Classical theorems for a Gould type integral

ALINA GAVRILUŢ
Faculty of Mathematics
University "Al.I. Cuza" Iaşi
Romania
gavrilut@uaic.ro

ANCA CROITORU
Faculty of Mathematics
University "Al.I. Cuza" Iaşi,
Romania
croitoru@uaic.ro

Abstract: In this paper, we continue the study of the Gould type integral introduced in [30] which generalizes the results of [12, 13, 17, 28] and [29]. We obtain various classical properties, such as a mean type theorem, a Lebesgue (Fatou respectively) type theorem, Hölder and Minkowski inequalities etc. Other results concerning measurability, semi-convexity, diffusion and atoms are also established.

Key–Words: (multi)(sub)measure, semi-convex, Darboux property, diffused, atom, totally-measurable, Gould integral, Lebesgue theorem, Fatou lemma.

1 Introduction

In [20] G. G. Gould introduced an integral for bounded real functions with respect to finitely additive set functions taking values in a Banach space, integral which is more general that the Lebesgue one.

In the last years, the non-additive case and the set-valued case received a special attention because of their applications in mathematical economics, decision theory, artificial intelligence, statistics or theory of games.

A. Precupanu and A. Croitoru generalized Gould's results [20], studying in [28] a Gould type integral for multimeasures with values in $\mathcal{P}_{kc}(X)$, the family of all compact convex nonempty subsets of a real Banach space X. Also, Gould type integrals with respect to a (multi)submeasure were studied in [12]–[19]. In [30], A. Precupanu, A. Gavriluţ and A. Croitoru introduced and studied a Gould type integral for bounded real functions with respect to a set multifunction of finite variation with values in $\mathcal{P}_{bf}(X)$, the family of all bounded closed nonempty subsets of a real Banach space X.

On the other hand, notions as atoms, pseudo-atoms, Darboux property, non-atomicity (with different nonequivalent variants - see, for instance, [8, 9]), (finitely) purely atomicity, semi-convexity, diffusion were intensively studied in recent years, due to their applications in many classical measure theory problems, physics and convex analysis (see [1, 3, 4, 5, 6, 8, 9, 10, 11, 21, 23, 24, 25, 26]).

That is why, in this paper, we study these notions for the Gould type integral introduced in [30]. We prove that the Lebesgue theorem, Hölder and Minkowski inequalities, Fatou lemma have here a correspondent and our integral preserves properties like semi-convexity or diffusion. Results regarding measurability are also established.

2 Basic notions

Let $(X, \|\cdot\|)$ be a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_b(X)$ the family of all nonvoid bounded subsets of X, $\mathcal{P}_f(X)$ the family of all nonvoid closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid closed bounded subsets of X, $\mathcal{P}_{bfc}(X)$ the family of all nonvoid closed bounded convex subsets of X, $\mathcal{P}_{kc}(X)$ the family of all nonvoid compact convex subsets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$.

It is known that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where $e(M,N) = \sup\limits_{x\in M} d(x,N)$, for every $M,N \in \mathcal{P}_f(X)$ is the excess of M over N and $d(x,N)$ is the distance from x to N with respect to the distance induced by the norm of X.

We denote $|M| = h(M,\{0\}) = \sup\limits_{x\in M} \|x\|$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X.

For every $M,N \in \mathcal{P}_0(X)$ and every $\alpha \in \mathbb{R}$, let $M+N = \{x+y | x \in M, y \in N\}$ and $\alpha M = \{\alpha x | x \in M\}$. We denote by $\overline{M}$ the closure of M with respect to the topology induced by the norm of X.

On $\mathcal{P}_0(X)$ we consider the Minkowski addition "$\stackrel{\bullet}{+}$" [18], defined by:

$$M \stackrel{\bullet}{+} N = \overline{M+N}, \text{ for every } M,N \in \mathcal{P}_0(X).$$

Let T be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a ring of subsets of T.

By $i = \overline{1,n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

Some properties of h are presented in the following proposition (see Hu and Papageorgiu [22], Petruşel and Moţ [27]).

Proposition 1 *Let $A, B, C, D, A_n, B_n \in \mathcal{P}_0(X)$, for every $n \in \mathbb{N}^*$. Then:*

I) $(\alpha + \beta)A = \alpha A + \beta A$, for every $\alpha, \beta \in \mathbb{R}_+$ and convex A.

II) $(A \stackrel{\bullet}{+} B) \stackrel{\bullet}{+} C = A \stackrel{\bullet}{+} (B \stackrel{\bullet}{+} C)$.

III) $(A \stackrel{\bullet}{+} B) \stackrel{\bullet}{+} (C \stackrel{\bullet}{+} D) = (A \stackrel{\bullet}{+} C) \stackrel{\bullet}{+} (B \stackrel{\bullet}{+} D)$.

IV) $h(A, B) = h(\overline{A}, \overline{B})$.

V) $e(A, B) = 0$ if and only if $A \subseteq \overline{B}$.

VI) $h(A, B) = 0$ if and only if $\overline{A} = \overline{B}$.

VII) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$, for all $\alpha \in \mathbb{R}$.

VIII) $h(\sum_{i=1}^{n} A_i, \sum_{i=1}^{n} B_i) \le \sum_{i=1}^{n} h(A_i, B_i)$.

IX) $h(\alpha A, \beta A) \le |\alpha - \beta| \cdot |A|$, for all $\alpha, \beta \in \mathbb{R}$.

X) $h(\alpha A \stackrel{\bullet}{+} \beta B, \gamma A \stackrel{\bullet}{+} \delta B) \le |\alpha - \gamma| \cdot |A| + |\beta - \delta| \cdot |B|$, for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

XI) $h(A+C, B+C) = h(A, B)$, for every $A, B \in \mathcal{P}_{bfc}(X)$ and $C \in \mathcal{P}_b(X)$.

XII) If $A, A_n \in \mathcal{P}_b(X)$ and $\alpha, \alpha_n \in \mathbb{R}$, for every $n \in \mathbb{N}^*$, are so that $h(A_n, A) \to 0$ and $\alpha_n \to \alpha$, then $h(\alpha_n A_n, \alpha A) \to 0$.

We now recall some classical notions:

Definition 2 *A set function $m : \mathcal{C} \to \overline{\mathbb{R}}_+$, with $m(\emptyset) = 0$, is said to be:*

I) monotone if $m(A) \le m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

II) superadditive if $m(\bigcup\limits_{i \in I} A_i) \ge \sum\limits_{i \in I} m(A_i)$, for every sequence of pairwise disjoint sets $(A_i)_{i \in I} \subset \mathcal{C}$, with $\bigcup\limits_{i \in I} A_i \in \mathcal{C}$, $I \subseteq \mathbb{N}$.

III) subadditive if $m(A \cup B) \le m(A) + m(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

IV) a submeasure (in Drewnowski's sense [7]) if m is monotone and subadditive.

Example 3 *I) If $\nu : \mathcal{C} \to \mathbb{R}_+$ is a finitely additive set function, then $m : \mathcal{C} \to [0, 1]$ defined for every $A \in \mathcal{C}$ by $m(A) = \frac{\nu(A)}{1+\nu(A)}$ is a submeasure.*

II) ([8,9]) Let $m_n : \mathcal{C} \to \mathbb{R}_+$ be a submeasure for every $n \in \mathbb{N}$. Then the set function $m : \mathcal{C} \to \overline{\mathbb{R}}_+$ defined by $m(A) = \sup\limits_n m_n(A)$, for every $A \in \mathcal{C}$, is a submeasure, too.

Remark 4 *Suppose $m : \mathcal{C} \to \mathbb{R}_+$ is a submeasure of finite variation. If $\overline{m}$ denotes the variation of m on $\mathcal{P}(T)$, then:*

I) $\overline{m}$ is finitely additive on $\mathcal{C}$.

II) The following statements are equivalent:
i) m is o-continuous;
ii) m is σ-subadditive;
iii) $\overline{m}$ is σ-additive on $\mathcal{C}$;
iv) $\overline{m}$ is o-continuous on $\mathcal{C}$.

Definition 5 *For a set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, with $\mu(\emptyset) = \{0\}$, we consider:*

I) the extended real valued set function $|\mu|$ defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

II) the variation $\overline{\mu}$ of μ defined by $\overline{\mu}(A) = \sup\{\sum\limits_{i=1}^{n} |\mu(A_i)|\}$, for every $A \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$, with $A_i \subseteq A$, for every $i \in \{1, \ldots, n\}$.

μ is said to be of finite variation on $\mathcal{C}$ if $\overline{\mu}(A) < \infty$, for every $A \in \mathcal{C}$.

Definition 6 *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$. μ is said to be*

I) monotone if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

II) a multimeasure if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

III) a multisubmeasure if μ is monotone and $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$
(or, equivalently, for every $A, B \in \mathcal{C}$).

IV) h-σ-subadditive if $|\mu(\bigcup\limits_{n=1}^{\infty} A_n)| \le \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, with $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

V) null-additive if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{C}$, with $\mu(B) = \{0\}$.

VI) null–null-additive if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$.

VII) order-continuous (shortly, o-continuous) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, with $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$ (denoted by $A_n \searrow \emptyset$).

VIII) increasing convergent if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing se-

quence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$*, with* $\bigcup\limits_{n=1}^{\infty} A_n = A \in \mathcal{C}$ *(denoted by* $A_n \nearrow A$*).*

Remark 7 *If* μ *is* $\mathcal{P}_f(X)$*-valued, then in Definition 6-II), III) it usually appears the Minkowski addition instead of the classical addition because the sum of two closed sets is not, generally, a closed set.*

Remark 8 . *I)* $\overline{\mu}$ *is monotone and superadditive on* $\mathcal{P}(T)$*. Also (see [12]), if* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ *is a multi(sub)measure, then* $\overline{\mu}$ *is finitely additive on* $\mathcal{C}$ *and* $|\mu|$ *is a submeasure.*

II) Every monotone multimeasure is, particularly, a multisubmeasure. Also, any multisubmeasure is null-additive. Any null-additive set multifunction is null-null-additive. The converses are not valid.

III) Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ *be a multisubmeasure of finite variation. The following statements are equivalent:*

i) μ *is* h*-*σ*-subadditive;*
ii) μ *is order-continuous;*
iii) $\overline{\mu}$ *is* σ*-additive on* $\mathcal{C}$*.*

3 Semi-convexity, Darboux property, diffusion and atoms of set multifunctions

We present some properties regarding semi-convexity, Darboux property, diffusion and atoms for set multifunctions. These properties will be discussed in section 5 in relation with the Gould type set-valued integral.

The following notions are classical in measure theory, but they are extended to the set valued case (see for instance [2, 3, 4, 15, 16]).

Definition 9 *Let* $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ *be a set multifunction, with* $\mu(\emptyset) = \{0\}$*.*

I) We say that μ

i) is semi-convex if for every $A \in \mathcal{C}$*, with* $\mu(A) \supsetneq \{0\}$ *, there is a set* $B \in \mathcal{C}$ *such that* $B \subseteq A$ *and* $\mu(B) = \frac{1}{2}\,\mu(A)$*.*

ii) has the Darboux property if for every $A \in \mathcal{C}$*, with* $\mu(A) \supsetneq \{0\}$ *and every* $p \in (0, 1)$*, there exists a set* $B \in \mathcal{C}$ *such that* $B \subseteq A$ *and* $\mu(B) = p\,\mu(A)$*.*

iii) is diffused if for every $t \in T$*, with* $\{t\} \in \mathcal{C}$*, we have* $\mu(\{t\}) = \{0\}$*.*

II) A set $A \in \mathcal{C}$ *is said to be an atom of* μ *if* $\mu(A) \supsetneq \{0\}$ *and for every* $B \in \mathcal{C}$*, with* $B \subseteq A$*, we have* $\mu(B) = \{0\}$ *or* $\mu(A\backslash B) = \{0\}$*.*

III) We say that μ *is*

i) finitely purely atomic if there is a finite disjoint family $(A_i)_{i=\overline{1,n}} \subset \mathcal{C}$ *of atoms of* μ *so that* $T = \bigcup\limits_{i=1}^{n} A_i$*.*

ii) purely atomic if there is at most a countable number of atoms $(A_n)_n \subset \mathcal{C}$ *of* μ *so that* $\mu(T\backslash \bigcup\limits_{n=1}^{\infty} A_n) = \{0\}$ *(evidently, here* $\mathcal{C}$ *must be a* σ*-algebra).*

iii) non-atomic if it has no atoms.

IV) We say that $\mu : \mathcal{C} \to \mathcal{P}_{kc}(\mathbb{R})$ *is induced by a set function* $m : \mathcal{C} \to \mathbb{R}_+$*, with* $m(\emptyset) = 0$*, if* $\mu(A) = [0, m(A)]$*, for every* $A \in \mathcal{C}$*.*

Remark 10 *I) The Lebesgue measure* μ *is diffused. Also, the set functions* $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$ *defined for every* $A \in \mathcal{C}$ *by* $m_1(A) = \sqrt{\mu(A)}$ *and* $m_2(A) = \frac{\mu(A)}{1+\mu(A)}$ *are diffused submeasures. The same are the multisubmeasures induced by them.*

II) If $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_0(X)$ *are diffused multimeasures, then the same is the multimeasure* $\mu_1 + \mu_2$ *defined by* $(\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A)$*, for every* $A \in \mathcal{C}$*.*

III) Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ *be a set multifunction, with* $\mu(\emptyset) = \{0\}$*. Then the following statements are equivalent:*

a) μ *is diffused;*
b) $|\mu|$ *is diffused;*
c) $\overline{\mu}$ *is diffused on* $\mathcal{C}$*.*

The following result is obviously true.

Proposition 11 *If the set multifunction* $\mu : \mathcal{C} \to \mathcal{P}_0(X)$*, with* $\mu(\emptyset) = \{0\}$*, has the Darboux property, then it is semi-convex.*

Under some assumptions, the converse of Proposition 11 is also valid, as shown below:

Theorem 12 *Let* $\mathcal{C}$ *be a* σ*-ring and* $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ *a monotone increasing convergent multimeasure. Then* μ *has the Darboux property if and only if* μ *is semi-convex.*

Proof. The "*only if*" part results from Proposition 11.

The "*if*" part. Every $p \in (0, 1)$ has an expansion $p = \sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$, where $a_n \in \{0, 1\}$, for every $n \in \mathbb{N}^*$. Let $A \in \mathcal{C}$, so that $\mu(A) \supsetneq \{0\}$ and let $p \in (0, 1)$.

By the semi-convexity of μ, there is $B_1 \in \mathcal{C}$ so that $B_1 \subseteq A$ and $\mu(B_1) = \frac{a_1}{2}\mu(A)$.

Analogously, there is $B_2 \in \mathcal{C}$ so that $B_2 \subseteq A\backslash B_1$ and $\mu(B_2) = \frac{a_2}{2^2}\mu(A)$ and so on. Consider $B = \bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty}(\bigcup\limits_{k=1}^{n} B_k) \in \mathcal{C}$. We have

$\mu(B) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{2^k}\mu(A)$ (with respect to h). By Proposition 1-I and XII, it follows $\mu(B) = p\mu(A)$, as claimed. □

Remark 13 *I) If μ is monotone, then μ is non-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(A\backslash B) \supsetneq \{0\}$).*

II) Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $\nu(\emptyset) = 0$ and μ the set multifunction induced by ν. Then μ has the Darboux property if and only if ν has it.

III) [15] Suppose T is a locally compact Hausdorff space, $\mathcal{B}$ is the Borel δ-ring generated by the compact subsets of T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a multisubmeasure. Then μ is non-atomic if and only if it is diffused.

4 $\widetilde{\mu}$-totally-measurability

In this section we present some properties of $\widetilde{\mu}$-totally-measurable functions. In the sequel, $\mathcal{A}$ is an algebra of subsets of T, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a set multifunction so that $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ an arbitrary function.

Definition 14 *I) A partition of a set $A \in \mathcal{A}$ is a finite family $P = \{A_i\}_{i=\overline{1,n}}$ of pairwise disjoint sets of $\mathcal{A}$ such that $\bigcup_{i=1}^{n} A_i = A$.*

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A.

II) For a set multifunction $\mu : \mathcal{A} \to \mathcal{P}_0(X)$, we consider the extended real valued set function $\widetilde{\mu}$ defined by $\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}$, for every $A \in \mathcal{P}(T)$.

Remark 15 *I) $\widetilde{\mu}(A) = \overline{\mu}(A)$, for every $A \in \mathcal{A}$, $\widetilde{\mu}$ is monotone and if $\overline{\mu}$ is subadditive, then $\widetilde{\mu}$ is also subadditive.*

II) Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a multisubmeasure of finite variation. Then:

i) $\widetilde{\mu}$ is a submeasure.

ii) If, moreover, μ is h-σ-subadditive, then $\widetilde{\mu}$ is σ-subadditive.

Definition 16 *I) f is said to be $\widetilde{\mu}$-totally-measurable on $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:*

$$(*)\quad \begin{cases} a)\ \ \widetilde{\mu}(A_0) < \varepsilon \ \text{ and} \\ b)\ \ \sup\limits_{t,s\in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon, \\ \qquad \text{for every } i = \overline{1,n}. \end{cases}$$

II) f is said to be $\widetilde{\mu}$-totally-measurable on $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\widetilde{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\mu_B = \mu|_{\mathcal{A}_B}$.

One can easily observe that if f is $\widetilde{\mu}$-totally-measurable on T, then f is $\widetilde{\mu}$-totally-measurable on every $A \in \mathcal{A}$.

Definition 17 *We say that a property (P) holds μ-almost everywhere (shortly, μ-ae) if there is $A \in \mathcal{P}(T)$, with $\widetilde{\mu}(A) = 0$, so that the property (P) is valid on $T\backslash A$.*

Definition 18 *Let $f_n : T \to \mathbb{R}$ be a real function for every $n \in \mathbb{N}$. One says that the sequence (f_n)*

I) converges in submeasure to f (denoted by $f_n \xrightarrow{\widetilde{\mu}} f$) if for every $\delta > 0$, $\lim_{n\to\infty} \widetilde{\mu}(B_n(\delta)) = 0$, where

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}.$$

II) converges almost everywhere to f (denoted by $f_n \xrightarrow{a.e.} f$) if there is $A \in \mathcal{P}(T)$ so that $\widetilde{\mu}(A) = 0$ and (f_n) pointwise converges to f on $T\backslash A$.

III) (Li [23, 24]) is almost uniformly convergent on T (with respect to $\widetilde{\mu}$), denoted by $f_n \xrightarrow{au} f$, if there exists $(A_k)_{k\in\mathbb{N}^*} \subset \mathcal{A}$, with $\lim_{k\to\infty} \widetilde{\mu}(A_k) = 0$, such that f_n converges to f on $T\backslash A_k$ uniformly for any fixed $k \in \mathbb{N}^*$.

From now on, μ is supposed to be of finite variation.

Theorem 19 *Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ be a multisubmeasure.*

I) ([11]) If $f, g : T \to \mathbb{R}$ are bounded $\widetilde{\mu}$-totally-measurable functions, then:

i) $f + g$ is $\widetilde{\mu}$-totally-measurable;

ii) λf is $\widetilde{\mu}-$totally-measurable, for every $\lambda \in \mathbb{R}$;

iii) f^2 and fg are $\widetilde{\mu}$-totally-measurable;

iv) $|f|^p$ is $\widetilde{\mu}$-totally-measurable, for every $p \in [1, +\infty)$;

v) If $\inf_{t\in T} f(t) > 0$, then $\frac{1}{f}$ is $\widetilde{\mu}$-totally-measurable.

II) Suppose $f, g : T \to \mathbb{R}$ are bounded functions. If $|f|^p$ and $|g|^p$ are $\widetilde{\mu}$-totally-measurable for an arbitrary $p \in [1, +\infty)$, then $|f + g|^p$ is $\widetilde{\mu}$-totally-measurable.

III) ([13]) If for every $n \in \mathbb{N}, f_n : T \to \mathbb{R}$ is bounded $\widetilde{\mu}$-totally-measurable and (f_n) is convergent in submeasure to a bounded function $f : T \to \mathbb{R}$, then f is $\widetilde{\mu}$-totally-measurable.

Remark 20 *If $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $\varphi \circ f$ is $\widetilde{\mu}$-totally-measurable.*

Proposition 21 *Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ be a (multi)(sub)measure, $f : T \to \mathbb{R}$ a bounded function and $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$. Then f is $\widetilde{\mu}$-totally-measurable on $A \cup B$ if and only if it is $\widetilde{\mu}$-totally-measurable on A and $\widetilde{\mu}$-totally-measurable on B.*

Proof. The *if part* is straightforward. For the *only if part*, by the $\widetilde{\mu}$-totally-measurability of f on A and B, there are $P_\varepsilon^A = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}_A$ and $P_\varepsilon^B = \{B_j\}_{i=\overline{0,q}} \in \mathcal{P}_B$ satisfying the condition $(*)$. Since $\overline{\mu}$ is additive on $\mathcal{A}$, then $P_\varepsilon^{A\cup B} = \{A_0 \cup B_0, A_1, \dots, A_n, B_1, ..., B_q\} \in \mathcal{P}_{A\cup B}$ also satisfies condition $(*)$, so f is $\widetilde{\mu}$-totally-measurable on $A \cup B$. □

Remark 22 *I) In the above proposition, A and B need not to be disjoint. Indeed, if we take arbitrary $A, B \in \mathcal{A}$, since $A \cup B = (A \backslash B) \cup B$ and $\widetilde{\mu}$-totally-measurability is hereditary, the statement follows.*

II) Under the assumptions of the above proposition, if $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$, then f is $\widetilde{\mu}$-totally-measurable on $\overset{p}{\underset{i=1}{\cup}} A_i$ if and only if the same is f on every $A_i, i = \overline{1,p}$.

Proposition 23 *If $\mathcal{A}$ is a σ-algebra, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is an o-continuous (multi)(sub)measure, $f : T \to \mathbb{R}$ is a bounded function and $(A_n)_n \subset \mathcal{A}$ are pairwise disjoint, then f is $\widetilde{\mu}$-totally-measurable on every $A_n, n \in \mathbb{N}$ if and only if the same is f on $A = \overset{\infty}{\underset{n=1}{\cup}} A_n$.*

Proof. The *only if part* immediately follows. The *if part*: Since μ is an o-continuous (multi)(sub)measure of finite variation, then $\overline{\mu}$ is additive on $\mathcal{A}$, so $\overline{\mu}$ is also o-continuous on $\mathcal{A}$. We observe that $A \backslash \overset{n}{\underset{k=1}{\cup}} A_k \searrow \emptyset$, so for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$, with $\overline{\mu}(A \backslash \overset{n_0}{\underset{k=1}{\cup}} A_k) < \varepsilon$.

Since for every $l = \overline{1, n_0}$, f is $\widetilde{\mu}$-totally-measurable on A_l, let $\{B_j^1\}_{j=\overline{0,p_1}}$, $\{B_j^2\}_{j=\overline{0,p_2}}, ..., \{B_j^{p_{n_0}}\}_{j=\overline{0,p_{n_0}}}$ be the corresponding partitions satisfying $(*)$.

The partition $P_\varepsilon^A = \{(A \backslash \overset{n_0}{\underset{k=1}{\cup}} A_k), \{B_j^1\}_{j=\overline{1,p_1}}, \{B_j^2\}_{j=\overline{1,p_2}}, \dots, \{B_j^{p_{n_0}}\}_{j=\overline{1,p_{n_0}}}\} \in \mathcal{P}_A$ satisfies $(*)$, so f is $\widetilde{\mu}$-totally-measurable on $A = \overset{\infty}{\underset{n=1}{\cup}} A_n$. □

Theorem 24 *Suppose $\mathcal{A}$ is a σ-algebra, $\mu : \mathcal{A} \to \mathbb{R}_+$ is an o-continuous submeasure of finite variation and $(f_n)_{n\in\mathbb{N}^*}$ is a sequence of uniformly bounded $\widetilde{\mu}$-totally-measurable functions $f_n : T \to \mathbb{R}$. Then g defined for every $t \in T$ by $g(t) = \inf_{n\in\mathbb{N}^*} f_n(t)$, is $\widetilde{\mu}$-totally-measurable.*

Proof. One can easily check that for every $t, s \in T$, the following inequality holds:

$$(1) \qquad |g(t) - g(s)| \leq \sup_{n\in\mathbb{N}^*} |f_n(t) - f_n(s)|.$$

Since for every $n \in \mathbb{N}^*$, f_n is $\widetilde{\mu}$-totally-measurable, then for every $\varepsilon > 0$, there is a partition $P_\varepsilon^n = \{A_j^n\}_{j=\overline{0,p_n}} \in \mathcal{P}$ so that $\overline{\mu}(A_0^n) < \frac{\varepsilon}{2^{n+1}}$ and
(2)

$$\sup_{t,s\in A_j^n} |f_n(t) - f_n(s)| < \frac{\varepsilon}{2^{n+1}}, \text{ for every } j = \overline{1, p_n}.$$

Let $A_0 = \overset{\infty}{\underset{n=1}{\cup}} A_0^n \in \mathcal{A}$. Because μ is an o-continuous submeasure of finite variation, then, by Remark 4-II, $\overline{\mu}$ is σ-additive on $\mathcal{A}$, so,

$$\overline{\mu}(A_0) \leq \sum_{n=1}^{\infty} \overline{\mu}(A_0^n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} cA_0 &= \overset{\infty}{\underset{n=1}{\cap}} cA_0^n = \overset{\infty}{\underset{n=1}{\cap}} (A_1^n \cup A_2^n \cup ... \cup A_{p_n}^n) = \\ &= (A_1^1 \cup A_2^1 \cup ... \cup A_{p_1}^1) \cap (A_1^2 \cup A_2^2 \cup ... \cup A_{p_2}^2) \cap ... \\ &= \bigcup_{(i_n)\in\prod_{n=1}^{\infty} I_n} (A_{i_1}^1 \cap A_{i_2}^2 \cap \dots \cap A_{i_n}^n \cap \dots), \end{aligned}$$

where $I_n = \{1, 2, \dots, p_n\}$, for every $n \in \mathbb{N}^*$. Denote the last reunion by $\bigcup_{n=1}^{\infty} B_n$. Now let $C_n = \bigcup_{k=1}^{n} B_k$ and $D_n = cA_0 \backslash C_n$, for every $n \in \mathbb{N}^*$. We observe that $B_n \cap B_m = \emptyset$ whenever $n \neq m$, $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} B_n = cA_0$ and $D_n \searrow \emptyset$.

Since $\overline{\mu}$ is o-continuous, there is $n_0(\varepsilon) = n_0 \in \mathbb{N}^*$ such that $\overline{\mu}(cA_0 \backslash (\overset{n_0}{\underset{i=1}{\cup}} B_i)) < \frac{\varepsilon}{2}$. Because $\overline{\mu}(A_0) < \frac{\varepsilon}{2}$, we get $\overline{\mu}(c(\overset{n_0}{\underset{i=1}{\cup}} B_i)) < \varepsilon$.

From (1) and (2), we have:

$$\sup_{t,s\in B_i} |g(t) - g(s)| \leq \sup_{t,s\in B_i} \{ \sup_{n\in\mathbb{N}^*} |f_n(t) - f_n(s)|\} < \frac{\varepsilon}{2},$$
$\forall i \in \{1, \dots, n_0\}$.

If we now consider the partition $P_\varepsilon = \{c(\overset{n_0}{\underset{i=1}{\cup}} B_i), B_1, \dots, B_{n_0}\}$, we obtain that g is $\widetilde{\mu}$-totally-measurable. □

Corollary 25 *Under the assumptions of Theorem 24, the function* h *defined for every* $t \in T$ *by* $h(t) = \sup_{n\in\mathbb{N}^*} f_n(t)$*, is* $\widetilde{\mu}$*-totally-measurable. Moreover, supposing there exists* $\lim_{n\to\infty} f_n(t) = f(t)$*, for every* $t \in T$*, then* f *is* $\widetilde{\mu}$*-totally-measurable.*

Theorem 26 *Suppose* (T, ρ) *is a compact metric space,* B *is the Borel* δ*-ring generated by the compact subsets of* T*,* $f : T \to \mathbb{R}$ *is continuous on* T *and* $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ *is a finitely purely atomic multisubmeasure. Then* f *is* $\widetilde{\mu}$*-totally-measurable on* T*.*

Proof. According to Remark 22, it is sufficient to establish the $\widetilde{\mu}$-totally-measurability of f on an arbitrary, fixed atom A_0 of μ. Since μ is a multisubmeasure, by [15], there is an unique $a_0 \in A_0$ so that $\mu(A_0\backslash\{a_0\}) = \{0\}$.

Let $\varepsilon > 0$. Since f is continuous in a_0, there is $\delta_\varepsilon > 0$ so that for every $t \in A_0$, with $\rho(t, a_0) < \delta_\varepsilon$, we have $|f(t) - f(a_0)| < \frac{\varepsilon}{3}$.

Let $B_\varepsilon = \{t \in A_0; \rho(t, a_0) < \delta_\varepsilon\} = A_0 \cap B(a_0, \delta_\varepsilon)$, where $B(a_0, \delta_\varepsilon)$ is the open ball of center a_0 and radius δ_ε. It results $B_\varepsilon \in \mathcal{B}$ and since A_0 is an atom, we have $\mu(B_\varepsilon) = \{0\}$ or $\mu(A_0\backslash B_\varepsilon) = \{0\}$.

If $\mu(B_\varepsilon) = \{0\}$, then since $a_0 \in B_\varepsilon$, we get $\mu(\{a_0\}) = \{0\}$. But $\mu(A_0\backslash\{a_0\}) = \{0\}$, so $\mu(A_0) = \{0\}$, a contradiction. So, we have $\mu(A_0\backslash B_\varepsilon) = \{0\}$. Now, one can easily observe that the partition $P_{A_0} = \{A_0\backslash B_\varepsilon, B_\varepsilon\}$ assures the $\widetilde{\mu}$-totally-measurability of f. □

5 Semi-convexity, diffusion, atoms and purely atomicity for a Gould type set-valued integral

In this section, we establish results concerning semi-convexity, diffusion, atoms and purely atomicity for the Gould type set-valued integral introduced and studied in [30].

In what follows, without any special assumptions, we suppose $\mathcal{A}$ is an algebra of subsets of T, X is a Banach space, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ is a bounded function. We now recall the following notions and results (see [12, 13, 28, 29]).

Remark 27 *If* $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ *is of finite variation, then* μ *takes its values in* $\mathcal{P}_{bf}(X)$*.*

Definition 28 *I) Let* $P = \{A_i\}_{i=\overline{1,n}}$ *and* $P' = \{B_j\}_{j=\overline{1,m}}$ *be two partitions of* T*.* P' *is said to be finer than* P*, denoted* $P \leq P'$ *(or* $P' \geq P$*) if for every* $j = \overline{1,m}$*, there exists* $i_j = \overline{1,n}$ *so that* $B_j \subseteq A_{i_j}$*.*

II) The common refinement of two partitions $P = \{A_i\}_{i=\overline{1,n}}$ *and* $P' = \{B_j\}_{j=\overline{1,m}}$ *is the partition* $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n}\\ j=\overline{1,m}}}$*.*

Definition 29 *([30]) For every partition* $P = \{A_i\}_{i=\overline{1,n}}$ *of* T *and every* $t_i \in A_i$*,* $i = \overline{1,n}$*, let* $\sigma_{f,\mu}(P)$ *(or, if there is no doubt,* $\sigma_f(P), \sigma_\mu(P), \sigma(P)$*) be:*

$$\sigma(P) \,{=}^{\bullet}\! \sum_{i=1}^{n} f(t_i)\mu(A_i) = \overline{f(t_1)\mu(A_1) + \ldots + f(t_n)\mu(A_n)}.$$

I) f *is said to be* μ*-integrable on* $(T, \mathcal{A}, \mu)$ *if the net* $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ *is convergent in* $(\mathcal{P}_f(X), h)$*, where* $\mathcal{P}$ *is ordered by the relation "* $\leq$ *" given in Definition 4.2.*

If $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ *is convergent, then its limit is called the integral of* f *on* T *with respect to* μ*, denoted by* $\int_T f d\mu$*.*

II) For an arbitrary $B \in \mathcal{A}$*,* f *is said to be* μ*-integrable on* B *if the restriction* $f|_B$ *of* f *to* B *is* μ*-integrable on* $(B, \mathcal{A}_B, \mu_B)$*.*

Remark 30 *I)* f *is* μ*-integrable on* T *if and only if there exists a set* $I \in \mathcal{P}_{bf}(X)$ *such that for every* $\varepsilon > 0$*, there exists a partition* P_ε *of* T*, so that for every other partition of* T*,* $P = \{A_i\}_{i=\overline{1,n}}$*, with* $P \geq P_\varepsilon$ *and every choice of points* $t_i \in A_i, i = \overline{1,n}$*, we have* $h(\sigma(P), I) < \varepsilon$*.*

II) If μ *is a multimeasure (multisubmeasure, submeasure, monotone set multifunction, respectively), we obtain the corresponding definitions of [28, 12, 17, 29], respectively).*

III) If μ *is a multimeasure and* $f = 1$*, then* $\int_T f d\mu = \mu(T)$*.*

IV) If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$*, then* $\int_T f d\mu \in \mathcal{P}_{kc}(X)$*.*

V) Suppose $m : \mathcal{A} \to \mathbb{R}_+$ *is an arbitrary set function of finite variation with* $m(\emptyset) = 0$ *and consider the set multifunction* $\mu : \mathcal{A} \to \mathcal{P}_f(\mathbb{R})$ *defined by* $\mu(A) = \{m(A)\}$*, for every* $A \in \mathcal{A}$*. Then, by I),* f *is* m*-integrable on* T *if and only if there is* $I \in \mathbb{R}$ *such that for every* $\varepsilon > 0$*, there exists a partition* P_ε *of* T*, so that for every other partition of* T*,* $P = \{A_i\}_{i=\overline{1,n}}$*, with* $P \geq P_\varepsilon$ *and every choice of points* $t_i \in A_i, i = \overline{1,n}$*, we have* $|\sigma(P) - I| = |\sum_{i=1}^{n} f(t_i)m(A_i) - I| < \varepsilon$*.*

Here, $I = \int_T f dm$*.*

Moreover, if m *is finitely additive and* $f = 1$*, then* $\int_T f dm = m(T)$*.*

VI) Our integral, if it exists, is unique and has the following properties: homogeneity and additivity with *respect to the function f and the set multifunction μ, additivity with respect to the set, monotonicity with respect to the function f, to the set multifunction μ, and to the set (see [28]–[30] for details. The assumption of monotonicity is not necessary in [29], as observed in [30]).*

VII) Let $m : \mathcal{A} \to [0,1]$ be a submeasure of finite variation. One can easily check that the set function $m_1 : \mathcal{A} \to [0,1]$ defined for every $A \in \mathcal{A}$ by $m_1(A) = \sin m(A)$ is also a submeasure of finite variation (since $\overline{m_1}(A) \leq \overline{m}(A)$, for every $A \subseteq T$). Suppose $f : T \to \mathbb{R}$ is bounded. Since, according to [17], m-integrability of f is equivalent to its $\widetilde{m}$-totally-measurability and because $\frac{2}{\pi}t \leq \sin t \leq t$, for every $t \in [0, \frac{\pi}{2})$, then f is m-integrable if and only if f is m_1-integrable.

Theorem 31 *I) Let $f : T \to \mathbb{R}$ be a μ-integrable function. Then*

$$\Big| \int_T f d\mu \Big| \leq \sup_{t\in T} |f(t)| \cdot \overline{\mu}(T).$$

II) Let $f : T \to \mathbb{R}$ and $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$. If f is μ-integrable on A and μ-integrable on B, then f is μ-integrable on $A \cup B$ and $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$.

III) Suppose $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$. If $f : T \to \mathbb{R}$ is μ-integrable on T, then f is μ-integrable on every $B \in \mathcal{A}$.

IV) If $f : T \to \mathbb{R}$ is μ-integrable on every $A \in \mathcal{A}$, then the set multifunction $M : \mathcal{A} \to \mathcal{P}_f(X)$, defined by

$$(**)\ M(A) = \int_A f d\mu, \text{for every } A \in \mathcal{A},$$

is a monotone multimeasure, $M \ll \mu$ and M is strongly absolutely continuous with respect to μ.

V) If $f, g : T \to \mathbb{R}$ are bounded functions so that f is μ-integrable on T and $f = g$ μ-a.e, then g is μ-integrable on T and $\int_T f d\mu = \int_T g d\mu$.

Remark 32 *By Theorem 31-I and Remark 10-III), we immediately get that if $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ is diffused, then the same is M defined in $(**)$. Also, by Remark 30-I, if $\inf_{t\in T} f(t) > 0$, then the converse is also valid. So, in this case, μ is diffused if and only if the same is M.*

Proposition 33 *Let $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$ be set functions of finite variation, so that $m_1 \leq m_2$ and $m_1(\emptyset) = m_2(\emptyset) = 0$, $f : T \to \mathbb{R}$ and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R})$ the set multifunction defined by $\mu(A) = [m_1(A), m_2(A)]$, for every $A \in \mathcal{A}$. Then f is μ-integrable on T if and only if f is m_1-integrable on T and m_2-integrable on T and, in this case,*

$$\int_T f d\mu = [\int_T f dm_1, \int_T f dm_2].$$

Proof. f is m_1-integrable on T and m_2-integrable on T if and only if for every $\varepsilon > 0$, there exists a partition P_ε of T so that for every other partitions of T, $P' = \{A_i\}_{i=\overline{1,n}}$, $P'' = \{B_j\}_{j=\overline{1,p}}$, so that $P' \geq P_\varepsilon$, $P'' \geq P_\varepsilon$ and every $t_i \in A_i, i = \overline{1,n}, s_j \in B_j, j = \overline{1,p}$, we have

$$\Big|\sum_{i=1}^n f(t_i) m_k(A_i) - \sum_{j=1}^p f(s_j) m_k(B_j)\Big| < \varepsilon, \quad k = 1, 2.$$

Since

$$\begin{aligned} & h\Big(\sum_{i=1}^n f(t_i)\mu(A_i), \sum_{j=1}^p f(s_j)\mu(B_j)\Big) = \\ & = h\Big([\sum_{i=1}^n f(t_i)m_1(A_i), \sum_{i=1}^n f(t_i)m_2(A_i)], \\ & [\sum_{j=1}^p f(s_j)m_1(B_j), \sum_{j=1}^p f(s_j)m_2(B_j)]\Big) \\ & = \max\Bigg\{ \Big| \sum_{i=1}^n f(t_i)m_1(A_i) - \sum_{j=1}^p f(s_j)m_1(B_j)\Big|, \\ & \Big|\sum_{i=1}^n f(t_i)m_2(A_i) - \sum_{j=1}^p f(s_j)m_2(B_j)\Big| \Bigg\}, \end{aligned}$$

it follows that for every $\varepsilon > 0$, there exists a partition P_ε of T so that for every other partitions of T, $P' = \{A_i\}_{i=\overline{1,n}}$, $P'' = \{B_j\}_{j=\overline{1,p}}$, so that $P' \geq P_\varepsilon$, $P'' \geq P_\varepsilon$ and every $t_i \in A_i, i = \overline{1,n}, s_j \in B_j, j = \overline{1,p}$, we have

$$h(\sum_{i=1}^n f(t_i)\mu(A_i), \sum_{j=1}^p f(s_j)\mu(B_j)) < \varepsilon,$$

which means that f is μ-integrable on T.

Now, let us prove that $\int_T f d\mu = [\int_T f dm_1, \int_T f dm_2]$.

Since f is μ-integrable on T, m_1-integrable on T and m_2-integrable on T, it results that for every $\varepsilon > 0$, there exists a partition $\{C_k\}_{k=\overline{1,l}}$ of T so that for every

$s_k \in C_k, k = \overline{1,l}$, we have

$$h(\int_T f d\mu, \sum_{k=1}^{l} f(s_k)\mu(C_k)) < \frac{\varepsilon}{2} \text{ and}$$

$$\Big| \int_T f dm_i - \sum_{k=1}^{l} f(s_k)m_i(C_k)\Big| < \frac{\varepsilon}{2}, i = 1, 2.$$

Then

$$h(\int_T f d\mu, [\int_T f dm_1, \int_T f dm_2]) \leq$$
$$\leq h(\int_T f d\mu, \sum_{k=1}^{l} f(s_k)\mu(C_k))$$
$$+ h(\sum_{k=1}^{l} f(s_k)\mu(C_k), [\int_T f dm_1, \int_T f dm_2]) =$$
$$= h(\int_T f d\mu, \sum_{k=1}^{l} f(s_k)\mu(C_k))$$
$$+ \max\Big\{\Big| \int_T f dm_1 - \sum_{k=1}^{l} f(s_k)m_1(C_k)\Big|,$$
$$\Big| \int_T f dm_2 - \sum_{k=1}^{l} f(s_k)m_2(C_k)\Big|\Big\} < \varepsilon,$$

for every $\varepsilon > 0$ and this implies $\int_T f d\mu = [\int_T f dm_1, \int_T f dm_2]$. □

Taking $m_1 = 0$ in Proposition 33, we obtain the following result.

Corollary 34 *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a set function of finite variation with $m(\emptyset) = 0$, $\mu : \mathcal{C} \to \mathcal{P}_{kc}(\mathbb{R})$ the set multifunction defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$ and $f : T \to \mathbb{R}$. Then f is μ-integrable on T if and only if f is m-integrable on T and, in this case,*

$$\int_T f d\mu = [0, \int_T f dm].$$

Theorem 35 *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ be a semi-convex multimeasure and $f : T \to \mathbb{R}$ a $\widetilde{\mu}$-totally-measurable bounded function on T. Then M defined in $(**)$ is also semi-convex.*

Proof. The following statements, even they are established for T, remain valid for any arbitrary set $A \in \mathcal{A}$. Also, according to [28], f is μ-integrable on T and on every $A \in \mathcal{A}$. Consider arbitrary $\varepsilon > 0$ and let $M = \max\{\overline{\mu}(T), \sup_{t\in T} |f(t)|\}$.

By the μ-integrability of f on T, there is a partition $\{A_i\}_{i=\overline{1,n}}$ of T such that for every $s_i \in A_i$, $i = \overline{1,n}$, we have $h(\int_T f d\mu, \sum_{i=1}^{n} f(s_i)\mu(A_i)) < \frac{2\varepsilon}{3}$, so $h(\frac{1}{2}\int_T f d\mu, \sum_{i=1}^{n} f(s_i)\frac{1}{2}\mu(A_i)) < \frac{\varepsilon}{3}$.

Because μ is semi-convex, for every $i = \overline{1,n}$, there is $B_i \subset A_i$ so that $B_i \in \mathcal{A}$ and $\mu(B_i) = \frac{1}{2}\mu(A_i)$, which implies $h(\frac{1}{2}\int_T f d\mu, \sum_{i=1}^{n} f(s_i)\mu(B_i)) < \frac{\varepsilon}{3}$.

Since f is μ-integrable on $B = \bigcup_{i=1}^{n} B_i$, there exists a partition $\widetilde{P}_\varepsilon^B = \{D_k\}_{k=\overline{1,s}} \in \mathcal{P}_B$ so that for every partition $P \in \mathcal{P}_B$, with $P \geq \widetilde{P}_\varepsilon^B$, we have $h(\int_B f d\mu, \sigma(P)) < \frac{\varepsilon}{3}$.

On the other hand, because f is $\widetilde{\mu}$-totally-measurable on B, there is a partition $\widetilde{\widetilde{P}}_\varepsilon^B = \{E_l\}_{l=\overline{0,m}} \in \mathcal{P}_B$ such that $\overline{\mu}(E_0) < \frac{\varepsilon}{12M}$ and $\sup_{t,s\in E_l} |f(t) - f(s)| < \frac{\varepsilon}{6M}$, for every $l = \overline{1,m}$.

Consider $\{D_k \cap E_l\}_{k=\overline{1,s},\, l=\overline{0,m}} \in \mathcal{P}_B$ and denote it by $\{C_j\}_{j=\overline{1,q}}$. For instance, $C_1 = D_1 \cap E_0, C_2 = D_2 \cap E_0, ..., C_s = D_s \cap E_0, C_{s+1} = D_1 \cap E_1$ etc. We observe that

$$\overline{\mu}(\bigcup_{j=1}^{s} C_j) = \overline{\mu}(E_0) < \frac{\varepsilon}{12M} \text{ and}$$

$$\sup_{t_j,s_j\in C_j} |f(t_j) - f(s_j)| < \frac{\varepsilon}{6M}, \text{ for every } j = \overline{s+1,q}.$$

Let $P_\varepsilon^B = \{B_i \cap C_j\}_{i=\overline{1,n},\, j=\overline{1,q}} \in \mathcal{P}_B$. Since $P_\varepsilon^B \geq \widetilde{P}_\varepsilon^B$, then $h(\int_B f d\mu, \sigma(P_\varepsilon^B)) < \frac{\varepsilon}{3}$.

Now, we have:

$$h(\frac{1}{2}\int_T f d\mu, \int_B f d\mu) \leq h(\frac{1}{2}\int_T f d\mu, \sum_{i=1}^{n} f(s_i)\mu(B_i))$$
$$+ h(\int_B f d\mu, \sigma(P_\varepsilon^B))+$$
$$+ h(\sigma(P_\varepsilon^B), \sum_{i=1}^{n} f(s_i)\mu(B_i)) < \frac{2\varepsilon}{3}$$
$$+ h(\sigma(P_\varepsilon^B), \sum_{i=1}^{n} f(s_i)\mu(B_i)).$$

It only remains to prove that for every $\theta_{ij} \in B_i \cap C_j$, $i = \overline{1,n}$, $j = \overline{1,q}$,

$$h(\sigma(P_\varepsilon^B), \sum_{i=1}^{n} f(s_i)\mu(B_i))$$
$$= h(\sum_{i=1}^{n}\sum_{j=1}^{q} f(\theta_{ij})\mu(B_i \cap C_j), \sum_{i=1}^{n} f(s_i)\mu(B_i)) < \frac{\varepsilon}{3}.$$

Indeed, we have:

$$h(\sum_{i=1}^{n}\sum_{j=1}^{q} f(\theta_{ij})\mu(B_i\cap C_j), \sum_{i=1}^{n} f(s_i)\mu(B_i)) =$$
$$= h(\sum_{i=1}^{n}\sum_{j=1}^{q} f(\theta_{ij})\mu(B_i\cap C_j),$$
$$\sum_{i=1}^{n}\sum_{j=1}^{q} f(s_i)\mu(B_i\cap C_j)) \leq$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{q} |f(s_i)-f(\theta_{ij})|\cdot|\mu(B_i\cap C_j)| =$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{s} |f(s_i)-f(\theta_{ij})|\cdot|\mu(B_i\cap C_j)| +$$
$$+ \sum_{i=1}^{n}\sum_{j=s+1}^{q} |f(s_i)-f(\theta_{ij})|\cdot|\mu(B_i\cap C_j)| \leq$$
$$\leq 2M\sum_{j=1}^{s}\overline{\mu}(C_j) + \sum_{j=s+1}^{q} |f(s_i)-f(\theta_{ij})|\cdot\overline{\mu}(C_j) <$$
$$< 2M\overline{\mu}(\bigcup_{j=1}^{s} C_j) + \frac{\varepsilon}{6M}\overline{\mu}(\bigcup_{j=s+1}^{q} C_j)$$
$$< 2M\frac{\varepsilon}{12M} + \frac{\varepsilon}{6M}M = \frac{\varepsilon}{3}.$$

Consequently, $h(\frac{1}{2}\int_T f d\mu, \int_B f d\mu) < \varepsilon$, for every $\varepsilon > 0$, so $\frac{1}{2}\int_T f d\mu = \int_B f d\mu$. Therefore, M is semi-convex. □

Theorem 36 *Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is monotone, null-additive and finitely purely atomic. If f is $\widetilde{\mu}$ -totally-measurable on T, then f is μ -integrable on T.*

Proof. According to Theorem 31-II, it will be sufficient to prove that f is μ-integrable on every atom A of μ. First, we observe that, if A is an atom of μ and if $\{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$, then, there exists only one set, for instance, without any loss of generality, A_1, so that $\mu(A_1) \supsetneq \{0\}$ and $\mu(A_2) = ... = \mu(A_n) = \{0\}$.

Let $A \in \mathcal{A}$ be an atom of μ.

Since f is $\widetilde{\mu}$-totally-measurable on A, then for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of A such that:

$$(*)\begin{cases} i)\ \widetilde{\mu}(A_0) < \frac{\varepsilon}{2M} \text{ (where } M = \sup_{t\in T}|f(t)|) \text{ and} \\ ii)\ \sup_{t,s\in A_i} |f(t)-f(s)| < \frac{\varepsilon}{\overline{\mu}(T)}, \text{ for every } i = \overline{1,n}. \end{cases}$$

Let $\{B_j\}_{j=\overline{1,k}}, \{C_p\}_{p=\overline{1,s}} \in \mathcal{P}_A$ be two arbitrary partitions which are finer than P_ε and consider $s_j \in B_j, j = \overline{1,k}, \theta_p \in C_p, p = \overline{1,s}$.

We prove that

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) < \varepsilon.$$

We have two cases:

I. $\mu(A_0) \supsetneq \{0\}$. Then $\mu(A_1) = ... = \mu(A_n) = \{0\}$.

Suppose, without any loss of generality that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then $B_1 \subset A_0$ and $C_1 \subset A_0$. Consequently,

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p))$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)) \leq$$
$$\leq |f(s_1)||\mu(B_1)| + |f(\theta_1)||\mu(C_1)| \leq$$
$$\leq 2M\overline{\mu}(A_0) < \varepsilon.$$

II. $\mu(A_0) = \{0\}$. Then, without any loss of generality, $\mu(A_1) \supsetneq \{0\}$ and $\mu(A_i) = \{0\}$, for every $i = \overline{2,n}$. Suppose that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then $B_1 \subset A_1$ and $C_1 \subset A_1$, and, therefore,

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p))$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)).$$

Since A is an atom of μ and $\mu(B_1) \supsetneq \{0\}$, then $\mu(A\backslash B_1) = \{0\}$, so $\mu(C_1\backslash B_1) = \{0\}$. By the null-additivity of μ, we get $\mu(C_1) = \mu(B_1)$. Then

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p))$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1))$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(B_1)).$$

By Proposition 1, we have

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p))$$
$$\leq |\mu(B_1)||f(s_1) - f(\theta_1)| \leq \overline{\mu}(T)\frac{\varepsilon}{\overline{\mu}(T)} = \varepsilon.$$

Therefore, the net $(\sigma(P))_{P\in\mathcal{P}_A}$ is a Cauchy one in the complete metric space $(\mathcal{P}_{bf}(X), h)$, hence f is μ-integrable on A. □

In [8, 9], submeasures of the following type are studied. Here, we investigate the relationship between their Gould integrals.

Theorem 37 *Let $(m_n)_{n\in\mathbb{N}}$ be an uniformly bounded sequence of submeasures of finite variation, $m_n : \mathcal{A} \to \mathbb{R}_+$, $\forall n \in \mathbb{N}$ and $m : \mathcal{A} \to \mathbb{R}_+$ defined by $m(A) = \sup\limits_n m_n(A)$, for every $A \in \mathcal{A}$.*

Suppose $A_0 \in \mathcal{A}$ is an atom of m and $f : T \to \mathbb{R}$ is $\widetilde{m}$-totally-measurable on T. Then $\int_{A_0} f dm = \sup\limits_n \int_{A_0} f dm_n$.

Proof. By Example 3-II), m is a submeasure too. Since $\overline{m}_n(A) \le \overline{m}(A)$, for every $A \in \mathcal{A}$, then for every $n \in \mathbb{N}$, f is $\widetilde{m}_n$-totally-measurable on T. According to [17], f is m-integrable and m_n-integrable on T and on every $A \in \mathcal{A}$. By [17], $\int_{A_0} f dm_n \le \int_{A_0} f dm$, for every $n \in \mathbb{N}$.

Since $m(A_0) = \sup\limits_n m_n(A_0)$, we get that for every $\varepsilon > 0$, there is $n_0(\varepsilon, A_0) = n_0$ so that $m(A_0) < m_{n_0}(A_0) + \frac{\varepsilon}{2M}$, where $M = \sup\limits_{t\in T}|f(t)|$.

Because f is m-integrable and m_{n_0}-integrable on A_0, we have that for every $\varepsilon > 0$, there is a common partition $\{B_j\}_{j=\overline{1,k}} \in \mathcal{P}_{A_0}$ so that for every $t_j \in B_j$, $|\int_{A_0} f dm - \sum\limits_{j=1}^{k} f(t_j)m(B_j)| < \frac{\varepsilon}{4}$ and $|\int_{A_0} f dm_{n_0} - \sum\limits_{j=1}^{k} f(t_j)m_{n_0}(B_j)| < \frac{\varepsilon}{4}$.

Since $\{B_j\}_{j=\overline{1,k}} \in \mathcal{P}_{A_0}$, we observe that there can exist only one set, for instance, B_1, so that $m(B_1) > 0$ and $m(B_j) = 0$, for every $j = \overline{2,k}$. Then $m_{n_0}(B_j) = 0$, for every $j = \overline{2,k}$.

Consequently, because $m(B_1) = m(A_0)$ and $m_{n_0}(B_1) = m_{n_0}(A_0)$, we have

$$\begin{aligned}\int_{A_0} f dm &\le \Big| \int_{A_0} f dm - \sum_{j=1}^{k} f(t_j)m(B_j)\Big| \\ &+ \Big| \int_{A_0} f dm_{n_0} - \sum_{j=1}^{k} f(t_j)m_{n_0}(B_j)\Big| \\ &+ |f(t_1)| \cdot |m(B_1) - m_{n_0}(B_1)| \\ &+ \int_{A_0} f dm_{n_0} < \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} + \int_{A_0} f dm_{n_0} = \\ &= \varepsilon + \int_{A_0} f dm_{n_0},\end{aligned}$$

so $\int_{A_0} f dm = \sup\limits_n \int_{A_0} f dm_n$, as claimed. □

6 Classical results for the Gould type set-valued integral

In this section we obtain some classical theorems (such as Hölder inequality, Minkowski inequality, mean convergence theorem, Lebesgue theorem, Fatou lemma) for the Gould type set-valued integral introduced in [30].

Theorem 38 ***(Hölder Inequality)*** *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $f, g : T \to \mathbb{R}$ m -integrable bounded functions on T. Then*

$$\int_T |fg| dm \le \left(\int_T |f|^p dm\right)^{\frac{1}{p}} \cdot \left(\int_T |g|^q dm\right)^{\frac{1}{q}},$$

for every $p, q \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since (see [17]) for submeasures, m-integrability is equivalent to $\widetilde{m}$-totally-measurability, then by Theorem 19-I and Theorem 2.17 [17], $|f|, |g|$, $|fg|$, $|f|^p$ and $|g|^q$ are also m-integrable, so, for every $\varepsilon > 0$, there is a common partition $P_\varepsilon = \{A_i\}_{i=\overline{1.n}}$ such that for every $t_i \in A_i$, $i = \overline{1,n}$, we have:

$$\Big| \int_T |fg| dm - \sum_{i=1}^{n} |f(t_i)g(t_i)| m(A_i)\Big| < \frac{\varepsilon}{3},$$

$$\Big| \int_T |f| dm - \sum_{i=1}^{n} |f(t_i)| m(A_i)\Big| < \frac{\varepsilon}{3} \text{ and}$$

$$\Big| \int_T |g| dm - \sum_{i=1}^{n} |g(t_i)| m(A_i)\Big| < \frac{\varepsilon}{3}.$$

Since

$$\begin{aligned}&\sum_{i=1}^{n} |f(t_i)g(t_i)| m(A_i) \\ &= \sum_{i=1}^{n} \left[|f(t_i)| \left(m(A_i)\right)^{\frac{1}{p}} \cdot |g(t_i)| \left(m(A_i)\right)^{\frac{1}{q}}\right] \le \\ &\le \left(\sum_{i=1}^{n} |f(t_i)|^p m(A_i)\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |g(t_i)|^q m(A_i)\right)^{\frac{1}{q}},\end{aligned}$$

we immediately have the conclusion. □

Using the above theorem, we obtain the Minkowski inequality, by a classical proof.

Theorem 39 ***(Minkowski inequality)*** *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $f, g : T \to \mathbb{R}$ m -integrable bounded functions on T. Then*

$$\left(\int_T |f+g|^p dm\right)^{\frac{1}{p}} \le \left(\int_T |f|^p dm\right)^{\frac{1}{p}} + \left(\int_T |g|^p dm\right)^{\frac{1}{p}},$$

for every $p \in [1, +\infty)$.

If $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation, we consider the space $\mathcal{L}^p = \{f : T \to \mathbb{R};\ f$ is bounded on T and $|f|^p$ is m-integrable on $T\}$.

Remark 40 *From Theorem 19-II, it results that if $f, g \in \mathcal{L}^p$, then $f + g \in \mathcal{L}^p$. So, $\mathcal{L}^p$ is a linear space.*

Corollary 41 *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $p \in [1, +\infty)$. Then the function $||\cdot|| : \mathcal{L}^p \to \mathbb{R}_+$, defined for every $f \in \mathcal{L}^p$ by $||f|| = (\int_T |f|^p dm)^{\frac{1}{p}}$, is a semi-norm.*

Definition 42 *Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ be a set multifunction with $\mu(\emptyset) = \{0\}$. If for every $n \in \mathbb{N}$, $f_n : T \to \mathbb{R}$ is μ-integrable on T, then the sequence (f_n) is said to be mean convergent to f on T if $\lim\limits_{n\to\infty} \int_T (f_n - f)d\mu = \{0\}$ (with respect to h).*

Theorem 43 ***(Mean Convergence Theorem)*** *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ be a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f_n : T \to \mathbb{R}$, for every $n \in \mathbb{N}$. Suppose (f_n) is an uniformly bounded sequence of μ-integrable functions such that (f_n) is convergent in submeasure to a bounded function $f : T \to \mathbb{R}$. Then f is μ-integrable on T and on every $A \in \mathcal{A}$,*

$$\lim_{n\to\infty} \int_A (f_n - f)d\mu = \{0\}$$

(with respect to h)

Proof. Let $M' = \overline{\mu}(T)$, $M_1 = \sup\limits_{t\in T} |f(t)|$, $M_2 = \sup\limits_{t\in T, n\in\mathbb{N}} |f_n(t)|$ and $M = \max\{M_1, M_2\}$.

Since $f_n \xrightarrow{\widetilde{\mu}} f$, it results that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $\widetilde{\mu}(B_n\left(\frac{\varepsilon}{6M'}\right)) < \frac{\varepsilon}{4M}$, for every $n \geq n_0$.

Particularly, $\widetilde{\mu}(B_{n_0}\left(\frac{\varepsilon}{6M'}\right)) < \frac{\varepsilon}{4M}$. By the definition of $\widetilde{\mu}$, there is $C_{n_0} \in \mathcal{A}$ so that $B_{n_0}(\frac{\varepsilon}{6M'}) \subseteq C_{n_0}$ and $\widetilde{\mu}(C_{n_0}) = \overline{\mu}(C_{n_0}) < \frac{\varepsilon}{4M}$.

First, we prove that f is μ-integrable on C_{n_0}. Indeed, for every $\varepsilon > 0$, there is a partition $P_\varepsilon = \{C_{n_0}\} \in \mathcal{P}_{C_{n_0}}$ so that, for every other partition $P = \{D_l\}_{l=\overline{1,p}} \in \mathcal{P}_{C_{n_0}}$, with $P \geq P_\varepsilon$ and every $t_l \in D_l$, $l = \overline{1,p}$ and $c \in C_{n_0}$, we have:

$$h\left(\sum_{l=1}^{p} f(t_l)\mu(D_l), f(c)\mu(C_{n_0})\right)$$
$$\leq \sum_{l=1}^{p} |f(t)|\cdot|\mu(D_l)| +$$
$$+ \frac{\varepsilon}{4M}\cdot M_1 < \overline{\mu}(C_{n_0})\cdot M_1 + \frac{\varepsilon}{4M}\cdot M_1$$
$$< 2\cdot\frac{\varepsilon}{4M}\cdot M_1 = \frac{\varepsilon}{2}.$$

Consider another partition $P' = \{E_s\}_{s=\overline{1,q}} \in \mathcal{P}_{C_{n_0}}$, with $P' \geq P_\varepsilon$ and $r_s \in E_s$, $s = \overline{1,q}$, arbitrarily.

In a similar way we get $h\left(\sum\limits_{s=1}^{q} f(r_s)\mu(E_s), f(c)\mu(C_{n_0})\right) < \frac{\varepsilon}{2}$, whence, $h\left(\sum\limits_{l=1}^{p} f(t_l)\mu(D_l), \sum\limits_{s=1}^{q} f(r_s)\mu(E_s)\right) < \varepsilon$. Then f is μ-integrable on C_{n_0}.

Consequently, according to Theorem 31-II, in order to prove that f is μ-integrable on T, it is sufficient to establish the μ-integrability of f on $T\backslash C_{n_0}$.

Since for every $n \in \mathbb{N}$ f_n is μ-integrable on T, then f_{n_0} is μ-integrable on $T\backslash C_{n_0}$. Consequently, there is a partition $P_\varepsilon^{n_0} = \{A_i\}_{i=\overline{1,m_{n_0}}} \in \mathcal{P}_{T\backslash C_{n_0}}$ so that, for every other partition $P \in \mathcal{P}_{T\backslash C_{n_0}}$, with $P \geq P_\varepsilon^{n_0}$, we have $h(\sigma(P), \sigma(P_\varepsilon^{n_0})) < \frac{\varepsilon}{3}$.

Let $P = \{D_j\}_{j=\overline{1,l}} \in \mathcal{P}_{T\backslash C_{n_0}}$, with $P \geq P_\varepsilon^{n_0}$ be arbitrarily, but fixed. For every $t_j \in D_j$, $j = \overline{1,l}$ and every $c_i \in A_i$, $i = \overline{1, m_{n_0}}$, we have:

$$h(\sum_{j=1}^{l} f(t_j)\mu(D_j), \sum_{i=1}^{m_{n_0}} f(c_i)\mu(A_i))$$
$$\leq h\left(\sum_{j=1}^{l} f(t_j)\mu(D_j), \sum_{j=1}^{l} f_{n_0}(t_j)\mu(D_j)\right) +$$
$$+ h\left(\sum_{j=1}^{l} f_{n_0}(t_j)\mu(D_j), \sum_{i=1}^{m_{n_0}} f_{n_0}(c_i)\mu(A_i)\right) +$$
$$+ h\left(\sum_{i=1}^{m_{n_0}} f_{n_0}(c_i)\mu(A_i), \sum_{i=1}^{m_{n_0}} f(c_i)\mu(A_i)\right) \leq$$
$$\leq \overline{\mu}(T\backslash C_{n_0})\cdot \sup_{t\in cB_{n_0}} |f(t) - f_{n_0}(t)|$$
$$+ \frac{\varepsilon}{3} + \overline{\mu}(T\backslash C_{n_0})\cdot \sup_{t\in cB_{n_0}} |f(t) - f_{n_0}(t)| <$$
$$< M'\cdot\frac{\varepsilon}{6M'} + \frac{\varepsilon}{3} + M'\cdot\frac{\varepsilon}{6M'} = \varepsilon.$$

A similar inequality for every other partition $P' \in \mathcal{P}_{T\backslash C_{n_0}}$, with $P' \geq P_\varepsilon^{n_0}$, may analogously be obtained. Then, by the triangular inequality, f is μ-integrable on $T\backslash C_{n_0}$ and, according to Theorem 4.5-II, f is μ-integrable on T.

Now, we prove that $\lim\limits_{n\to\infty} \int_T (f_n - f)d\mu = \{0\}$ with respect to h. According to Theorem 31-III, there exist $\int_A f d\mu$ and $\int_A f_n d\mu$, for every $n \in \mathbb{N}$ and every $A \in \mathcal{A}$.

We shall use the same $B_n\left(\frac{\varepsilon}{6M'}\right)$, with $n \geq n_0$, as before. By the definition of $\widetilde{\mu}$, we get that for every $n \geq n_0$, there exists $C_n \in \mathcal{A}$ so that $B_n(\frac{\varepsilon}{6M'}) \subseteq C_n$ and $\widetilde{\mu}(C_n) = \overline{\mu}(C_n) < \frac{\varepsilon}{4M}$.

Then, for every $n \geq n_0$, we have:

$$\Big|\int_A (f_n - f)d\mu\Big| = \Big|\int_{A\setminus C_n} (f_n - f)d\mu$$
$$+ \int_{A\cap C_n} (f_n - f)d\mu\Big| \leq$$
$$\leq \sup_{t\in A\setminus C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A\setminus C_n)$$
$$+ \sup_{t\in A\cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \cap C_n) <$$
$$< \frac{\varepsilon}{6M'} \cdot M'$$
$$+ 2M \cdot \overline{\mu}(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon,$$

so $\lim\limits_{n\to\infty} \int_A (f_n - f)d\mu = \{0\}$ (with respect to h), for every $A \in \mathcal{A}$. □

Theorem 44 ***(Lebesgue type Theorem)*** *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ be a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f_n : T \to \mathbb{R}$, for every $n \in \mathbb{N}$. Suppose $(f_n)_n$ is an uniformly bounded sequence of μ-integrable functions such that $(f_n)_n$ is convergent in submeasure to a bounded function $f : T \to \mathbb{R}$. Then, f is μ -integrable on every $A \in \mathcal{A}$ and*

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu \text{ (with respect to } h).$$

Proof. By the proof of Theorem 43, it results that f is μ-integrable on every $A \in \mathcal{A}$. Using the same sets as before, we have for every $n \geq n_0$ and every $A \in \mathcal{A}$:

$$h(\int_A f_n d\mu, \int_A f d\mu)$$
$$= h(\int_{A\setminus C_n} f_n d\mu + \int_{A\cap C_n} f_n d\mu, \int_{A\setminus C_n} f d\mu$$
$$+ \int_{A\cap C_n} f d\mu) \leq h(\int_{A\setminus C_n} f_n d\mu, \int_{A\setminus C_n} f d\mu)$$
$$+ h(\int_{A\cap C_n} f_n d\mu, \int_{A\cap C_n} f d\mu) \leq$$
$$\leq \sup_{t\in A\setminus C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A\setminus C_n)$$
$$+ \sup_{t\in A\cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \cap C_n) < \frac{\varepsilon}{6M'} \cdot M'$$
$$+ 2M \cdot \overline{\mu}(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon,$$

and the conclusion follows. □

Theorem 45 ***(Fatou Lemma)*** *Suppose $\mathcal{A}$ is a σ-algebra, $\mu : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation so that $\widetilde{\mu}$ is o-continuous and $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded, $\widetilde{\mu}$-totally-measurable functions $f_n : T \to \mathbb{R}$. Then*

$$\int_T \liminf_n f_n d\mu \leq \liminf_n \int_T f_n d\mu.$$

Proof. For every $n \in \mathbb{N}$, consider g_n defined for every $t \in T$ by $g_n(t) = \inf\limits_{k\geq n} f_k(t)$. Let also be $f : T \to \mathbb{R}$, $f(t) = \lim\limits_{n\to\infty} g_n(t)$, for every $t \in T$. We observe that $g_n \overset{ae}{\to} f$ and $g_n \leq f_n$, for every $n \in \mathbb{N}$.

According to Theorem 24, $(g_n)_n$ is also a sequence of uniformly bounded, $\widetilde{\mu}$-totally-measurable functions, so, by Corollary 25, f is $\widetilde{\mu}$-totally-measurable on T.

By [17], f_n and f are μ-integrable on T, for every $n \in \mathbb{N}$.

Since $g_n \overset{ae}{\to} f$ and $\widetilde{\mu}$ is an o-continuous submeasure on $\mathcal{P}(T)$, then, according to Li [23], $g_n \overset{\widetilde{\mu}}{\to} f$, so, by [13],

$$\int_T \liminf_n f_n d\mu = \int_T f d\mu = \lim_{n\to\infty} \int_T g_n d\mu.$$

Consequently,

$$\int_T \liminf_n f_n d\mu = \liminf_n \int_T g_n d\mu$$
$$\leq \liminf_n \int_T f_n d\mu.$$

This completes the proof. □

References:

[1] G. Apreutesei, N.E. Mastorakis, A. Croitoru, A. Gavriluţ, On the translation of an almost linear topology, *WSEAS Transactions on Mathematics*, Vol.8, (2009), pp. 479–488.

[2] U. Bandyopadhyay, On vector measures with the Darboux property, *Quart. J. Oxford Math.*, (1974), pp. 57-61.

[3] I. Chiţescu, - *Finitely purely atomic measures and L_p-spaces*, An. Univ. Bucureşti Şt. Natur. 24 (1975), 23-29.

[4] I. Chiţescu, Finitely purely atomic measures: coincidence and rigidity properties, *Rend. Circ. Mat. Palermo*, vol. 50, (2001), No. 3, pp. 455-476.

[5] A. Croitoru, A. Gavriluţ, N. E. Mastorakis, Convergence theorems for totally-measurable functions, *WSEAS Trans. Math.*, Vol.8, (2009), p-p. 614–623.

[6] A. Croitoru, A. Gavriluţ,N.E. Mastorakis, G. Gavriluţ, On different types of non-additive set multifunctions, *WSEAS Trans. Math.*, Vol.8, (2009), pp. 246-257.

[7] L. Drewnowski, Topological rings of sets, continuous set functions, Integration, I, II, III, *Bull. Acad. Polon. Sci. Sér. Math. Astron. Phys.*, Vol. 20, (1972), pp. 269-276, pp. 277-286.

[8] L. Drewnowski,T. Luczak, On nonatomic submeasures on $\mathbb{N}$, *Arch. Math.*, vol. 91, (2008), pp. 76-85.

[9] L. Drewnowski, T. Luczak, On nonatomic submeasures on $\mathbb{N}$ II, *J. Math. Anal. Appl.*, vol.347, (2008), no 2, pp. 442-449.

[10] N. J. Fernández, – *Are diffused, Radon measures nonatomic?*, Papers in honor Pablo Bobillo Guerrero (Spanish), 261-265, Univ. Granada, 1992.

[11] R. J. Gardner, W. F. Pfeffer, Are diffused, regular, Radon measures σ-finite?, *J. London Math. Soc.*, vol.20, (1979), pp. 485-494.

[12] A. Gavriluţ, A Gould type integral with respect to a multisubmeasure, *Math. Slovaca*, vol.58, (2008), No. 1, pp. 1-20.

[13] A. Gavriluţ, On some properties of the Gould type integral with respect to a multisubmeasure, *An. Şt. Univ. Iaşi*, vol. 52, (2006), no 1, pp. 177-194.

[14] A. Gavriluţ, The general Gould type integral with respect to a multisubmeasure, *Math. Slovaca*, vol.60, (2010), No. 3, pp. 289-318.

[15] A. Gavriluţ, Non-atomicity and the Darboux property for fuzzy and non-fuzzy Borel/Baire multivalued set functions, *Fuzzy Sets and Systems*, vol.160, (2009), pp. 1308-1317.

[16] A. Gavriluţ, A. Croitoru, Non-atomicity for fuzzy and non-fuzzy multivalued set functions, *Fuzzy Sets and Systems*, vol.160, (2009), pp. 2106-2116.

[17] A. Gavriluţ, A. Petcu, A Gould type integral with respect to a submeasure, *An. Şt. Univ. Iaşi*, Tomul LIII, 2007, f. 2, pp. 351-368.

[18] A. Gavriluţ, A. Croitoru, N. E. Mastorakis, G. Gavriluţ, Measurability and Gould integrability in finitely purely atomic multisubmeasure spaces, *WSEAS Trans. Math.*, Vol.8, (2009), pp. 435-444.

[19] A. Gavriluţ, A. Croitoru, N. E. Mastorakis, Diffusion and semi-convexity of fuzzy set multifunctions, *WSEAS Trans. Math.*, vol.9, (2010), pp. 561-570.

[20] G. G. Gould, On integration of vector-valued measures, *Proc. London Math. Soc.*, vol.15, (1965), pp. 193-225.

[21] P. Holický, C. E. Weil, L. Zajiček, A note on the Darboux property of Fréchet derivatives, *Real Anal. Exchange*, vol.32, (2007), pp. 489-494.

[22] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis*, vol. I, Kluwer Acad. Publ., Dordrecht, 1997.

[23] J. Li, Order continuous of monotone set function and convergence of measureable functions sequence, *Applied Mathematics and Computation*, vol.135, (2003), pp. 211-218.

[24] J. Li, A note on the null-additivity of the fuzzy measure, *Fuzzy Sets and Systems*, vol.125, (2002), pp. 269-271.

[25] N.E. Mastorakis, A. Gavriluţ, A. Croitoru, G. Apreutesei, On Darboux property for fuzzy multimeasures, *Proceedings of the 10th WSEAS International Conference on Fuzzy Systems [FS'09]*, Prague, Czech Republic, March 23-25, 2009, pp. 54-58.

[26] E. Pap, - *Null-additive Set Functions*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[27] A. Petruşel, G. Moţ, – *Multivalued Analysis and Mathematical Economics,* House of the Book of Science, Cluj Napoca, 2004.

[28] A. Precupanu, A. Croitoru, A Gould type integral with respect to a multimeasure, I, *An. Şt. Univ. Iaşi*, vol.48, (2002), pp. 165-200.

[29] A. Precupanu, A. Gavriluţ, A. Croitoru, A fuzzy Gould type integral, *Fuzzy Sets and Systems*, vol.161, (2010), pp. 661-680.

[30] A. Precupanu, A. Gavriluţ, A. Croitoru, On a generalized Gould type set valued integral, submitted for publication.