_1 = \begin{pmatrix} -a & -2ey & 0 \\ kz & b & kx \\ -my & -mx & -c \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \tag{2}$$

These eigenvalues of the Jacobian matrix computed at equilibrium $E_1(0, 0, 0)$ are given by

$$\lambda_1 = -1, \lambda_2 = 2.5, \lambda_3 = -5$$

Here $\lambda_2$ is a positive real number, $\lambda_1$ and $\lambda_3$ are two negative real numbers. Therefore, the equilibrium $E_1(0, 0, 0)$ is a saddle point. So, this equilibrium point is unstable.

At equilibrium $E_2(-1.250, -1.118, -0.559)$, the Jacobian matrix of system (1) is equal to

$$J_2 = \begin{pmatrix} -1 & 2.236 & 0 \\ -2.236 & 2.5 & -5 \\ 2.236 & 2.5 & -5 \end{pmatrix} \tag{3}$$

These eigenvalues of the Jacobian matrix computed at equilibrium $E_2(-1.250, -1.118, -0.559)$ are given by

$$\lambda_1 = -4.38776, \lambda_2 = 0.443881 + 3.34638j$$

$$\lambda_3 = 0.443881 - 3.34638j$$

Here $\lambda_1$ is a negative real number, $\lambda_2$ and $\lambda_3$ are a pair of complex conjugate eigenvalues with positive real parts. Therefore, the equilibrium $E_2(-1.250, -1.118, -0.559)$ is a saddle-focus point. So, this equilibrium point $E_2(-1.250, -1.118, -0.559)$ is unstable.

At equilibrium $E_3(-1.250, 1.118, 0.559)$, the Jacobian matrix of system (1) is equal to

$$J_3 = \begin{pmatrix} -1 & -2.236 & 0 \\ 2.236 & 2.5 & -5 \\ -2.236 & 2.5 & -5 \end{pmatrix} \tag{4}$$

These eigenvalues of the Jacobian matrix computed at equilibrium $E_3(-1.250, 1.118, 0.559)$ are given by

$$\lambda_1 = -4.38776, \lambda_2 = 0.443881 + 3.34638j$$

$$\lambda_3 = 0.443881 - 3.34638j$$

Here $\lambda_1$ is a negative real number, $\lambda_2$ and $\lambda_3$ are a pair of complex conjugate eigenvalues with positive real parts. Therefore, the equilibrium $E_3(-1.250, 1.118, 0.559)$ is also a saddle-focus point, and the equilibrium point $E_3(-1.250, 1.118, 0.559)$ is also unstable.

For dynamical system (1), we can obtain

$$\nabla V = \frac{\partial}{\partial x}(\frac{dx}{dt}) + \frac{\partial}{\partial y}(\frac{dy}{dt}) + \frac{\partial}{\partial z}(\frac{dz}{dt}) = -a+b-c = p \tag{5}$$

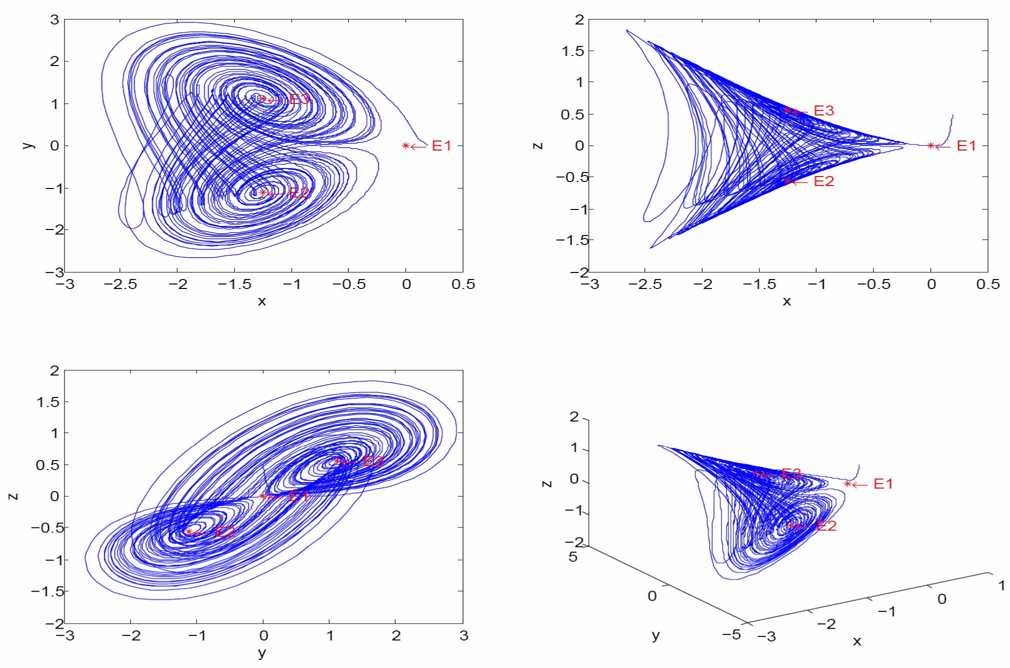


Figure 1: The chaotic attractor and the equilibria of system (1).

With $p = -a + b - c = -3.5$, here $p$ is a negative constant, dynamical system described by (1) is one dissipative system, and an exponential contraction of the system (1) is

$$\frac{dV}{dt} = e^{pt} = e^{-3.5t} \tag{6}$$

In dynamical system (1), a volume element $V_0$ is apparently contracted by the flow into a volume element $V_0 e^{pt} = V_0 e^{-3.5t}$ in time $t$. It means that each volume containing the trajectory of this dynamical system shrinks to zero as $t \to +\infty$ at an exponential rate $p$. So, all this dynamical system orbits are eventually confined to a specific subset that have zero volume, the asymptotic motion settles onto an attractor of the system (1) [16].

## 3 Fractional derivatives and chaos in the fractional-order chaotic autonomous system

### 3.1 Fractional derivatives and the fractional-order autonomous system

There are several definitions for the fractional-order differential operator, but the following definition is most used:

$$D^{\alpha} y(x) = J^{m-\alpha} y^{(m)}(x), \alpha > 0 \tag{7}$$

Where $m = \lceil \alpha \rceil$, i.e., $m$ is the first integer which is not less than $\alpha$, $y^{(m)}$ is the general $m$-order derivative, $J^{\beta}$ is the $\beta$-order Riemann-Liouville integral operator [17], which is expressed as follows:

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} z(t) dt, \beta > 0 \tag{8}$$

The operator $D^{\alpha}$ is generally called "$\alpha$-order Caputo differential operator". If the initial value $a = 0$, $D^{\alpha}$ is denoted by $D_*^{\alpha}$.

The fractional-order autonomous system is described by the following nonlinear fractional-order differential equations, called system (9):

$$\begin{cases} \frac{d^{q_1} x}{dt^{q_1}} = -ax - ey^2 \\ \frac{d^{q_2} y}{dt^{q_2}} = by + kxz \\ \frac{d^{q_3} z}{dt^{q_3}} = -cz - mxy \end{cases} \tag{9}$$

where the fractional differential operator is the Caputo differential operator; i.e., $d^{q_i} x/dt^{q_i} = D^{q_i}$, $i = 1,2,3$, $0 < q_1, q_2, q_3 \le 1$, and its order is denoted by $q = (q_1, q_2, q_3)$, $D^q = (d^{q_1} x/dt^{q_1}, d^{q_2} x/dt^{q_2}, d^{q_3} x/dt^{q_3})$ here. For comparing with the integer-order system (1), we also let $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$ and $m = 2$, with initial state $(0.2, 0, 0.5)$.

### 3.2 Chaos in the fractional-order autonomous system

Consider the following fractional-order differential equation:

$$\begin{aligned} D^q y(x) &= f(t, y(t)), 0 \le t \le T \\ y^{(k)}(0) &= y_0^{(k)} k = 0, 1, \cdots, m-1, m = \lceil q \rceil \end{aligned} \tag{10}$$

It is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds \tag{11}$$

Diethelm et al. have given a predictor-correctors scheme [18, 19], based on the Adams-Bashforth-Moulton algorithm to integrate Eq.(11). By applying this scheme to the fractional-order autonomous system, and setting $h = T/N$, $t_n = nh$, $n = 0, 1, \cdots, N \in Z^+$, system (9) can be discredited as follows:

$$x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-ax_{n+1}^p - ey_{n+1}^p y_{n+1}^p)$$

$$+\frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^{n} \alpha_{1,j,n+1}(-ax_j - ey_j y_j)$$

$$y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}(by_{n+1}^p + kx_{n+1}^p z_{n+1}^p)$$

$$+\frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^{n} \alpha_{2,j,n+1}(by_j + kx_j z_j)$$

$$z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}(-cz_{n+1}^p - mx_{n+1}^p y_{n+1}^p)$$

$$+\frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^{n} \alpha_{3,j,n+1}(-cz_j - mx_j y_j)$$

where

$$x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} \beta_{1,j,n+1}(-ax_j - ey_j y_j)$$

$$y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} \beta_{2,j,n+1}(by_j + kx_j z_j)$$

$$z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^{n} \beta_{3,j,n+1}(-cz_j - mx_j y_j)$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n - n^{q_i})(n+1)^{q_i}, & j = 0 \\ \begin{array}{l}(n-j+2)^{q_i+1} \\ +(n-j)^{q_i+1} \\ -2(n-j+1)^{q_i+1},\end{array} & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i}((n-j+2)^{q_i} + (n-j)^{q_i}), 0 \le j \le n.$$

The error estimate of the above scheme is $\max_{j=0,1,\cdots,N}\{|y(t_j) - y_h(t_j)|\} = O(h^p)$, in which $p = \min(2, 1+q_i)$ and $q_i > 0$, $i = 1, 2, 3$.

Using predictor-correctors scheme, when $q_1 = q_2 = q_3 = \alpha$, the simulation results demonstrate that the lowest order for the system (9) to remain chaotic is $\alpha = 0.91$. When $\alpha = 0.89$, $\alpha = 0.90$ and $\alpha = 0.91$, the $x - y$ phase portrait is show in Fig.2 to Fig.4.

## 4 Stability analysis of the fractional-order chaotic autonomous system

### 4.1 Fractional-order Routh-Hurwitz conditions

Consider a three-dimensional fractional-order system

$$\begin{cases} \frac{d^{q_1}x(t)}{dt^{q_1}} = f(x, y, z) \\ \frac{d^{q_2}y(t)}{dt^{q_2}} = g(x, y, z) \\ \frac{d^{q_3}z(t)}{dt^{q_3}} = h(x, y, z) \end{cases} \tag{12}$$

Where $q_1, q_2, q_3 \in (0, 1)$. The Jacobian matrix of the system (12) at the equilibrium points is

$$\begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} \tag{13}$$

The eigenvalues equation of the equilibrium point is given by the following polynomial:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{14}$$

$a_1$, $a_2$ and $a_3$ is the coefficients of the polynomial and its discriminant $D(P)$ is given as:

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2 \tag{15}$$

**Lemma 1** *If the eigenvalues of the Jacobian matrix (13) satisfy*

$$|\arg(\lambda)| \ge \pi\alpha/2, \alpha = \max(q_1, q_2, q_3) \tag{16}$$

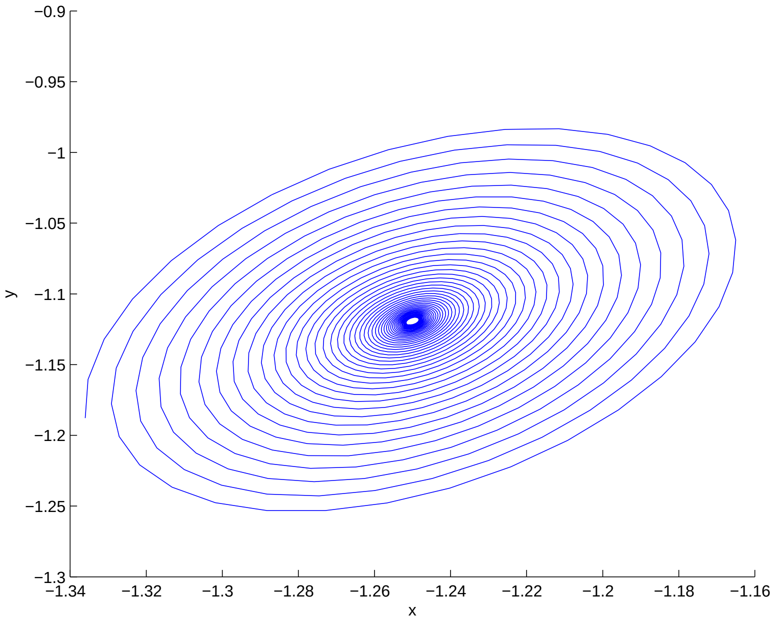


Figure 2: The $x-y$ phase portrait of system (9) when $\alpha = 0.89$.

*i.e. all the roots of the polynomial Eq.(14) satisfy Eq.(16), then the system is asymptotically stable at the equilibrium points.*

The stability region of the fractional-order system is illustrated in Fig.5, in which $\sigma$, $\omega$ refer to the real and imaginary parts of the eigenvalues, respectively. It is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case. Using the results of Ref. [13, 19], we have the following fractional Routh-Hurwitz conditions:

(i) If $D(P) > 0$, then the necessary and sufficient condition for the equilibrium point to be locally asymptotically stable, is $a_1 > 0$, $a_3 > 0$, $a_1a_2 - a_3 > 0$.

(ii) If $D(P) < 0$, $a_1 \geq 0$, $a_2 \geq 0$, $a_3 > 0$, then the equilibrium point is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0$, $a_1 < 0$, $a_2 < 0$, $\alpha > 2/3$, then all roots of Eq.(14) satisfy the condition .

(iii) If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 - a_3 = 0$, then the equilibrium point is locally asymptotically stable for all $\alpha \in [0, 1)$.

(iv) The necessary condition for the equilibrium point, to be locally asymptotically stable, is $a_3 > 0$. The proof for above conditions can be seen in detail from Ref. [12, 13].

### 4.2 The stability of equilibrium points

We assume $q_1 = q_2 = q_3 = \alpha$, and the fractional system (9) has the same equilibria $E_1(0, 0, 0)$, $E_2(-1.25, -1.118, -0.559)$ and $E_3(-1.25, 1.118, 0.559)$ as integer system (1). When $\alpha \geq 0.91$system (9) exhibits chaotic behavior, and the attractor and the equilibria are shown in Fig.6, when $\alpha = 0.91$.

If the equilibrium point is $E(x*, y*, z*)$, the Jacobian matrix of system (9) is

$$J = \begin{pmatrix} -a & -2ey & 0 \\ kz & b & kx \\ -my & -mx & -c \end{pmatrix}$$

The characteristic equation of the Jacobian matrix $J$ is

$$\lambda^3+(a-b+c)\lambda^2+(-ab-bc+ac+kmx*^2+2eky*z*)\lambda$$
$$+(-abc+akmx*^2-2ekmx*y*^2+2ceky*z*) = 0 \quad (17)$$

**Theorem 2** *For the parameters $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$ and $m = 2$, the equilibrium point $E_1$ of system.(9) is unstable for any $\alpha \in (0, 1)$ .*

**Proof:** When $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$ and $m = 2$, substituting coordinate of $E_1$ into Eq.(17), we can get the characteristic polynomial as follows

$$\lambda^3 + 3.5\lambda^2 - 10\lambda - 12.5 = 0 \quad (18)$$

The eigenvalues of Eq.(18) is $\lambda_1 = -5$, $\lambda_2 = -1$, $\lambda_3 = 2.5$. Here $\lambda_3$ is a positive real number. Therefore, according to Lemma 1 the equilibrium point $E_1$ is unstable for any $\alpha \in (0, 1)$. □

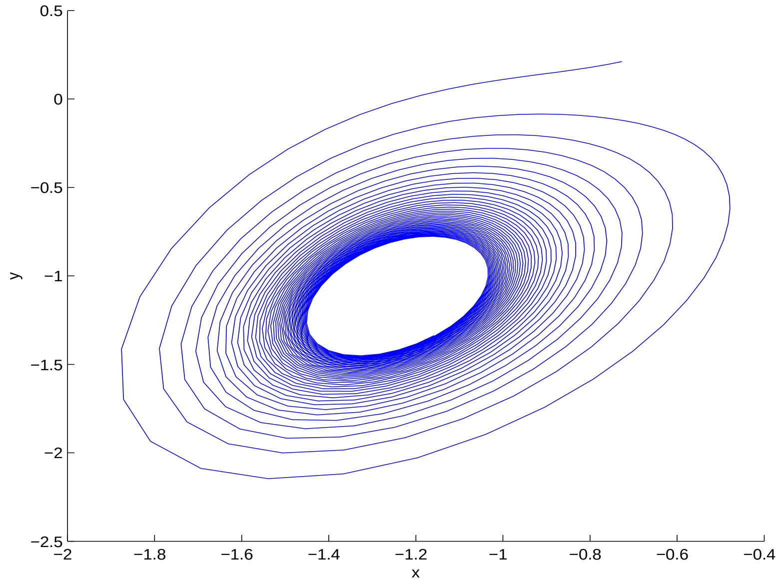


Figure 3: The $x-y$ phase portrait of system (9) when $\alpha = 0.90$.

**Theorem 3** *When the parameters $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$ and $m = 2$, if $\alpha < 0.91$, then equilibrium points $E_2$ and $E_3$ of system (9) are stable.*

**Proof:** When $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$ and $m = 2$, substituting coordinate of $E_2$ or $E_3$ into Eq.(17), we can get the characteristic polynomial as follows

$$\lambda^3 + 3.5\lambda^2 + 7.49998\lambda + 49.9997 = 0 \quad (19)$$

The eigenvalues of Eq.(18) is $\lambda_1 = -4.388$, $\lambda_2 = 0.444 - 3.346j$, $\lambda_3 = 0.444 + 3.346j$. $\lambda_1 = -4.388$ is a negative real number, and the arguments of $\lambda_2 = 0.444 - 3.346j$, $\lambda_3 = 0.444 + 3.346j$ are -1.43892 and 1.43892, respectively. $|\arg(\lambda_{2,3})| = 1.43892 > \pi\alpha/2 = 1.42942$. Therefore, according to Lemma 1, if $\alpha < 0.91$, then system (9) is stable at equilibrium points $E_2$ and $E_3$, although the real part of the eigenvalues is positive. □

# 5 Chaos control of the fractional-order autonomous system

## 5.1 Chaos control of the fractional-order autonomous system

The controlled fractional-order autonomous system is given by:

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -ax - ey^2 - k_1(x - x*) \\ \frac{d^{q_2}y}{dt^{q_2}} = by + kxz - k_2(y - y*) \\ \frac{d^{q_3}z}{dt^{q_3}} = -cz - mxy - k_3(z - z*) \end{cases} \quad (20)$$

where $E(x*, y*, z*)$ represents an arbitrary equilibrium point of system (9). The parameters $k_1$, $k_2$, $k_3 \geq 0$ are feedback control gains which can make the eigenvalues of the linearized equation of the controlled system (20) satisfy one of the above-mentioned Routh-Hurwitz conditions, then the trajectory of the controlled system (20) asymptotically approaches the unstable equilibrium point $E(x*, y*, z*)$.

Substituting the coordinate of $E(x*, y*, z*)$ into system (20), we get the Jacobian matrix refers to $E(x*, y*, z*)$ as follows

$$J = \begin{pmatrix} -k_1 - a & -2ey* & 0 \\ kz* & -k_2 + b & kx* \\ -my* & -mx* & -k_3 - c \end{pmatrix} \quad (21)$$

and the corresponding characteristic equation is

$$\begin{aligned} &P(\lambda) = \lambda^3 + (a - b + c + k_1 + k_2 + k_3)\lambda^2 \\ &+(-ab - bc + ac - bk_1 + ck_1 + ak_2 + ck_2 + k_1k_2 \\ &+ak_3 - bk_3 + k_1k_3 + k_2k_3 + kmx*^2 + 2eky * z*)\lambda \\ &+(-abc - bck_1 + ack_2 + ck_1k_2 - abk_3 - bk_1k_3 \\ &+ak_2k_3 + k_1k_2k_3 + akmx*^2 + kk_1mx*^2 \\ &-2ekmx * y*^2 + 2ceky * z * + 2ekk_3y * z*) = 0 \end{aligned} \quad (22)$$

and its discriminant is

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2$$

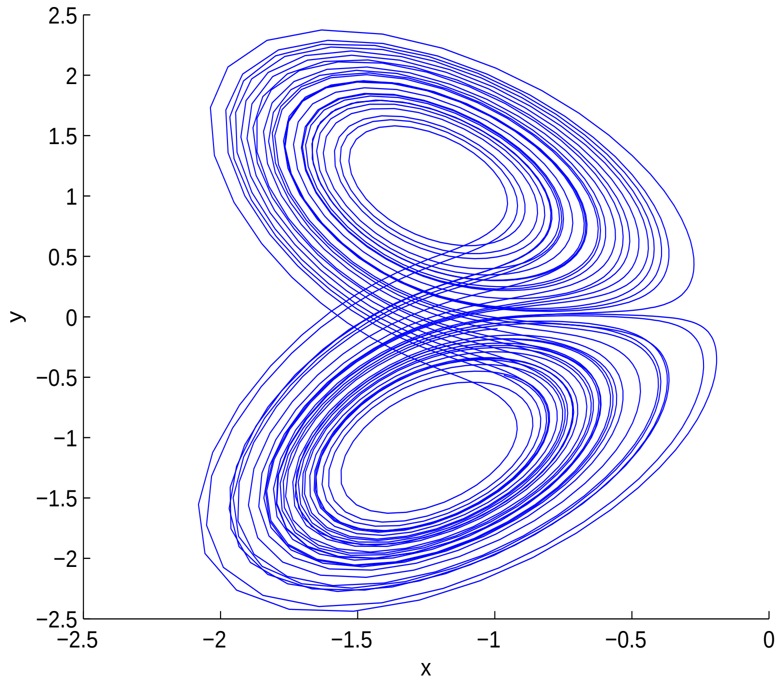


Figure 4: The $x-y$ phase portrait of system (9) when $\alpha=0.91$.

where

$$\begin{cases} a_1 = a-b+c+k_1+k_2+k_3 \\ \\ a_2 = -ab-bc+ac-bk_1+ck_1+ak_2+ck_2+k_1k_2 \\ +ak_3-bk_3+k_1k_3+k_2k_3+kmx*^2+2eky*z* \\ \\ a_3 = -abc-bck_1+ack_2+ck_1k_2-abk_3-bk_1k_3 \\ +ak_2k_3+k_1k_2k_3+akmx*^2+kk_1mx*^2 \\ -2ekmx*y*^2+2ceky*z*+2ekk_3y*z* \end{cases} \tag{23}$$

### 5.2 Some conditions for stabilizing the equilibrium point

When $a=1$, $b=2.5$, $c=5$, $e=1$, $k=4$ and $m=2$, because of the complexity of the condition, we only consider the situation $k_1=k_2=k_3=\beta$. When $\beta>2.5$, we have $D(P)>0$, $a_1>0$, $a_3>0$, $a_1a_2>a_3$ at the equilibrium point $E_1$, and all the eigenvalues are real and located in the left half of the s-plane. So Routh-Hurwitz conditions are the necessary and sufficient conditions for the fulfillment of Lemma 1.

When $a=1$, $b=2.5$, $c=5$, $e=1$, $k=4$, $m=2$ and $k_1=k_2=k_3=\beta\in R$, we can obtain $D(P)<0$, $a_1>0$, $a_2>0$, $a_3>0$, then the equilibrium point $E_2$ and $E_3$ is locally asymptotically stable for any $\alpha<2/3$.

### 5.3 Simulation results

The parameters are chosen as $\alpha=0.92$, to ensure the existence of chaos in the absence of control. The initial state is taken as $(0.2,0,0.5)$, the time step is $0.1(S)$ , and the control is active when $t\geq 80(s)$ in order to make a comparison between the behavior before activation of control and after it.

We simulate the process of which system (20) stabilizes to the equilibrium point $E_1(0,0,0)$. When $k_1=k_2=k_3=\beta=3$, we have $D(P)>0$, $a_1>0$, $a_3>0$ and $a_1a_2>a_3$, and the three roots are $\lambda_1=-8$, $\lambda_2=-4$, $\lambda_3=-0.5$, so system (20) is asymptotically stable at $E_1(0,0,0)$, Fig.7 displays the simulation result.

We have also simulated the process of which system (20) stabilizes to equilibrium points $E_2(-1.250,-1.118,-0.559)$ and $E_3(-1.250,1.118,0.559)$, using the feedback control method. When $k_1=k_2=k_3=\beta=0.1$, $\alpha=0.6$, we have $D(P)<0$, $a_1\geq 0$, $a_2\geq 0$, $a_3>0$ and the roots of $P(\lambda)=0$ are $\lambda_1=-4.488$, $\lambda_2=0.344+3.346j$, $\lambda_3=0.344-3.346j$. In this case, the integer-order system is unstable because the real part of $\lambda_2$ and $\lambda_3$ is positive, however, the arguments of the right half-plane eigenvalues are $84.1°$ , indicating that the eigenvalues are located in the stable region of the fractional-order autonomous system as shown in Fig.5, and the system's cone makes an angle $54°$. Hence,

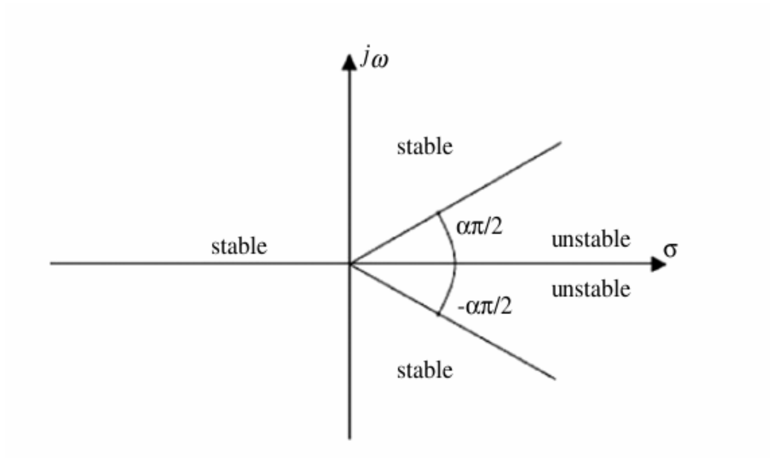


Figure 5: Stability region of the fractional-order system.

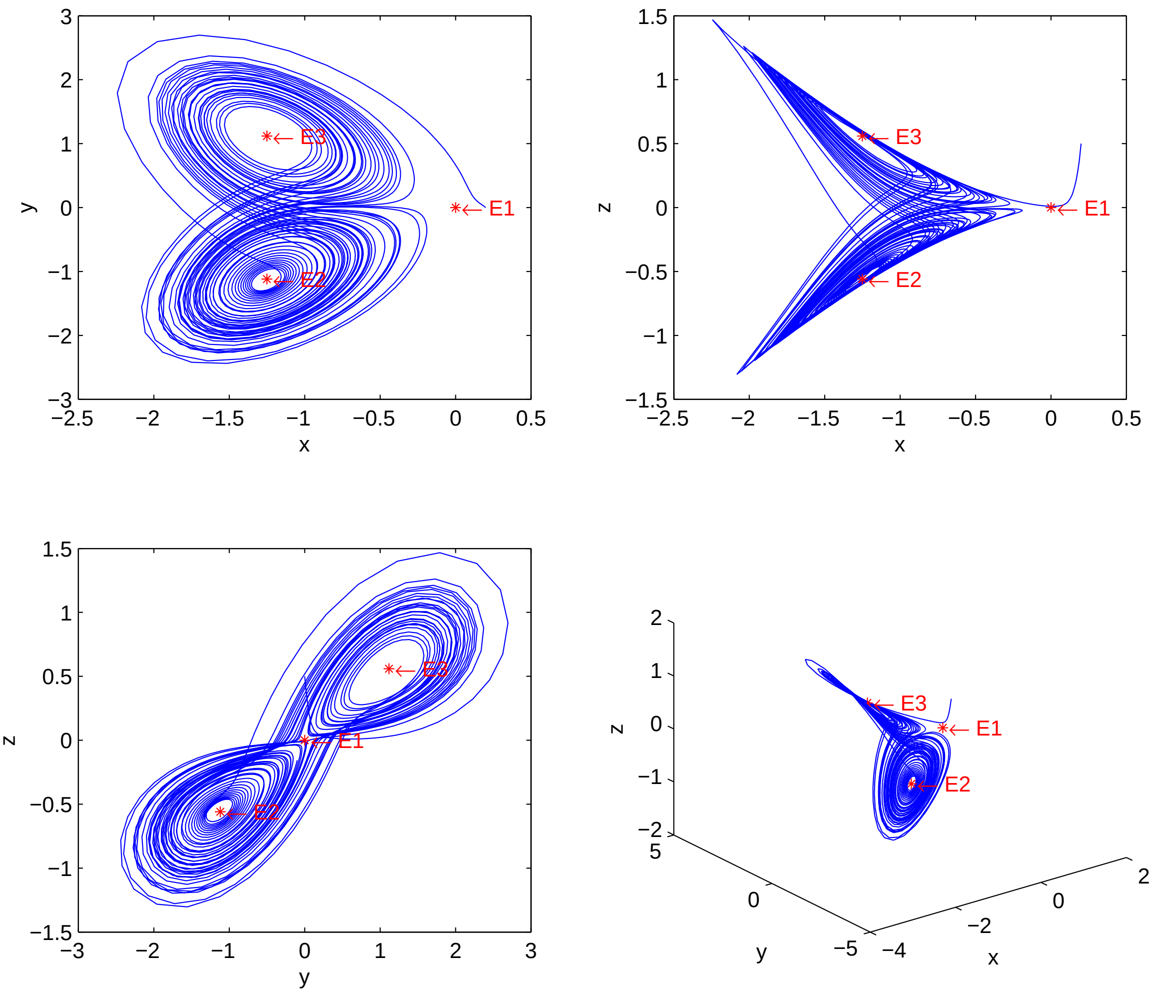


Figure 6: The chaotic attractor and the equilibria of system (9).

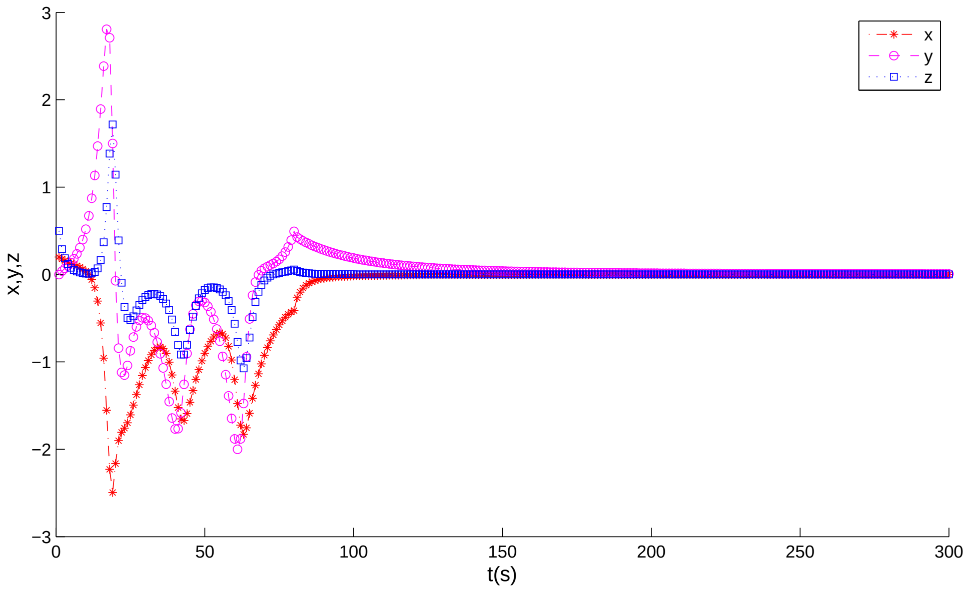


Figure 7: Stabilizing the equilibrium point $E_1$.

the controlled fractional-order system (20) is asymptotically stable at $E_2$ and $E_3$, as shown in Fig.8. and Fig.9.

## 6 Synchronization between two different fractional-order systems

The synchronization of chaotic fractional-order systems have attracted increasing attention, because of its application in secure communication and control processing [20, 21]. In this section, we focus on investigating two different systems: the novel fractional-order autonomous system and the fractional-order Liu system, using the active control technique in Ref. [?].

Let the novel fractional-order autonomous system (9) be the drive system:

$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}} = -ax_1 - ey_1^2 \\ \frac{d^{q_2}y_1}{dt^{q_2}} = by_1 + kx_1z_1 \\ \frac{d^{q_3}z_1}{dt^{q_3}} = -cz_1 - mx_1y_1 \end{cases} \tag{24}$$

and the response system is the fractional-order Liu system, which given as

$$\begin{cases} \frac{d^{q_1}x_2}{dt^{q_1}} = \alpha(y_2 - x_2) + u_1 \\ \frac{d^{q_2}y_2}{dt^{q_2}} = \beta x_2 - \theta x_2 z_2 + u_2 \\ \frac{d^{q_3}z_2}{dt^{q_3}} = -\delta z + hx_2^2 + u_3 \end{cases} \tag{25}$$

Where $U = (u_1, u_2, u_3)^T$ is the active control function. Here, the aim is to determinate the controller $U$ which is required for the drive system (24) to synchronize with the response system (25). For this purpose, we define the error states between (24) and (25) as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$. The error dynamic system can be expressed by

$$\begin{cases} \frac{d^{q_1}e_1}{dt^{q_1}} = \alpha(e_2 - e_1) + (a - \alpha)x_1 + \alpha y_1 + ey_1^2 + u_1 \\ \frac{d^{q_2}y_2}{dt^{q_2}} = \beta e_1 + \beta x_1 - by_1 - \theta x_2 z_2 - kx_1z_1 + u_2 \\ \frac{d^{q_3}z_2}{dt^{q_3}} = -\delta e_3 + (c - \delta)z_1 + hx_2^2 + mx_1y_1 + u_3 \end{cases} \tag{26}$$

Define the active control function $U = (u_1, u_2, u_3)^T$ as follows:

$$\begin{cases} u_1 = -(a - \alpha)x_1 - \alpha y_1 - ey_1^2 + V_1(t) \\ u_2 = -\beta x_1 + by_1 + \theta x_2 z_2 + kx_1z_1 + V_2(t) \\ u_3 = -(c - \delta)z_1 - hx_2^2 - mx_1y_1 + V_3(t) \end{cases} \tag{27}$$

and $V_1(t)$, $V_2(t)$ and $V_3(t)$ are the control inputs. Substituting (27) into (26) gives

$$\begin{cases} \frac{d^{q_1}e_1}{dt^{q_1}} = \alpha(e_2 - e_1) + V_1(t) \\ \frac{d^{q_2}y_2}{dt^{q_2}} = \beta e_1 + V_2(t) \\ \frac{d^{q_3}z_2}{dt^{q_3}} = -\delta e_3 + V_3(t) \end{cases} \tag{28}$$

The error system (28) is a linear system with the control input function $[V_1, V_2, V_3]^T = H[e_1, e_2, e_3]^T$, where $H$ is a $3 \times 3$ matrix. Let $J$ denote the Jacobi matrix of system (28) at equilibrium point $e_i = 0 (i = 1, 2, 3)$.

$$J = H + \begin{pmatrix} -\alpha & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix}$$

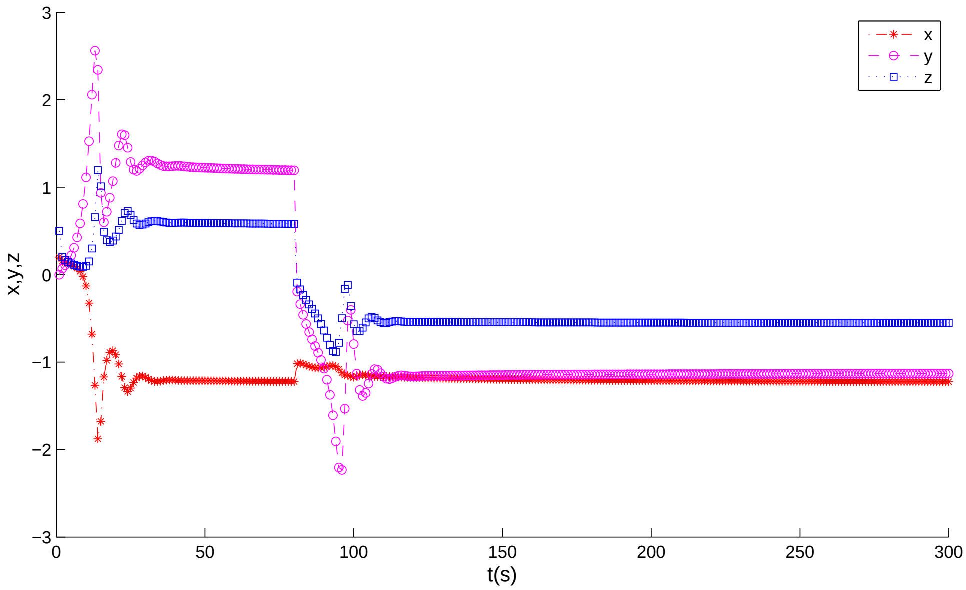


Figure 8: Stabilizing the equilibrium point $E_2$.

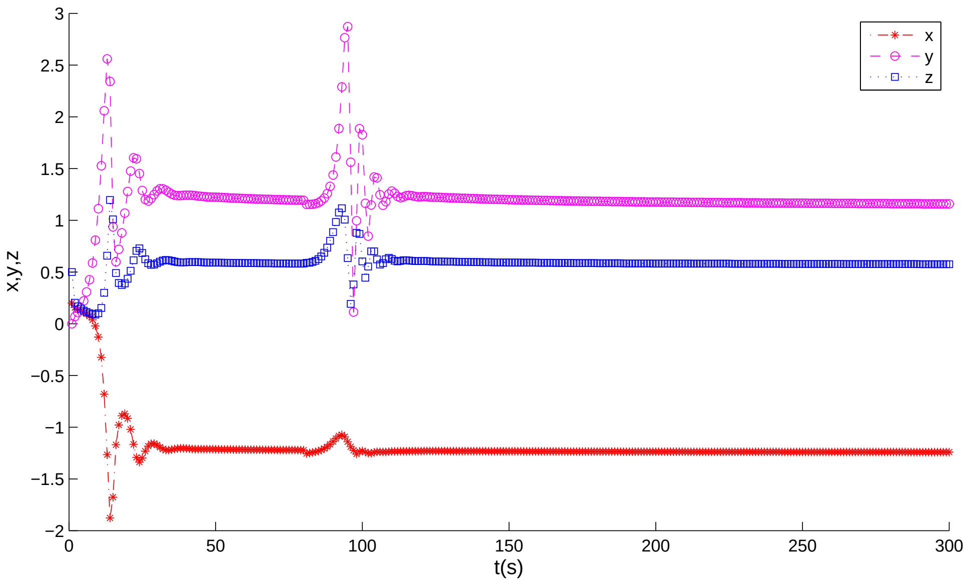


Figure 9: Stabilizing the equilibrium point $E_3$.

We can choose the suitable matrix $H$, and let the eigenvalues of $J$ satisfy the Lemma 1:$|\arg(\lambda)| \geq \pi\alpha/2$,$\alpha = \max(q_1, q_2, q_3)$. Thus synchronization between the fractional-order drive system (24) and the fractional-order response system (25) is achieved. For example, we can choose

$$H = \begin{pmatrix} \alpha - 1 & -\alpha & 0 \\ -\beta & -1 & 0 \\ 0 & 0 & \delta - 1 \end{pmatrix}.$$

In the following numerical simulation, the parameters are chosen as $a = 1$, $b = 2.5$, $c = 5$, $e = 1$, $k = 4$, $m = 2$, $\alpha = 10$, $\beta = 40$, $\delta = 2.5$, $\theta = 1$ and $h = 4$. The fractional order is set as $q_1 = q_2 = q_3 = 0.98$, and the initial states of the drive system (24) and the response system (25) are taken as $(0.2, 0, 0.5)$ and $(2.2, 2.4, 38)$ respectively. Simulation results of the synchronization between the fractional systems (24) and (25) are displayed in Fig.10.From Fig.10, we can find the novel autonomous system and Liu system are globally synchronized asymptotically for $t > 198$.

# 7 Conclusion

In this paper, chaos control of a fractional-order autonomous chaotic system is studied. Using predictor-correctors scheme the lowest order for the fractional-order autonomous system to remain chaotic is found via numerical simulation. Furthermore, we have studied the local stability of the equilibria using the fractional Routh-Hurwitz conditions. Analytical conditions for linear feedback control have been implemented, showing the effect of the fractional order on controlling chaos in this system. The synchronization of the fractional-order autonomous system and the fractional-order Liu system is also investigated using the active control method. Simulation results have illustrated the effectiveness of the proposed controlled method.

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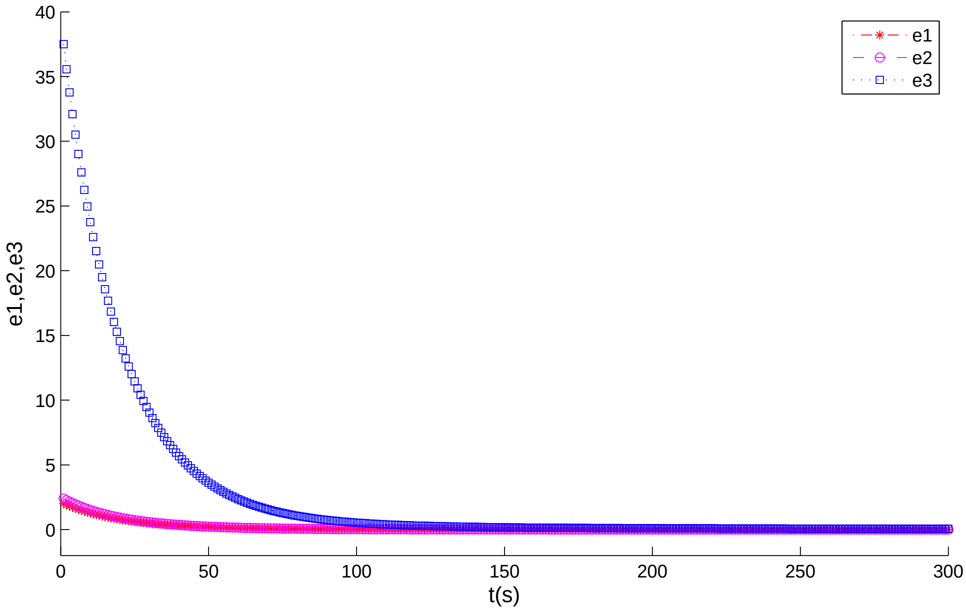


Figure 10: Simulation results of synchronization between system (24) and (25).

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