A Filled Function Algorithm for Multiobjective Optimization

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Abstract: In this paper a filled function algorithm is applied to compute one of the nonisolated Pareto optimal points of an unconstrained multiobjective optimization problem. Firstly, the original problem is converted into an equivalent global optimization problem. Subsequently, a novel filled function algorithm is presented for solving the corresponding global optimization problem. The implementation of the algorithm on several test problems is reported with numerical results.

Key–Words: Multiobjective optimization, Filled function method, Global optimization, Local minimizer, Global minimizer, Pareto optimal.

1 Introduction

In multiobjective optimization problems, several objective functions $f_1, \ldots, f_m : R^n \to R$ have to be minimized simultaneously. However, in many practical problems, there usually does not exist a point that is a simultaneous minimizer of all the objective functions. Thus the concept of *Pareto optimality* was introduced. A point is called Pareto optimal, if there does not exist a different point with the same or smaller objective function values, such that there is a decrease in at least one objective function value. This type of problem (denoted by (MOP)) has attracted much attention due to its various applications in engineering [7] (especially truss optimization [5, 22], design [7, 14, 21, 32, 33], space exploration [34]), statistics [3], management sciences [1, 9, 17, 30, 36], environmental analysis [11, 23], cancer treatment planning [18], etc.

In the literature, many methods have been proposed to find a Pareto optimal point of multiobjective optimization problems. The most notable among them are the steepest descent based methods [12], the scalarization approach [6], Newton methods [13], evolutionary algorithms [41], approximation methods [31], Levenberg-Marquardt algorithms [10], etc. To determine Pareto optimal points, a prevalent approach is to convert the multiobjective problem into a scalar optimization problem and minimize a convex combination of the different objectives (see, e.g., [6, 33]). In other words, m weights λ_i are chosen such that $\lambda_i \geq 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$ and the following problem is solved

$$\min f_\lambda(x) := \sum_{i=1}^{m} \lambda_i f_i(x). \tag{1}$$

It follows immediately that the global minimizer x^* of problem (1) is a Pareto optimal point for (MOP). But most of the methods for solving the problem (1) can only find the stationary points or local minimizers, which is not desirable. In this paper, we propose a filled function algorithm, which can find the global minimizer of the problem (1).

Over the last four decades, different approaches and algorithms have been developed for solving global optimization problems [4, 15, 16, 19, 24, 25, 26, 38, 39]. Recently, the filled function methods have attracted much attention from both the theoretical and the algorithmic points of view. The filled function algorithm is an efficient deterministic global optimization algorithm (see, e.g., [2, 15, 16, 35, 37, 40]).

The filled function was firstly introduced by Ge in [15]. The definition of the filled function is as follows.

Definition 1 *Let x_1^* be a current minimizer of $f(x)$. A function $P(x)$ is called* ***a filled function*** *of $f(x)$ at x_1^* if $P(x)$ has the following properties:*

(1) x_1^* is a maximizer of $P(x)$ and the whole basin B_1^* of $f(x)$ at x_1^* becomes a part of a hill of $P(x)$;

(2) $P(x)$ has no minimizers or saddle points in any higher basin of $f(x)$ than B_1^*;

(3) *If $f(x)$ has a lower basin than B_1^*, then there is a point x' in such a basin that minimizes $P(x)$ on the line through x and x_1^*.*

The definitions of a basin and a hill are given in [15].

The filled function given at x_1^* in the paper [15] has the following form:

$$P(x, x^*, r, \rho) = \frac{1}{r + f(x)} \exp(-\frac{\|x - x^*\|^2}{\rho^2}) \quad (2)$$

where the parameters r and ρ need to be chosen appropriately. By adopting the concept of the filled functions, a global optimization problem can be solved via a two-phase cycle. In Phase 1, we start from a given point and use any local minimization method to find a local minimizer x_1^* of $f(x)$. In Phase 2, we construct a filled function (2) at x_1^* and minimize the filled function in order to identify a point x' with that $f(x') < f(x_1^*)$. If it is identified, x' is certainly contained in a lower basin than the whole basin B_1^* of $f(x)$ at x_1^*. We can then use x' as the initial point in Phase 1 again, and hence we can find a better minimizer x_2^* of $f(x)$ with $f(x_2^*) < f(x_1^*)$. This process repeats until the minimization of a filled function could not yield a better solution. The current local minimizer will be taken as a global minimizer of $f(x)$.

However, the filled function algorithm proposed in the paper [15] has some drawbacks:

1. The efficiency of the filled function algorithm strongly depends on two parameters r and ρ. It is not easy to adjust the parameters to make them satisfy that r is small and the ration $\rho^2/[r + f(x_1^*)]$ is large.

2. When the domain is large or ρ is small, the factor $\exp(-||x-x^*||^2/\rho^2)$ will be approximately zero. Since smoothing increases with this factor, the filled function (2) will become very flat. This makes the filled function algorithm less efficient.

3. The stopping criterion is not efficient because it requires a large computational effort to find a global minimizer.

Although some other filled functions [16, 25, 26, 38] have been proposed later, all of them are still no satisfactory for global optimization due to the drawbacks above. In this paper, a new filled function with one parameter is proposed. An algorithm based on the proposed filled function for solving the multiobjective optimization problem is presented. The objective function value can be reduced by half in each iteration of our algorithm. The implementation of the algorithm on several test problems is reported with numerical results.

The rest of the paper is organized as follows. A filled function with one parameter for the global optimization problem is proposed in Section 2. Some properties of the filled function are investigated and discussed. In Section 3, an algorithm based on the proposed filled function for solving the multiobjective optimization problem is presented. Application of the filled function algorithm on several test problems is reported in Section 4 with numerical results. Finally, some conclusions are drawn in Section 5.

2 Filled Function and Its Some Properties

In the following, basic definitions and notations used in this paper are given.

The multiobjective optimization problem (MOP) is formulated as

$$\begin{array}{ll} \min & f(x) = (f_1(x), \ldots, f_m(x))^T \\ \text{s.t.} & x \in R^n, \end{array}$$

where $f : R^n \to R^m$ is composed of m real-valued objective functions, and R^n and R^m are finite-dimensional Euclidean vector spaces. For the special case $m = 2$, we refer to this problem as the biobjective problem (BOP).

Throughout the rest of the paper, we always assume that the following assumptions for (MOP) hold.

Assumption 2 $f_1, \ldots, f_m : R^n \to R$ *are continuously differentiable functions.*

Assumption 3 $f_1, \ldots, f_m$ *are coercive, i.e.,* $f_i(x) \to +\infty$ *for each* i *as* $||x|| \to +\infty$, *where* $||\cdot||$ *is the usual Euclidean norm.*

By Assumption 3, there exists a closed bounded box $X \subset R^n$ containing all Pareto optimal points of (MOP) in its interior.

In this paper, $\lambda_i (i = 1, \ldots, m)$ is defined as follows:

$$\lambda_i = \frac{f_i^{\max} - f_i(x)}{f_i^{\max} - f_i^0} \geq 0,\ i = 1, \ldots, m, \quad (3)$$

where $f_i(x)$ is the ith objective function, f_i^0 and $f_i^{\max}$ are defined as follows:

$$f_i^0 = \min_x \{f_i(x) | x \in R^n\},$$

$$f_i^{\max} = \max_{1 \leq j \leq m, i \neq j} f_i(x_j^*),$$

where x_j^* is the point that minimizes the jth objective function.

Consequently, problem (1) is equivalent to the following problem:

$$\begin{array}{ll} \min & h(x) = f_\lambda(x) \\ s.t. & x \in X, \\ & \sum_{i=1}^{m} \lambda_i = 1, \end{array} \tag{4}$$

where λ_i is defined by (3).

The following theorem shows that the global minimizer of problem (4) is a Pareto optimal point for (MOP), the proof of which can be found in [33].

Theorem 4 *If x^* is the global minimizer of problem (4), then x^* is a Pareto optimal point of (MOP).*

The set of local minimizers of problem (4) is denoted by L(P), and the set of global minimizers is denoted by G(P).

Assumption 5 *The set L(P) is nonempty and finite.*

For a given $x^* \in X$, the definition of the filled function is given as follows.

Definition 6 ([37]) *Suppose that x^* is a current local minimizer of problem (4). A continuously differentiable function $p(x, x^*)$ is called **a filled function** of problem (4) at x^*, if it satisfies the following conditions:*

(1) x^* is a strict local maximizer of $p(x, x^*)$;

(2) $\nabla p(x, x^*) \neq 0$ when $x \in S_1$, where $S_1 = \{x | h(x) \geq \frac{h(x^*)}{2}, x \in X \backslash \{x^*\}\}$;

(3) If x^* is not a global minimizer and $S_2 = \{x | h(x) < \frac{h(x^*)}{2}, x \in X\}$ is nonempty, then there exists a point $\bar{x} \in S_2$ such that $\bar{x}$ is a local minimizer of $p(x, x^*)$.

Remark: With the definition above, we know that these conditions of the filled function ensure that when a descent method, for example, the steepest descent method, is employed to minimize the constructed filled function, the sequence of iteration points will not terminate at any point where the objective function value is larger than $\frac{h(x^*)}{2}$; If x^* is not a global minimizer of problem (4), then there must exist a minimizer of the filled function at which the objective function value is less than the half of $h(x^*)$, namely any local minimizer of the filled function $p(x, x^*)$ must belong to the set $S_2 = \{x \mid h(x) < \frac{h(x^*)}{2}, x \in X\}$. Therefore the present local minimizer of the objective function escapes and a better minimizer can be found by a local search algorithm starting from the minimizer of the filled function.

In the following, a one-parameter filled function satisfying Definition 6 is introduced. To begin with, we design a continuously differentiable function $\varphi_r(t)$ with the following properties: it is equal to 1 when $t \geq r$ and increasing on R.

More specifically, for any given $r > 0$, $\varphi_r(t)$ is constructed as follows

$$\varphi_r(t) = \begin{cases} 1, & t \geq r, \\ \frac{r-2}{r^3} t^3 + \frac{3-2r}{r^2} t^2 + t, & 0 < t < r, \\ t, & t \leq 0. \end{cases} \tag{5}$$

Note that the requirement for continuous differentiability of $\varphi_r(t)$ justifies the use of cubic polynomial in constructing $\varphi_r(t)$.

Obviously, we have

$$\varphi_r'(t) = \begin{cases} 0, & t \geq r, \\ \frac{3(r-2)}{r^3} t^2 + \frac{2(3-2r)}{r^2} t + 1, & 0 < t < r, \\ 1, & t \leq 0. \end{cases} \tag{6}$$

It is easy to see that $\varphi_r(t)$ is continuously differentiable and increasing on R.

Given an $x^* \in L(P)$, the following filled function with one parameter is constructed

$$F(x, x^*, q) = \frac{1}{1 + \|x - x^*\|} + q\varphi_{\frac{h(x^*)}{4}}(h(x) - \frac{h(x^*)}{4}), \tag{7}$$

where the only parameter $q > 0$. Clearly, $F(x, x^*, q)$ is continuously differentiable on R^n.

The following theorems show that $F(x, x^*, q)$ satisfies Definition 6 when the positive parameter q is sufficiently large.

Theorem 7 *Let $x^* \in L(P)$, $q > 0$. Then x^* is a strict local maximizer of $F(x, x^*, q)$.*

Proof: Since $x^* \in L(P)$, there exists a neighborhood $N(x^*, \sigma)$ of x^* with $\sigma > 0$ such that $h(x) \geq h(x^*)$ for all $x \in N(x^*, \sigma)$, where $N(x^*, \sigma) = \{x | \|x - x^*\| < \sigma\}$. Then for any $x \in N(x^*, \sigma)$, $x \neq x^*$ and $q > 0$, we have

$$F(x, x^*, q) = \frac{1}{1 + \|x - x^*\|} + 1 < 1 + q = F(x^*, x^*, q).$$

Thus, x^* is a strict local maximizer of $F(x, x^*, q)$. □

Theorem 7 reveals that the proposed filled function satisfies condition (1) of Definition 6.

Theorem 8 *Let $x^* \in L(P)$, $q > 0$ and $\bar{x}$ be a point such that $\bar{x} \in S_1 = \{x | h(x) \geq \frac{h(x^*)}{2}, x \in X \backslash \{x^*\}\}$. Then $\nabla F(\bar{x}, x^*, q) \neq 0$ holds.*

Proof: Assume that $\bar{x} \in S_1$, namely $h(\bar{x}) \geq \frac{h(x^*)}{2}$ and $\bar{x} \neq x^*$. Then, we have

$$\nabla F(\bar{x}, x^*, q) = \frac{-(\bar{x} - x^*)}{(1 + \|\bar{x} - x^*\|)^2 \, \|\bar{x} - x^*\|},$$

$$\nabla F(\bar{x}, x^*, q) \frac{(\bar{x} - x^*)}{\|\bar{x} - x^*\|} = \frac{-1}{(1 + \|\bar{x} - x^*\|)^2} < 0.$$

It implies that when $\bar{x} \in S_1$, $\nabla F(\bar{x}, x^*, q) \neq 0$. □

Theorem 9 *Let $x^* \in L(P)$, $q > 0$. Suppose that x_1 and x_2 are two points in X such that $\|x_1 - x^*\| < \|x_2 - x^*\|$ and $\frac{h(x^*)}{2} \leq \min\{h(x_1), h(x_2)\}$. Then we have*

$$F(x_2, x^*, q) < F(x_1, x^*, q) < 1 + q = F(x^*, x^*, q).$$

Proof: Obviously, by the condition above, we have

$$F(x_1, x^*, q) = \frac{1}{1 + ||x_1 - x^*||} + q,$$

$$F(x_2, x^*, q) = \frac{1}{1 + ||x_2 - x^*||} + q.$$

So, the conclusion can be obtained immediately. □

Theorem 10 *Let $x^* \in L(P)$, $q > 0$. Suppose that $\bar{x} \in int(X)$ is a point such that $h(\bar{x}) \geq \frac{h(x^*)}{2}$, $\bar{x} \neq x^*$. Then for any small $\varepsilon_1 > 0$, there exists d_1 such that*

$$0 < ||d_1|| \leq \varepsilon_1,$$

$$\bar{x} - d_1, \bar{x} + d_1 \in X,$$

$$||\bar{x} - d_1 - x^*|| < ||\bar{x} - x^*|| < ||\bar{x} + d_1 - x^*||,$$

$$h(\bar{x} \pm d_1) \geq \frac{h(x^*)}{2},$$

$$\begin{aligned} F(\bar{x} + d_1, x^*, q) \quad &< F(\bar{x}, x^*, q) < F(\bar{x} - d_1, x^*, q) \\ &< 1 + q = F(x^*, x^*, q). \end{aligned}$$

Proof: For a given $\varepsilon_1 > 0$, let

$$d_1 = \varepsilon_2 \frac{\bar{x} - x^*}{||\bar{x} - x^*||},$$

where $0 < \varepsilon_2 \leq \varepsilon_1$. Then $0 < ||d_1|| \leq \varepsilon_1$.

Furthermore, if ε_1 is sufficiently small, then

$$\|\bar{x} + d_1 - x^*\| = (1 + \varepsilon) \|\bar{x} - x^*\| > \|\bar{x} - x^*\|,$$

$$\|\bar{x} - d_1 - x^*\| = (1 - \varepsilon) ||\bar{x} - x^*|| < \|\bar{x} - x^*\|,$$

$$h(\bar{x} \pm d_1) \geq \frac{h(x^*)}{2},$$

where $\varepsilon = \varepsilon_2 \backslash ||\bar{x} - x^*||$.

Hence, by Theorem 9,

$$\begin{aligned} F(\bar{x} + d_1, x^*, q) \quad &< F(\bar{x}, x^*, q) < F(\bar{x} - d_1, x^*, q) \\ &< 1 + q = F(x^*, x^*, q). \end{aligned}$$

□

The implication of Theorem 10 is clear: For any point $\bar{x} \in \text{int}(X)$ with $h(\bar{x}) \geq \frac{h(x^*)}{2}$ and $\bar{x} \neq x^*$, it will never be a local minimizer of $F(x, x^*, q)$.

Theorem 11 *Let $x^* \in L(P)$, $q > 0$. Then any local minimizer or saddle point of $F(x, x^*, q)$ must belong to the set $S_2 = \{x \in X | h(x) < \frac{h(x^*)}{2}\}$.*

Proof: Suppose that the theorem is not true. Then there is a local minimizer or saddle point of $F(x, x^*, q)$, x_1^*, such that $x_1^* \notin S_2$ and $h(x_1^*) \geq \frac{h(x^*)}{2}$. Since x^* is a strict local maximizer of $F(x, x^*, q)$ and x_1^* is a local minimizer or saddle point of $F(x, x^*, q)$, thus $x^* \neq x_1^*$. If x_1^* is a local minimizer of $F(x, x^*, q)$, it contradicts Theorem 10. Similarly, if x_1^* is a saddle point of $F(x, x^*, q)$, it contradicts Theorem 8. Consequently, the theorem is true. □

Condition (2) in definition 6 is satisfied for the proposed filled function $F(x, x^*, q)$ due to Theorem 8 and 10.

Theorem 12 *Let $x^* \in L(P)$, but $x^* \notin G(P)$. Then $F(x, x^*, q)$ does have a local minimizer in the region $S_2 = \{x \in X | h(x) < \frac{h(x^*)}{2}\}$ when $q > 0$ is sufficiently large.*

Proof: Since $x^* \in L(P)$, but $x^* \notin G(P)$. Without loss of generality, suppose that $S_3 = \{x \in X | h(x) \leq \frac{h(x^*)}{4}\}$ is nonempty, there exists an $\bar{x}^* \in L(P)$ such that $h(\bar{x}^*) \leq \frac{h(x^*)}{4}$. As to the case when S_3 is empty, the proof is the same. By the continuity of $h(x)$ and Assumption 3, there exists an ε small enough, it holds $h(\bar{x}^*) + \varepsilon < \frac{h(x^*)}{4}$ and

$$\{x \in X | h(x) = h(\bar{x}^*) + \varepsilon\} \subset \text{int}(X).$$

Define that

$$S^{=} = \{x \in X | h(x) = h(\bar{x}^*) + \varepsilon\} \subset \text{int}(X),$$

where for all $x \in S^{=}$, it holds

$$h(x) - \frac{h(x^*)}{4} = h(\bar{x}^*) + \varepsilon - \frac{f(x^*)}{4} < 0.$$

Therefore, for each $x \in S^{=}$, there are two cases:

(1) $\|\bar{x}^* - x^*\| \geq \|x - x^*\|$;

(2) $\|\bar{x}^* - x^*\| < \|x - x^*\|$.

For case (1), by

$$h(\bar{x}^*) - \frac{f(x^*)}{4} < h(x) - \frac{h(x^*)}{4} < 0$$

and $||\bar{x}^* - x^*|| \geq ||x - x^*||$, we have

$$\begin{aligned} F(\bar{x}^*, x^*, q) &= \tfrac{1}{1+\|\bar{x}^*-x^*\|} + q(h(\bar{x}^*) - \tfrac{h(x^*)}{4}) \\ &< \tfrac{1}{1+\|x-x^*\|} + q(h(x) - \tfrac{h(x^*)}{4}) \\ &= F(x, x^*, q). \end{aligned}$$

For case (2), $F(\bar{x}^*, x^*, q) < F(x, x^*, q)$ if and only if

$$\begin{aligned} \tfrac{1}{1+\|\bar{x}^*-x^*\|} + q(h(\bar{x}^*) - \tfrac{h(x^*)}{4}) &< \tfrac{1}{1+\|x-x^*\|} \\ &\quad + q(h(x) - \tfrac{h(x^*)}{4}), \end{aligned}$$

if and only if

$$\frac{1}{1+\|\bar{x}^*-x^*\|} - \frac{1}{1+\|x-x^*\|} < q(h(x) - h(\bar{x}^*)),$$

which is equivalent to

$$q > \frac{\|x-x^*\| - \|\bar{x}^*-x^*\|}{(1+\|x-x^*\|)(1+\|\bar{x}^*-x^*\|)(h(x)-h(\bar{x}^*))}.$$

Let

$$q_0 = \frac{\|x-x^*\| - \|\bar{x}^*-x^*\|}{(1+\|x-x^*\|)(1+\|\bar{x}^*-x^*\|)(h(x)-h(\bar{x}^*))}.$$

Thus, there exists sufficiently large $q_0 > 0$ as function $h(x)$ approaches $h(\bar{x}^*)$. Consequently, we must have that $F(\bar{x}^*, x^*, q) < F(x, x^*, q)$ for all $x \in S^=$ and $q \in (q_0, +\infty)$. Now we denote

$$S^{\leq} = \{x \in X \mid h(x) \leq h(\bar{x}^*) + \varepsilon\} \subset \text{int}(X),$$

and we have

$$\min_{x \in S^{\leq}} F(x, x^*, q) = F(x_0^*, x^*, q).$$

Since $S^=$, $S^{\leq}$ are compact sets, it is obvious that $F(x_0^*, x^*, q) \leq F(\bar{x}^*, x^*, q)$. Since

$$\min_{x \in S^{\leq}} F(x, x^*, q) = F(x_0^*, x^*, q) = \min_{x \in S^{\leq} \setminus S^=} F(x, x^*, q).$$

and $S^{\leq} \setminus S^=$ is an open set, thus $x_0^* \in S^{\leq} \setminus S^= \subset \text{int}(X)$ is a local minimizer of $F(x, x^*, q)$, we have $\nabla F(x_0^*, x^*, q) = 0$. By Theorem 8, $h(x_0^*) < \frac{h(x^*)}{2}$ holds. □

Theorem 12 clearly states that the proposed filled function satisfies condition (3) of Definition 6. Therefore three conditions of the filled function definition are satisfied by the proposed filled function.

Theorem 13 *If $x^* \in G(P)$, then $F(x, x^*, q) > 0$ for all $x \in X$.*

Proof: Since $x^* \in G(P)$, we have $h(x) \geq h(x^*)$ for all $x \in X$. Thus,

$$F(x, x^*, q) = \frac{1}{1+\|x-x^*\|} + q > 0.$$

□

Remark: In the phase of minimizing the filled function, Theorems 7-12 guarantee that the present local minimizer x^* of the objective function is escaped and the minimum of the filled function will be always achieved at a point where the objective function value is not greater than the half of the current minimum of the objective function. Moreover, the proposed filled function doesn't include exponential terms or logarithmic terms. A continuously differentiable function is used in the construction of the filled function, which possesses many good properties and is efficient in numerical implementation.

3 Filled Function Algorithm

In this section, a global optimization algorithm for solving problem (4) is presented based on the constructed filled function (7), which leads to a Pareto optimal point or an approximate Pareto optimal point for (MOP).

The general idea of the global optimization method is as follows. Let $x_0 \in X$ be a given initial point. Starting from this initial point, a local minimizer x_0^* of problem (4) is obtained with a local minimization method. The main task is to find deeper local minimizers of problem (4) if x_0^* is not a global minimizer.

Consider the following filled function problem (for short, (FFP))

$$\text{(FFP)} \quad \min_{x \in X} F(x, x_k^*, q),$$

where $F(x, x_k^*, q)$ is given by (7).

Let $\bar{x}_0^*$ be a obtained local minimizer of problem (FFP) on X. Then by theorem 12, we have $h(\bar{x}_0^*) < \frac{h(x_0^*)}{2}$ and $\bar{x}_0^* \in \text{int}(X)$. Starting from this initial point $\bar{x}_0^*$, we can obtain a local minimizer x_1^* of problem (4). If x_1^* is a global minimizer, then x_1^* is a Pareto optimal point of (MOP); Otherwise locally solve problem (FFP). Let $\bar{x}_1^*$ be the obtained local minimizer. Then we have that $h(\bar{x}_1^*) < \frac{h(x_0^*)}{2^2}$ and $\bar{x}_1^* \in \text{int}(X)$. Repeating this process, we can finally obtain a Pareto optimal point of (MOP) or a sequence $\{\bar{x}_k^*\} \subset \text{int}(X)$ with $h(\bar{x}_k^*) < \frac{h(x_0^*)}{2^k}, k = 1, 2, \ldots$. For such a sequence $\{\bar{x}_k^*\}$, $k = 1, 2, \ldots$, when k is sufficiently large, $\bar{x}_k^*$ can be regarded as an approximate Pareto optimal point of (MOP).

Let $x^* \in X$ and $\varepsilon > 0$. Then x^* is called a ε-approximate Pareto optimal point of (MOP) if $x^* \in X$ and $h(x^*) \leq \varepsilon$.

The corresponding filled function algorithm for the problem (4) is described as follows.

Algorithm 3.1

Step 0: Choose small positive numbers ε, λ, a large positive number q^U, and an initial value q_0 for the parameters q. (e.g., $\varepsilon = 10^{-8}$, $\lambda = 10^{-5}$, $q^U = 10^{15}$ and $q_0 = 10^{12}$). Choose a positive integer number K (e.g., $K = 2n$) and directions e_i, $i = 1, \ldots, K$, are the coordinate directions. Choose an initial point $x_0 \in X$. Set $k := 0$.

If $h(x_0) \leq \varepsilon$, then let $x_k^* := x_0$ and go to Step 6; Otherwise, let $q := q_0$ and go to Step 1.

Step 1: Find a local minimizer x_k^* of the problem (4) by local search methods starting from x_k. If $h(x_k^*) \leq \varepsilon$, go to Step 6.

Step 2: Let

$$\begin{aligned} F(x, x_k^*, q) &= \frac{1}{1+\|x-x_k^*\|} \\ &+ q\varphi_{\frac{h(x_k^*)}{4}}\left(h(x) - \frac{h(x_k^*)}{4}\right), \end{aligned} \tag{8}$$

where $\varphi_r(t)$ is defined by (5). Set $l = 1$ and $u = 0.1$.

Step 4: (a) If $l > K$, then set $q := 10q$, and go to Step 5; Otherwise, go to (b).

(b) If $u \geq \lambda$, then set $y_k^l := x_k^* + ue_l$, and go to (c); Otherwise, set $l := l + 1$, $u = 0.1$, go to (a).

(c) If $y_k^l \in X$, then go to (d); Otherwise, set $u := u/10$, go to (b).

(d) If $h(y_k^l) < \frac{h(x_k^*)}{2}$, then set $x_{k+1} := y_k^l$, $k := k + 1$, and go to Step 1; Otherwise, go to Step 4.

Step 4: Search for a local minimizer of the following filled function problem starting from y_k^l

$$\min_{x \in R^n} F(x, x_k^*, q). \tag{9}$$

Once a point $y_k^* \in \mathrm{int}(X)$ with $h(y_k^*) < \frac{h(x_k^*)}{2}$ is obtained in the process of searching, set $x_{k+1} := y_k^*$, $k := k + 1$ and go to Step 1; Otherwise continue the process. Let $\bar{x}_k^*$ be an obtained local minimizer of problem (9). If $\bar{x}_k^*$ satisfies $h(\bar{x}_k^*) < \frac{h(x_k^*)}{2}$ and $\bar{x}_k^* \in \mathrm{int}(X)$, then set $x_{k+1} := \bar{x}_k^*$, $k := k + 1$ and go to Step 1; Otherwise, set $u := u/10$, and go to Step (3b).

Step 5: If $q \leq q^U$, then go to Step 2.

Step 6: Let $x_s = x_k^*$ and stop.

From Theorems 7-12, it can be seen that if λ is small enough, q^U is large enough, and the direction set $\{e_1, \ldots, e_K\}$ is large enough, x_s can be obtained from Algorithm 3.1 within finite steps.

4 Numerical Experiment

In this section, five test problems using Algorithm 3.1 are solved to illustrate the efficiency of Algorithm 3.1. All the numerical experiments are implemented in Matlab2010b. In our programs, the local minimizers of problem (FFP) and problem (4) are obtained by the SQP method. $||\nabla h(x)|| \leq 10^{-8}$ is used as the terminate condition.

The number of variables n (column 1), the initial point x_0 (column 2), the number of iterations (NI, column 3), the final function values obtained ($h(x_k^*)$, column 4), the approximation Pareto optimal point of (MOP) (x_k^*, column 5) are shown in Table 1-5.

Example 1. The first problem is taken from [20] and is known as JOS. It is biobjective and f is given by

$$\begin{aligned} f_1(x) &:= \frac{1}{n}\sum_{i=1}^{n} x_i^2, \\ f_2(x) &:= \frac{1}{n}\sum_{i=1}^{n} (x_i - 2)^2. \end{aligned}$$

Both of the objective functions of JOS are strongly convex. The Pareto-optimal set of JOS is the set $\{(a, a, \ldots, a) \in R^n | a \in [0, 2]\}$. The computational results are summarized in Table 1.

Example 2. Consider the biobjective, convex MOP taken from [42] where the objective functions $f_1, f_2 : R^2 \to R$ are given by

$$\begin{aligned} f_1(x) &= x_1, \\ f_2(x) &= g(x)\left(1 - \sqrt{\frac{x_1}{g(x)}}\right), \end{aligned}$$

where $g(x) = 1 + \frac{9}{n-1}\sum_{i=2}^{n} x_i^2$, $x_1 > 0$.

The Pareto-optimal solutions are $x_1 > 0$, $x_i = 0$, $i = 2, 3, \ldots, n$. The computational results are summarized in Table 2.

Example 3. This problem is taken from [13] and is called FDS, a problem of variable dimension defined by

$$f_1(x) := \frac{1}{n^2}\sum_{k=1}^{n} k(x_k - k)^4,$$
$$f_2(x) := \exp\left(\sum_{k=1}^{n} x_k/n\right) + \|x\|_2^2,$$
$$f_3(x) := \frac{1}{n(n+1)}\sum_{k=1}^{n} k(n-k+1)e^{-x_k}.$$

These objectives have been designed as a convex problem with three criteria whose numerical difficulty is sharply increasing in the dimension n. The computational results are summarized in Table 3.

Example 4. Consider an academic nonconvex problem having three objective functions, which is taken from [8]. The objective function $f_i : R^2 \to R$ $(i = 1,2,3)$ are constructed using function $\varphi(x_1,x_2) = \psi(x_1,x_2) - \exp(-50\psi(x_1,x_2))$, where we have $\psi(x_1,x_2) = x_1^2 + x_2^2$. The actual unconstrained problem is expressed in the following form

$$f_1 = \varphi(x_1,x_2),$$
$$f_2 = \varphi(x_1,x_2-1),$$
$$f_3 = \varphi(x_1-1,x_2).$$

The computational results are summarized in Table 4.

Example 5. Consider a pollution problem of a river involving four objective functions and two variables in the following form. This problem considered has been described in [29] and also considered in [27] and briefly described in [28].

$$\begin{aligned}
f_1(x) &= -4.07 - 2.27x_1,\\
f_2(x) &= -2.60 - 0.03x_1 - 0.02x_2\\
&\quad + \tfrac{0.01}{x_1^2-1.39} + \tfrac{0.30}{x_2^2-1.39},\\
f_3(x) &= -8.21 + \tfrac{0.71}{1.09-x_1^2},\\
f_4(x) &= -0.96 + \tfrac{0.96}{1.09-x_2^2}.\\
&0.3 \le x_1, x_2 \le 1.0.
\end{aligned}$$

The computational results are summarized in Table 5.

Table 1: Numerical results of Example 1

n	x_0	NI	$h(x_k^*)$	x_k^*
2	$\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$	1	$1.0023e-16$	$\begin{pmatrix} 0.0000 \\ 0.0000 \end{pmatrix}$
5	$\begin{pmatrix} 1.5000 \\ \vdots \\ 1.5000 \end{pmatrix}$	1	$9.0908e-17$	$\begin{pmatrix} 2.0000 \\ \vdots \\ 2.0000 \end{pmatrix}$

Table 2: Numerical results of Example 2

n	x_0	NI	$h(x_k^*)$	x_k^*
2	$\begin{pmatrix} 0.1000 \\ 0.1000 \end{pmatrix}$	1	0.3914	$\begin{pmatrix} 0.1906 \\ -0.0000 \end{pmatrix}$
5	$\begin{pmatrix} 0.0000 \\ \vdots \\ 0.0000 \end{pmatrix}$	1	0.3915	$\begin{pmatrix} 0.2018 \\ 0.0000 \\ \vdots \\ 0.0175 \end{pmatrix}$

Table 3: Numerical results of Example 3

n	x_0	NI	$h(x_k^*)$	x_k^*
2	$\begin{pmatrix} 0.0000 \\ 0.0000 \end{pmatrix}$	1	$4.7699e-14$	$\begin{pmatrix} 0.0331 \\ 0.0547 \end{pmatrix}$
3	$\begin{pmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix}$	1	$2.7071e-15$	$\begin{pmatrix} 0.0017 \\ 0.0522 \\ 0.0175 \end{pmatrix}$

Table 4: Numerical results of Example 4

n	x_0	NI	$h(x_k^*)$	x_k^*
2	$\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$	1	-3.0023	$\begin{pmatrix} 0.0000 \\ 0.0000 \end{pmatrix}$

Table 5: Numerical results of Example 5

n	x_0	NI	$h(x_k^*)$	x_k^*
2	$\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$	2	-9.8050	$\begin{pmatrix} 0.8502 \\ 0.7899 \end{pmatrix}$

5 Conclusion

In this paper, the filled function $F(x, x^*, q)$ with one parameter is constructed for solving unconstrained multiobjective optimization problem and it has been proved that it satisfies the basic characteristics of the filled function definition. Promising computation results have been observed from our numerical experiments. In the future, the filled function method can be considered to solve the general constrained multiobjective optimization problem.

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