On the Routh reduction of variational integrals. Part 1: The classical theory.

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Abstract: A geometrical approach to the reductions of one-dimensional first order variational integrals with respect to a Lie symmetry group is discussed. The method includes both the Routh reduction of cyclic variables and the Jacobi-Maupertuis reduction to the constant energy level. In full generality, it may be applied even to the Lagrange variational problems with higher order symmetries.

Key–Words: Variational integral, Poincaré-Cartan form, conservation law, Routh reduction, Jacobi-Maupertuis principle, infinitesimal symmetry, orbit space.

1 Introduction

Let us recall the universal aspect of symmetry reduction: a certain part of a symmetrical structure is represented by a *factorstructure without symmetries* on the orbit space. We will deal with the calculus of variations of one-dimensional integrals that admit the Lie symmetries. Then some particular cases of the reduction principle appear in classical mechanics: the Routh theorem on cyclic variables and the Jacobi-Maupertuis principle for constant energy systems [7], [2]. Warning: the symmetry reduction of Hamiltonian systems [3], [6] belong to quite different area of mathematics, namely to the symplectical geometry, and should not be confused with the reduction of variational integrals.

In more details, the classical calculus of variations consists of two ingredients, *the variational integral* and *the differential constraints* (represented by a system of differential equations or by a Pfaffian system) and so we simultaneously deal with two closely related reduction problems. Even in the subcase of trivial constraints (empty differential systems, all jets are admitted) the factorsystem may be rather involved. For the convenience of exposition, we begin with the point symmetries of first order variational integrals in this Part 1. Then the final result is well known [1] but we state a much better approach here. After this preparation, the reduction of higher order and constrained variational integrals will be quite analogously treated in subsequent Part 2. Alas, we shall see that the relatively simple generalization of Routh and Jacobi-Maupertuis results look as a mere lucky accident – the reduction of a *Lagrange variational problem need not* be a variational problem in the classical setting.

2 The Routh theorem anew

Let us consider the variational integral

$$\int \varphi$$

$$(\varphi = f(t, y, z, \dot{y}, \dot{z})dt,\ y = y(t),\ z = z(t),\ \dot{} = \frac{d}{dt})$$

and assume that the variable z is *cyclic* in the sense $\partial f/\partial z = 0$ hence $f = f(t, y, \dot{y}, \dot{z})$. Then the second equation of the *Euler-Lagrange system*

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right), \qquad 0 = \frac{\partial f}{\partial z} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right),$$

reads $\partial f/\partial \dot{z} = c$ ($c \in \mathbb{R}$). Assuming the *normal case* $\partial^2 f/\partial \dot{z}^2 \neq 0$, this can be resolved as $\dot{z} = g(t, y, \dot{z}, c)$ by using the implicit function theorem. Let us recall the Poincaré-Cartan form

$$\breve{\varphi} := f dt + \frac{\partial f}{\partial \dot{y}}(dy - \dot{y}dt) + \frac{\partial f}{\partial \dot{z}}(dz - \dot{z}dt)$$

and its restriction

$$\begin{aligned}\breve{\varphi}|_{\dot{z}=g} &= (f|_{\dot{z}=g} - cg)\, dt + \frac{\partial f}{\partial \dot{y}}|_{\dot{z}=g}(dy - \dot{y}dt) + cdz \\ &= \tilde{\varphi} + cdz\end{aligned}$$

to the level set $\dot{z} = g$ hence $\partial f/\partial \dot{z} = c$ (fixed $c \in \mathbb{R}$). Some direct calculation (or a theory bellow, see Lemma 1) implies that this $\tilde{\varphi}$ again is a Poincaré-Cartan form but for the variational problem restricted to the level set. It follows that extremals of the Routh variational integral

$$\int \tilde{\varphi} \qquad \left(\tilde{\varphi} := \tilde{f}dt,\ \tilde{f} = f|_{\dot{z}=g} - cg\right)$$

regarded on the level set (coordinates $t, y, \dot{y}$) are identical with those extremals of the original integral $\int \varphi$ which are lying in the given level set. This is *the Routh reduction*.

Unlike the common method based on direct introduction of *Routh function* $\tilde{f}$ and subsequent mechanical calculus, we have just indicated the crucial role of the Poincaré-Cartan form which will be systematically developed here.

3 Preliminaries

We limit ourselves to C^∞ smooth and local theory in the (universal for all considerations to follow) *infinite-order jet space* $M(m)$ with coordinates

$$x,\ w_r^i \qquad (i = 1, \ldots, m;\ r = 0, 1, \ldots) \tag{1}$$

equipped moreover with module $\Omega(m)$ of *contact forms*

$$\omega = \sum a_r^i \omega_r^i \qquad (\omega_r^i = dw_r^i - w_{r+1}^i dx,\ \text{finite sum}) \tag{2}$$

where coefficients a_r^i (and all functions to follow) depend C^∞-smoothly on a *finite* number of coordinates.

Assuming coordinates (1), we will deal only with x-parametrized curves in $\mathbb{M}(m)$. Other curves need an alternative choice of coordinates and contact forms related within (rather complicated) intertwining change on the overlap of coordinate systems. However, the *module* $\Omega(m)$ *makes the absolute sense*.

We recall the *total derivative* vector field

$$D := \frac{\partial}{\partial x} + \sum_{i=1}^{m} \sum_{r=1}^{\infty} w_{r+1}^i \frac{\partial}{\partial w_r^i} \quad \text{(infinite sum)}, \tag{3}$$

and the formula

$$\begin{aligned} da &= \frac{\partial a}{\partial x} dx + \sum_{i=1}^{m} \sum_{r \geq 0} \frac{\partial a}{\partial w_r^i} dw_r^i \\ &= Da\, dx + \sum_{i=1}^{m} \sum_{r \geq 0} \frac{\partial a}{\partial w_r^i} \omega_r^i \end{aligned} \tag{4}$$

valid for all functions a on $\mathbb{M}(m)$. This formula and the equations

$$\begin{gathered} \Omega(m)(D) = D \lrcorner \Omega(m) = 0, \\ d\omega_r^i = dx \wedge \omega_{r+1}^i,\ L_D \omega_r^i = \omega_{r+1}^i \end{gathered} \tag{5}$$

(the Lie derivative L) will be of frequent use.

We are interested in the first order variational integral

$$\int \varphi \tag{6}$$

$$(\varphi = f(x, w_0^1, \ldots, w_0^m, w_1^1, \ldots, w_1^m)dx,\ w_r^i = \frac{d^r w^i}{dx^r})$$

with the Poincaré-Cartan form

$$\breve{\varphi} := f dx + \sum_{i=1}^{m} \frac{\partial f}{\partial w_1^i} \omega_0^i. \tag{7}$$

Clearly $d\breve{\varphi} \sim \sum e^i \omega_0^i \wedge dx \pmod{\Omega(m) \wedge \Omega(m)}$. Here

$$e^i = \frac{\partial f}{\partial w_0^i} - D\frac{\partial f}{\partial w_1^i} \qquad (i = 1, \ldots, m) \tag{8}$$

are the *Euler-Lagrange coefficients*. A curve $\mathbb{P} : I \to \mathbb{M}(m)$ (where $I \subset \mathbb{R}$ is an interval) satisfying

$$\begin{gathered} \mathbb{P}^* w_{r+1}^i = \frac{d\mathbb{P}^* w_r^i}{d\mathbb{P}^* x}, \\ \mathbb{P}^* e^i = \mathbb{P}^* \frac{\partial f}{\partial w_0^i} - \frac{d\mathbb{P}^* \partial f/\partial w_1^i}{d\mathbb{P}^* x} = 0 \\ (i = 1, \ldots, m) \end{gathered} \tag{9}$$

is the classical *extremal*. In coordinate-free transcription, conditions (3) read

$$\mathbb{P}^*\Omega(m) = 0,\ \mathbb{P}^*(X \lrcorner d\breve{\varphi}) = 0 \quad \text{(all vector fields } X). \tag{10}$$

Recall that (3) implies the prolongation equations $\mathbb{P}^* D^r e^i = 0$ ($i = 1, \ldots, m; r = 0, 1, \ldots$) by applying identities (5) and (10).

Lemma 1. *A one-form $\psi = adx + \sum a^i dw_0^i$ is Poincaré-Cartan form of an integral (6) if and only if $d\psi \sim 0 \pmod{\omega_0^1, \ldots, \omega_0^m}$.*

Lemma 2. *A closed two-form Ψ is differential of a Poincaré-Cartan form (7) if and only if $\Psi \sim 0$ $\pmod{\omega_0^1, \ldots, \omega_0^m}$.*

Remark 3. Hint for proofs: *Clearly $\psi = fdx + \sum a^i \omega_0^i$ (where $f := a + \sum a^i dw_1^i$) and Lemma 1 directly follows by applying (5). Moreover $\Psi = d\gamma$ by using the Poincaré lemma. Here $d\gamma = \Psi \sim 0$ $(dx, dw_0^1, \ldots, dw_0^m)$ whence $\gamma = dg + \psi$ ($\psi = adx + \sum a^i dw_0^i$) again by the Poincaré lemma (with parameters $x, w_0^1, \ldots, w_0^m$). Therefore $\Psi = d\gamma = d\psi$ and Lemma 2 follows by applying Lemma 1.*

Definition 4. *A vector field*

$$Z = z\frac{\partial}{\partial x} + \sum_{i=1}^{m}\sum_{r=0}^{\infty} z_r^i \frac{\partial}{\partial w_r^i} \qquad (\textit{infinite sum}) \quad (11)$$

is called generalized infinitesimal symmetry *of the variational integral (6) if*

$$L_Z\Omega(m) \subset \Omega(m),\ L_Z\varphi \in \Omega(m)\ . \qquad (12)$$

In terms of coordinates, we have conditions

$$z_{r+1}^i = Dz_r^i - w_{r+1}^i Dz,\ Zf + fDz = 0 \qquad (13)$$

for the coefficients z, z_r^i. The second condition in (12) can be obviously replaced by the requirement $L_Z\breve{\varphi} \in \Omega(m)$ for the Poincaré-Cartan form.

Theorem 5 (Noether)**.** *Let Z be a generalized infinitesimal symmetry. Then $\breve{\varphi}(Z)$ is a conservation law, i.e. function $\mathbb{P}^*\breve{\varphi}(Z)$ is a constant for every extremal $\mathbb{P}$.*

Proof. We have

$$\begin{aligned} 0 = \mathbb{P}^*L_Z\breve{\varphi} &= \mathbb{P}^*(Z\lrcorner d\breve{\varphi} + d\breve{\varphi}(Z)) \\ &= \mathbb{P}^*d\breve{\varphi}(Z) = d\mathbb{P}^*\breve{\varphi}(Z), \end{aligned}$$

by virtue of (10) and (12). □

4 The pointwise symmetry

In accordance with classical setting, we shall deal with *pointwise vector field* (11) from now on. Any such field is defined by the requirement

$$z = z(x, w_0^1, \dots, w_0^m),\ z_0^i = z_0^i(x, w_0^1, \dots, w_0^m),$$
$$(i = 1, \dots, m)$$

for the coefficients. This ensures the existence of (local) zeroth-order first integrals

$$W^j = W^j(x, w_0^1, \dots, w_0^m), \qquad (j = 0, \dots, m-1);$$
$$\operatorname{rank}\left[\frac{\partial W^j}{\partial x}, \frac{\partial W^j}{\partial w_0^i}\right] = m \qquad (14)$$
$$(j = 0, \dots, m-1;\ i = 1, \dots, m),$$

where $ZW^j = 0$ ($j = 0, \dots, m-1$) by definition, and also the existence of a "complementary" function

$$W^m = W^m(x, w_0^1, \dots, w_0^m),\ ZW^m = 1\ . \qquad (15)$$

The somewhat strange choice of the range of indices will be soon clarified. One can observe that

$$\operatorname{rank}\left[\frac{\partial W^k}{\partial x}, \frac{\partial W^k}{\partial w_0^i}\right] = m+1 \qquad (16)$$
$$(k = 0, \dots, m;\ i = 1, \dots, m)\ .$$

Let the pointwise vector field (11) be moreover infinitesimal symmetry. Then

$$z_r^i = z_r^i(x, w_0^1, \dots, w_0^m, \dots, w_r^1, \dots, w_r^m)$$
$$(i = 1, \dots, m;\ r = 0, 1, \dots)$$

by virtue of recurrence (13) and there exist higher order first integrals. They can be obtained by the "prolongation" as follows. Assuming (13), one can verify the commutativity

$$[\mathscr{D}, Z] = 0 \qquad (17)$$
$$(\mathscr{D} := \frac{1}{DW}D,\ \text{where } ZW = const, DW \neq 0)$$

of vector fields $\mathscr{D}$, Z for every function W mentioned in (17). It follows that

$$W_r^i := \mathscr{D}^r W^i \qquad (i = 1, \dots, m;\ r = 1, 2, \dots)$$

are first integrals of order r. They are *not* functionally independent and we will continue with a somewhat tricky reasoning.

Due to (16), systems of zeroth-order functions are "equivalent", symbolically

$$\begin{aligned} \{x, w_0^1, \dots, w_0^m\} &\approx \{W^0, \dots, W^m\} \\ &= \{W_0^0, \dots, W_0^m\} \end{aligned}$$

in the sense that the left-hand functions may be represented as composed functions of the right-hand functions and conversely. (Alternatively speaking, the families of all composed functions $A(x, w_0^1, \dots, w_0^m)$ and $B(W_0^0, \dots, W_0^m)$ are identical.) On the first-order level, analogously

$$\{\mathscr{D}x, \mathscr{D}w_0^1, \dots, \mathscr{D}w_0^m\} \approx \{\mathscr{D}W_0^0, \dots, \mathscr{D}W_0^m\}$$

(modulo zeroth order functions, they are kept fixed for this moment) and moreover trivially

$$\begin{aligned} \{w_1^1, \dots, w_1^m\} &\approx \{\frac{1}{DW}, \frac{w_1^1}{DW}, \dots, \frac{w_1^m}{DW}\} \\ &= \{\mathscr{D}x, \mathscr{D}w_0^1, \dots, \mathscr{D}w_0^m\}\ . \end{aligned}$$

Altogether

$$\begin{aligned} \{w_1^1, \dots, w_1^m\} &\approx \{\mathscr{D}W_0^0, \dots, \mathscr{D}W_0^m\} \\ &= \{W_1^0, \dots, W_1^m\}\ . \end{aligned}$$

Continuing in this way, we obtain the equivalence

$$\{w_r^1, \dots, w_r^m\} \approx \{W_r^0, \dots, W_r^m\} \qquad (r = 1, 2, \dots)$$

modulo lower order functions. (Alternatively speaking, the families of all composed functions $A(w^1_r,\dots,w^m_r;w^1_0,\dots,w^m_{r-1})$ and $B(W^0_r,\dots,W^m_r;w^1_0,\dots,w^m_{r-1})$ are identical.)

Functions on the right-hand are *not* functionally independent, of course. Let us permanently chose $W = W^0(= W^0_0)$ in (17) from now on. Then the right-hand functions $W^0_1 = \mathscr{D}W^0 = 1$, $W^0_r = \mathscr{D}W^0_{r-1} = 0$ $(r > 1)$ may be omitted and therefore

$$\{w^1_r,\dots,w^m_r\} \approx \{W^1_r,\dots,W^m_r\} \qquad (18)$$
$$(r = 1, 2, \dots)\ .$$

The result can be rephrased as follows.

Lemma 6. *Functions*

$$W^0, W^i_r = \mathscr{D}^r W^i \qquad (19)$$
$$(i = 1,\dots,m;\ r = 0, 1, \dots;\ \mathscr{D} := \frac{1}{DW^0}D)$$

are functionally independent and may be taken for alternative coordinates on $\mathbb{M}(m)$. *All functions* (18) *except* $W^m = W^m_0$ *are first integrals.*

Proof. The Jacobi matrix of functions (19) is a lower block-triangular matrix with square blocks

$$A_0 = \left[\frac{\partial W^k_0}{\partial x}, \frac{\partial W^k_0}{\partial w^j_0},\right], A_r = \left[\frac{\partial W^i_r}{\partial w^j_r}\right]$$
$$(k = 0,\dots,m;\ i, j = 1,\dots,m;\ r = 1, 2, \dots)$$

on the diagonal. Then $\det A_0 \neq 0$ due to (16) and $\det A_r \neq 0$ due to (18). □

Corollary 7. *In terms of coordinates* (19)*, we have*

$$Z = \frac{\partial}{\partial W^m_0},\ \mathscr{D} = \frac{\partial}{\partial W^0} + \sum W^i_{r+1}\frac{\partial}{\partial W^i_r} \qquad (20)$$

and contact forms (2) *are represented as*

$$\omega = \sum A^i_r \Omega^i_r \qquad (\Omega^i_r = dW^i_r - W^i_{r+1}dW^0) \quad (21)$$

where

$$\Omega^i_r = \sum\left(\frac{\partial W^i_r}{\partial w^j_s} - W^i_{r+1}\frac{\partial W^0}{\partial w^j_s}\right)\omega^j_s,$$
$$A^i_r = \sum\left(\frac{\partial W^i_r}{\partial w^j_s} - W^i_{r+1}\frac{\partial W^0}{\partial w^j_s}\right)a^j_s\ .$$

We see that W^0 plays the role of the independent variable here and functions W^i_r correspond to original coordinates w^i_r. We again have a lower-triangular substitution.

We are passing to the factorobject, namely the orbit space. The vector field Z generates a local flow $\mathcal{F}^t_Z$. In more details

$$\frac{d}{dt}\mathcal{F}^t_Z(P) = Z(\mathcal{F}^t_Z(P)),\ \mathcal{F}^0_Z(P) = P$$
$$(P \in U, -\varepsilon < t < \varepsilon, \varepsilon > 0)$$

where $U \subset \mathbb{M}(m)$ is an appropriate open subset. Clearly

$$\mathcal{F}^{t*}_Z W^0 = W^0,\ \mathcal{F}^{t*}_Z W^i_r = W^i_r\ (i \neq m \text{ or } r \neq 0),$$
$$\mathcal{F}^{t*}_Z W^m_0 = W^m_0 + t,$$

and therefore $\mathcal{F}^{t*}_Z\Omega^i_r = \Omega^i_r$. Any point (locally) belongs to one orbit

$$\mathcal{F}_Z(P) := \{\mathcal{F}^t_Z(P) : -\varepsilon(P) < t < \varepsilon(P)\}\ .$$

We (locally, on a given $U \subset \mathbb{M}(m)$) introduce the *orbit space* $\mathbb{M}(m)_{orbU}$. This U will be kept fixed and systematically omitted for brevity. Then $\mathbb{M}(m)$ can be locally regarded as a fibered space with natural (local) projection

$$p : (U \subset)\mathbb{M}(m) \to \mathbb{M}(m)_{orb}, \qquad (22)$$
$$p : P \mapsto \mathcal{F}_Z(P)\ .$$

If V is a (local) function on the base space $\mathbb{M}(m)_{orb}$, then $W = p^*V$ is a first integral and conversely. In accordance with common convention, we formally identify $W = p^*V$: first integrals are simultaneously regarded as functions on the orbit space.

Corollary 8. *Functions*

$$W^0, W^j_r = \mathscr{D}^r W^j (j = 1,\dots,m-1; r = 0, 1, \dots),$$
$$W^m_s = \mathscr{D}^s W^m (s = 1, 2, \dots) \qquad (23)$$

provide coordinates on $\mathbb{M}(m)_{orb}$. *The vector field* $\mathscr{D}$ *is p-projectable and*

$$p_*\mathscr{D} = \frac{\partial}{\partial W^0} + \sum_{i=1}^{m}\sum_{r=0}^{\infty} W^j_{r+1}\frac{\partial}{\partial W^j_r} + \sum_{s=1}^{\infty} W^m_{s+1}\frac{\partial}{\partial W^m_s}\ . \qquad (24)$$

Contact forms Ω^i_r $(i \neq m$ *or* $r \neq 0)$ *also make good sense on* $\mathbb{M}(m)_{orb}$.

5 Symmetry of variational integral

The variational integral (6) was not taken into account as yet. Let us suppose (12) from now on.

Lemma 9. *Let* Z *be a pointwise infinitesimal symmetry and* $\breve{\varphi}$ *the Poincaré-Cartan form. Then* $L_Z\breve{\varphi} = 0$.

Proof. In accordance with (12) we put $L_Z\breve{\varphi} = \sum a_r^i\omega_r^i$. Then

$$dL_Z\breve{\varphi} \sim \sum(Da_r^i dx \wedge \omega_r^i + a_r^i dx \wedge \omega_{r+1}^i) \quad (\bmod \Omega(m)\wedge\Omega(m))$$

by applying (3) and (5). On the other hand clearly

$$L_Z d\breve{\varphi} \sim L_Z \sum e^i\omega_0^i \wedge dx \sim 0 \quad (\bmod \omega_0^1,\ldots,\omega_0^m,\ \Omega(m)\wedge\Omega(m)),$$

in the pointwise case. It follows that $a_r^i = 0$ identically. □

Lemma 10. *Functions*

$$G_{r+1} = \mathscr{D}^r G \qquad (r = 0,1,\ldots;\ G = G_1 = \breve{\varphi}(Z))$$

are first integrals.

Proof. Clearly

$$ZG = Z\breve{\varphi}(Z) = (L_Z\breve{\varphi})(Z) + \breve{\varphi}([Z,Z]) = 0$$

by virtue of Lemma 9. Then $ZG_{r+1} = Z\mathscr{D}G_r = \mathscr{D}ZG_r = 0$ $(r = 1,2,\ldots)$ by induction. □

Definition 11. *We speak of a* normal case *if coordinates W_s^m $(s = 1,2,\ldots)$ in the system (19) may be replaced by functions G_s $(s = 1,2,\ldots)$. In other words, we suppose that functions*

$$\begin{gathered} W^0, W_r^j = \mathscr{D}^r W^j \\ (j = 1,\ldots,m-1;\ r = 0,1,\ldots), \\ W_0^m, G_s\ (s = 1,2,\ldots) \end{gathered} \tag{25}$$

provide (local) coordinates on $\mathbb{M}(m)$.

Lemma 12. *The normal case takes place if and only if*

$$\sum_{i=1}^m\sum_{j=1}^m \frac{\partial^2 f}{\partial w_1^i\partial w_1^j}(z_0^i - w_1^i z)(z_0^j - w_1^j z) \neq 0. \tag{26}$$

Proof. The Jacobi matrix of the system (11) is lower block triangular matrix with the diagonal square blocks

$$A_0 = \left[\frac{\partial W_0^k}{\partial x}, \frac{\partial W_0^k}{\partial w_0^j}\right],\ B_r = \left[\frac{\partial W_r^i}{\partial w_r^j}, \frac{\partial G_r}{\partial w_r^j}\right],$$
$$(k = 0,\ldots,m;\ i = 1,\ldots,m-1;\ j = 1,\ldots,m).$$

Here $\det A_0 \neq 0$ and B_r differ from A_r in Lemma 6 only in the change $G_r \leftrightarrow W_r^m$. We have $\det A_r \neq 0$ whence $\operatorname{rank}[\partial W_r^i/\partial w_r^j] = m-1$. One can then see that

$$\frac{\partial W_r^i}{\partial w_r^j} = \frac{1}{(DW^0)^{r-1}}\frac{\partial W_1^i}{\partial w_1^j},\ \frac{\partial G_r}{\partial w_r^j} = \frac{1}{(DW^0)^{r-1}}\frac{\partial G_r}{\partial w_1^j}$$
$$(i,j = 1,\ldots m; r = 1,2,\ldots)$$

whence it is sufficient to prove that $\det B_1 \neq 0$. Using the explicit formulae

$$\frac{\partial W_1^i}{\partial w_1^j} = \frac{1}{(DW^0)^2}\left(\frac{\partial W^i}{\partial w_0^j}DW^0 - \frac{\partial W^0}{\partial w_0^j}DW^i\right)$$

identities

$$\sum \frac{\partial W_1^i}{\partial w_1^j}(z_0^j - w_1^j z) = 0 \quad (i = 1,\ldots,m-1)$$

follows by direct verification. Therefore $\det B_1 \neq 0$ is ensured if and only if

$$\sum \frac{\partial G}{\partial w_1^j}(z_0^j - w_1^j z) \neq 0\,.$$

However $G = \breve{\varphi}(Z) = fz + \sum \partial f/\partial w_1^i(z_0^i - w_1^i z)$, whence

$$\frac{\partial G}{\partial w_1^j} = \sum \frac{\partial^2 f}{\partial w_1^i\partial w_1^j}(z_0^i - w_1^i z)$$

and the proof is done. □

Corollary 13. *Functions*

$$\begin{gathered} W^0, W_r^j = \mathscr{D}^r W^j \quad (j = 1,\ldots,m-1;\ r = 0,1,\ldots), \\ G_s \quad (s = 1,2,\ldots) \end{gathered} \tag{27}$$

provide (local) coordinates on $\mathbb{M}(m)_{orb}$ in the normal case.

6 The restriction to subspaces

Definition 14. *Assuming the normal case, we introduce the subspaces $\mathbb{M}(m,c) \subset \mathbb{M}(m)$ defined by the conditions $G = G_1 = c$ $(c \in \mathbb{R})$, $G_r = 0$ $(r = 2,3,\ldots)$. By using the convention $G_s = p^*G_s$ $(s = 1,2,\ldots)$, we moreover introduce the subspace $\mathbb{M}(m,c)_{orb} \subset \mathbb{M}(M)_{orb}$ defined by the same conditions. In other words, the subspace $\mathbb{M}(m,c)_{orb} \subset \mathbb{M}(m)_{orb}$ consists of all orbits lying in the subspace $\mathbb{M}(m,c) \subset \mathbb{M}(m)$.*

Functions

$$W^0, W_r^j \quad (j = 1,\ldots,m-1;\ r = 0,1,\ldots),\ W^m \tag{28}$$

may be taken for coordinates on $\mathbb{M}(m,c)$. Functions

$$W^0, W^j_r \quad (j = 1, \ldots, m-1;\ r = 0, 1, \ldots), \quad (29)$$

provide coordinates on $\mathbb{M}(m,c)_{orb}$. It follows that vector fields $D, \mathscr{D}$ and Z are tangent to the subspace $\mathbb{M}(m,c) \subset \mathbb{M}(m)$, i.e., they may be applied to the functions defined only on $\mathbb{M}(m,c)$. The projection $p_*\mathscr{D}$ is analogously tangent to the subspace $\mathbb{M}(m,c)_{orb} \subset \mathbb{M}(m)_{orb}$.

As the contact forms Ω^j_r are concerned, they provide the contact module $\Omega(m-1)$ on the space $\mathbb{M}(m,c)_{orb}$. Every extremal is locally contained in a certain subspace $\mathbb{M}(m,c) \subset \mathbb{M}(m)$, however, the variational problem *cannot* be restricted to this subspace due to the "bad" coordinate W^m. Nevertheless, restriction to the orbit space $\mathbb{M}(m,c)_{orb}$ is reasonable as follows.

Theorem 15 (Routh). *Let a variational integral*

$$\int \varphi$$
$$(\varphi = f(x, w^1_0, \ldots, w^m_0, w^1_1, \ldots, w^m_1)dx,$$
$$w^i_r := \frac{d^r w^i_0}{dx^r})$$

admits a pointwise infinitesimal symmetry

$$Z = z\frac{\partial}{\partial x} + \sum_{i=1}^{m}\sum_{r=0}^{\infty} z^i_r \frac{\partial}{\partial w^i_r}$$
$$(z = z(x, w^1_0, \ldots, w^m_0), z^i_0 = z^i_0(x, w^1_0, \ldots, w^m_0))$$

such that

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f}{\partial w^i_1 \partial w^j_1}(z^i_0 - w^i_1 z)(z^j_0 - w^j_1 z) \neq 0.$$

Let $c \in \mathbb{R}$ *and a function* $W^m = W^m(x, w^1_0, \ldots, w^m_0)$ *satisfies* $ZW^m = 1$. *Then the variational integral*

$$\int \tilde\varphi \qquad (\tilde\varphi = (f - cDW^m)dx|_{G=c}), \qquad (30)$$

where

$$G = fz + \sum_{i=1}^{m} \frac{\partial f}{\partial w^i_1}(z^i_0 - w^i_1 z),$$

may be interpreted as a variational integral on the space $\mathbb{M}(m,c)_{orb}$. *The extremals of the integral* $\int \tilde\varphi$ *are just the natural projections of those extremals of the primary integral* $\int \varphi$ *which are lying in the subspace* $\mathbb{M}(m,c) \subset \mathbb{M}(m)$.

Proposition 16. *A certain function* $\tilde F$ *on* $\mathbb{M}(m,c)_{orb}$ *exists such that*

$$\tilde\varphi \sim \tilde F(W^0, W^1_0, \ldots, W^{m-1}_0, W^1_1, \ldots, W^{m-1}_1)dW^0 \qquad (31)$$
$$(\bmod\ \Omega^1_0, \ldots, \Omega^{m-1}_0)$$

and therefore the equality $\int \tilde\varphi = \int \tilde F dW^0$ *of the variational integrals is true.*

Proposition 17. *Let* $\breve\varphi$ *and* $\bar\varphi$ *be the Poincaré-Cartan forms of the variational integrals* $\int \varphi$ *and* $\int \tilde\varphi$, *respectively. Then*

$$i^*\breve\varphi = p^*\bar\varphi + cdW^m,\ i^*d\breve\varphi = p^*d\bar\varphi, \qquad (32)$$

where $i^* : \mathbb{M}(m,c) \subset \mathbb{M}(m)$ *is the natural inclusion and* $p : \mathbb{M}(m,c) \to \mathbb{M}(m,c)_{orb}$ *is the natural projection.*

Proof. The Poincaré-Cartan form (7) expressed in alternative coordinates (19) clearly satisfies

$$\breve\varphi = AdW^0 + \sum A^i dW^i_0,\ d\breve\varphi \sim 0$$
$$(\bmod\ \Omega^1_0, \ldots, \Omega^m_0)\ .$$

Applying Lemma 1 it follows that this $\breve\varphi$ is a Poincaré-Cartan form again, that is,

$$\breve\varphi = FdW^0 + \sum \frac{\partial F}{\partial W^i_1}\Omega^i_0, \qquad (FDW^0 = f),$$

even when it is expressed in alternative coordinates (19). It follows that

$$d\breve\varphi \sim \sum E^i \Omega^i_0 \wedge dW^0$$
$$(E^i = \frac{\partial F}{\partial W^i_0} - \mathscr{D}\frac{\partial F}{\partial W^i_1};\ i = 1, \ldots, m)\ .$$

Using (20) and (13) in coordinates (19), we have

$$ZF = \frac{\partial F}{\partial W^m_0} = 0,$$
$$G = \breve\varphi(Z) = \frac{\partial F}{\partial W^m_1}, E^m|_{G=c} = -\mathscr{D}c = 0\ .$$

Let us consider the restriction

$$i^*\breve\varphi = F|_{G=c}dW^0 + \sum^{m-1} \frac{\partial F}{\partial W^i_1}|_{G=c}\Omega^i_0$$
$$+ c(dW^m_0 - W^m_1|_{G=c}dW^0)$$
$$=: \tilde\varphi + cdW^m_0\ .$$

Here

$$\bar{\varphi} := (F - cW_1^m)|_{G=c} dW^0 + \sum^{m-1} \frac{\partial F}{\partial W_1^i}|_{G=c}\Omega_0^i$$
$$= \tilde{F} dW^0 + \sum^{m-1} \frac{\partial \tilde{F}}{\partial W_1^i}\Omega_0^i$$
$$(\tilde{F} := (F - cW_1^m)|_{G=c})$$

is a Poincaré-Cartan form. To see it it is sufficient to apply Lemma 1 to the form $\breve{\varphi} = \tilde{\varphi} = p^*\bar{\varphi}$ on the space $\mathbb{M}(m,c)_{orb}$. Then

$$\tilde{F} dW^0 \sim (F - cW_1^m) DW^0 dx|_{G=c}$$
$$= (f - cDW^m) dx|_{G=c}$$

modulo contact forms which gives (30), we use here $W_1^m = \mathscr{D}W_0^m = DW^m/DW^0$. Moreover

$$d\bar{\varphi} = i^* d\breve{\varphi} \sim \sum^m E^i \Omega_0^i|_{G=c} \wedge dW^0$$
$$= \sum^{m-1} E^i|_{G=c}\Omega_0^i \wedge dW^0 \quad (\mathrm{mod}\ \Omega(m) \wedge \Omega(m))$$

in term of coordinates (29). The Euler-Lagrange system of integrals $\int \varphi$ reads $E^i = 0$ $(i = 1, \ldots, m-1)$, $E^m = \mathscr{D}G = 0$ (hence $G = c$) while the Euler-Lagrange system of the reduced integral $\int \bar{\varphi}$ is $E^i|_{G=c} = 0$ $(i = 1, \ldots, m-1)$ and therefore it provides the natural projection of original extremals. The proof is done. □

7 On the abelian symmetry group

We will study the case of a Lie algebra $\mathscr{G}$ of pointwise symmetries. For this aim, let us introduce the vector fields

$$Z(k) = z(k)\frac{\partial}{\partial x} + \sum z(k)_r^i \frac{\partial}{\partial w_r^i} \quad (k = 1, \ldots, K) \tag{33}$$

$$z(k) = z(k)(x, w_0^1, \ldots, w_0^m),$$
$$z(k)_r^i = z(k)_r^i (w_0^1, \ldots, w_0^m),$$

for the generators of the algebra $\mathscr{G}$ with identities

$$[Z(l), Z(k)] = \sum c_{lk}^r Z(r) \tag{34}$$
$$(l, k = 1, \ldots, K;\ c_{lk}^r \in \mathbb{R})$$

We moreover suppose the symmetry requirements

$$L_{Z(k)}\Omega(m) \subset \Omega(m),\ L_{Z(k)}\varphi \in \Omega(m) \tag{35}$$
$$(k = 1, \ldots, K)$$

Then $L_{Z(k)}\breve{\varphi} = 0$ and, denoting $G(k) := \breve{\varphi}(Z(k))$, clearly

$$Z(l)G(k) = (L_{Z(l)}\breve{\varphi})(Z(k)) + \breve{\varphi}([Z(l), Z(k)])$$
$$= \sum c_{lk}^r G(r)\ . \tag{36}$$

Therefore functions $G(r)$ are first integrals of vector fields (33) in the *abelian subcase* $c_{lk}^r = 0$. Let us suppose $c_{lk}^r = 0$ identically from now on. Then there exist zeroth-order first integrals and "complementary" functions

$$W^j = W^j(x, w_0^1, \ldots, w_0^m)$$
$$(Z(k)W^j = 0;\ j = 0, \ldots, m-K;\ k = 1, \ldots, K)$$
$$W^{m-K+k} = W^{m-K+k}(x, w_0^1, \ldots, w_0^m)$$
$$(Z(l)W^{m-K+k} = \delta_l^k;\ k, l = 1, \ldots, K)$$

(where $\delta_l^k = 0$ if $k \neq l$, $\delta_k^k = 1$) satisfying (16). Moreover the functions

$$W_r^i = \mathscr{D}^r W^i$$
$$(i = 0, \ldots, m;\ r = 1, 2, \ldots;$$
$$\mathscr{D} := D/DW;\ Z(k)W = const(k))$$

are first integrals of order r for every function W mentioned. They are *not* functionally independent. Let us permanently chose $W = W^0$ from now on. Then Lemma 6 holds true without any change. Every vector field $Z(k)$ $(k = 1, \ldots, K)$ generates a local flow $\mathcal{F}_{Z(k)}^t$ and (due to Frobenius theorem) we obtain the orbit space $\mathbb{M}(m)_{orb}$ for the Lie algebra $\mathscr{G}$ with K-dimensional fibers, the orbits of $\mathscr{G}$. Functions

$$W^0 = W_0^0,\ W_r^j = D^r W^j$$
$$(j = 1, \ldots, m-K;\ r = 0, 1, \ldots),$$
$$W_s^{m-K+k} = D^s W^{m-K+k}$$
$$(k = 1, \ldots, K;\ s = 1, 2, \ldots),$$

provide (local) coordinates on $\mathbb{M}(m)_{orb}$.

Definition 18. *We speak of a* normal case *if functions W_s^{m-K+k} $(k = 1, \ldots, K;\ s = 1, 2, \ldots)$ in the above coordinate systems may be replaced by functions $G(k)_s = \mathscr{D}^s G(k)$.*

Definition 19. *Assuming the normal case, we introduce the subspaces $\mathbb{M}(m,c) \subset \mathbb{M}(m)$ $(c = (c(1), \ldots, c(K)) \in \mathbb{R}^K)$ defined by the conditions*

$$G_1(k) = G(k) = c(k)\ (k = 1, \ldots, K),$$
$$G(k)_r = 0\ (r = 2, 3, \ldots)$$

and the subspaces $M(m,c)_{orb} \subset \mathbb{M}(m)_{orb}$ $(c = (c(1), \ldots, c(K)) \in \mathbb{R}^K)$ formally defined by the same conditions.

Functions

$$W^0, W^j_r \ (j = 1, \ldots, m-K;\ r = 0, 1, \ldots),$$
$$W^{m-K+k} \ (k = 1, \ldots, K)$$

may be taken for coordinates on $\mathbb{M}(m, c)$. Functions

$$W^0, W^j_r \ (j = 1, \ldots, m-K;\ r = 0, 1, \ldots),$$

provide coordinates on $\mathbb{M}(m, c)_{orb}$. Vector fields $D, \mathscr{D}$ and $Z(k)$ $(k = 1, \ldots, K)$ are tangent to the subspace $M(m, c) \subset \mathbb{M}(m)$ and (the projection of) $\mathscr{D}$ is tangent to the subspace $\mathbb{M}(m, c)_{orb} \subset \mathbb{M}(m)_{orb}$. Contact forms Ω^j_r $(j = 1, \ldots, m-K;\ r = 0, 1, \ldots)$ make good sense on $\mathbb{M}(m)_{orb}$ and generate the contact module $\Omega(m-K)$.

Theorem 20 (Routh). *Let a variational integral*

$$\int \varphi \tag{37}$$

$$(\varphi = f(x, w^1_0, \ldots, w^m_0, w^1_1, \ldots, w^m_1)dx,\ w^i_r := \frac{d^r w^i_0}{dx^r})$$

admits an abelian Lie group $\mathscr{G}$ of infinitesimal pointwise symmetries with generators (33) *such that*

$$\det\Big[\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f}{\partial w^i_1 \partial w^j_1}(z(l)^i_0 - w^i_1 z(l)) \times \times (z(k)^j_0 - w^j_1 z(k))\Big] \neq 0 \tag{38}$$

$(k, l = 1, \ldots, K)$. *Let* $c = (c(1), \ldots, c(K)) \in \mathbb{R}^K$ *and functions* $W^{m-K+k} = W^{m-K+k}(x, w^1_0, \ldots, w^m_0)$ *satisfy the system*

$$Z(l)W^{m-K+k} = \begin{cases} 0 & (l \neq k) \\ 1 & (l = k)\ . \end{cases} \tag{39}$$

Then the variational integral

$$\int \tilde{\varphi} \tag{40}$$

$$(\tilde{\varphi} := (f - \sum c(k) D W^{m-K+k})dx)|_{G(1)=c(1),\ldots,G(K)=c(K)}$$

may be interpreted as a variational integral on the space $\mathbb{M}(m, c)_{orb}$. Moreover the extremals of the integral $\int \tilde{\varphi}$ are just the natural projections of those extremals of the primary integral (37) *which are lying in the subspace $\mathbb{M}(m, c) \subset \mathbb{M}(m)$.*

Proof. The proof closely follows the proof of Theorem 15 and may be omitted. □

8 A retrospective and perspectives

We have discussed a very particular reduction problem, the first order variational integral without differential constraints. Some special tools, especially the contact forms ω^i_r and Ω^i_r, were advantageously employed in order to simplify the reasoning. Let us therefore comment the true conceptual mechanisms of our procedure, which are latently present.

1. *Underlying space.* The jet space $\mathbb{M}(m)$ equipped with the contact module $\Omega(m)$ and the classical jet coordinates x, w^i_r are well-known. They correspond to the trivial differential constraints, the jet coordinates are free.

2. *Variational integral* $\int \varphi$ together with the contact module $\Omega(m)$ determines the variational problem. The first order integrals are thoroughly investigated in all textbooks.

3. *Poincaré-Cartan form* $\breve{\varphi}$ in the total jet space $\mathbb{M}(m)$ is a classical tool as well.

4. *Infinitesimal symmetry* Z together with $\breve{\varphi}$ immediately gives the conservation law $\breve{\varphi}(Z) = const$ and first integrals $G_r = \mathscr{D}^r \breve{\varphi}(Z)$. We obtain the fibration $\mathbb{M}(m, c) \subset \mathbb{M}(m)$ $(c \in \mathbb{R})$ with fibers the level sets. Every extremal is obviously lying in a certain level set.

5. *The pointwise case* of Z ensures the existence of large supply of first integrals and therefore the existence of alternative coordinates W^i_r. All but $W^m = W^m_0$ are first integrals of Z and provide coordinates on the orbit space $\mathbb{M}(m)_{orb}$.

6. *The normal case* ensures that the natural inclusions $i = i(c) : \mathbb{M}(m, c) \subset \mathbb{M}(m)$ (depending on $c \in \mathbb{R}$) of leaves are correctly related to the alternative coordinates W^i_r. We obtain module of Pfaffian forms $i^*\Omega(m)$ and the variational integral $\int i^*\breve{\varphi}$ on $\mathbb{M}(m, c)$. Alas the "exceptional" coordinate $W^m = W^m_0$ causes some difficulties since $i^*\mathscr{D}W^m = i^*W^m_1$ is *not* included into the coordinates on $\mathbb{M}(m, c)$ and therefore $i^*\Omega^m_0$ cannot be regarded as a "free" contact form.

7. *The Routh correction.* It follows that $i^*\breve{\varphi}$ is *not* a Poincaré-Cartan form on $\mathbb{M}(m, c)$, however, the form $di^*\breve{\varphi} = i^*d\breve{\varphi}$ is *differential* of a certain Poincaré-Cartan form $\tilde{\varphi}$ given by the correction (32). (Instead of explicit formula (32), the *existence* of correction $\tilde{\varphi}$ can be proved in coordinate-free manner by using Lemma 2.) Moreover this $\tilde{\varphi}$ is independent of the "poor" coordinate $W^m = W^m_0$ and therefore make good

sense on the space $\mathbb{M}(m,c)_{orb}$ of orbits contained in the leaf $\mathbb{M}(m,c)$.

8. *Final result.* The primary variational integral $\int\varphi$ on $\mathbb{M}(m)$ is reduced to the integral $\int\tilde\varphi$ on the space $\mathbb{M}(m,c)_{orb}$ depending on parameter $c\in\mathbb{R}$. The differential constraints remain trivial: the space $\mathbb{M}(m,c)_{orb}$ is equipped with contact forms $\Omega^i_r = dW^i_r - W^i_{r+1}dW^0$ $(i=1,\ldots,m-1;\ r=0,1,\ldots)$ where $W^0 = W^0_0$ stands for the new independent variable.

We intend to deal with the reduction on the general Lagrange problems in subsequent Part 2. Then the previous points will be adjusted as follow.

1. *Underlying space.* Differential constraints given by an undetermined system of differential equations is given. We prefer the *internal approach*: the system will be introduced as an infinite Pfaffian system Ω in a space $\mathbb{M}$ without any use of jet theory.

2. *Variational integral* $\int\varphi$ is given by a one-form φ on $\mathbb{M}$ and together with Ω determine the variational problem (the *Lagrange problem*) in a coordinate free manner.

3. *Poincaré-Cartan* form $\breve\varphi$ in the space $\mathbb{M}$ can be introduced and this provides the Euler-Lagrange system without any additional variables ([4]).

4. *Infinitesimal symmetry* Z determines the conservative law $\breve\varphi(Z) = const$ in the primary underlying space $\mathbb{M}$ exactly as above.

5. *The pointwise case.* The existence of many first integrals of the vector field Z is ensured if Z preserves a finite-dimensional subspace of $\mathbb{M}$ ([5]).

At this stage, all necessary technical tools for the subsequent generalized points 6)–8) are available. Alas, the extremals lying in the leaf $\mathbb{M}(m,c)\subset\mathbb{M}(m)$ analogous as above *cannot be* in general identified with *all* extremals of a variational problem. Roughly speaking, the Routh reduction of a Lagrange problem *need not* be a variational problem in the classical sense.

9 Examples

We conclude with several simple applications of general results

Example 9.1 (The multiparameter Routh theorem). Let us deal with integral (6) that admits infinitesimal symmetries

$$Z(k) = \frac{\partial}{\partial w^k_0}\quad (k=1,\ldots,K),\quad 1\le K\le m-1\ .$$

It follows that

$$f = f(x, w^{K+1}_0,\ldots,w^m_0, w^1_1,\ldots,w^m_1)$$

therefore $w^1_0,\ldots,w^K_0$ are "cyclic variables". We have the conservation laws

$$G(k) = \breve\varphi(Z(k)) = \frac{\partial f}{\partial w^k_1} = c(k)\quad (k=1,\ldots,K),$$

moreover the zeroth-order first integrals and "complementary" functions

$$W^0 = x,\ W^j = w^{K+j}_0\quad (j=1,\ldots,m-K),$$
$$W^{m-K+k} = w^k_0$$
$$(Z(l)W^{m-K+k} = \delta^k_l;\ k,l=1,\ldots,K)\ .$$

Theorem 20 may be applied. Assuming the normality $\det(\sum\partial^2 f/\partial w^i_1\partial w^j_1)\neq 0$ we have the Routh variational integral

$$\int\tilde\varphi$$
$$\left(\tilde\varphi := f - \sum c(k) w^k_1|_{G(1)=c(1),\ldots,G(K)=c(K)}dx\right)$$

on the space $\mathbb{M}(m,c)_{orb}$. Recall that functions

$$x,\ W^j_s := D^sW^j = w^{K+j}_s$$
$$(j=1,\ldots,m-K;\ s=0,1,\ldots)$$

provide coordinates on $\mathbb{M}(m,c)_{orb}$. The Routh classical theorem appears if $K=1$.

Example 9.2 (On the Jacobi-Maupertuis principle). We continue with integral (6) that admits the infinitesimal symmetry

$$Z = \frac{\partial}{\partial x} + \sum a^i\frac{\partial}{\partial w^i_0}\quad (a^i\in\mathbb{R})\ .$$

We have the conservation law

$$G = \breve\varphi(Z) = f + \sum\frac{\partial f}{\partial w^i_1}(a^i - w^i_1) = c,$$

moreover the zeroth-order first integrals and "complementary" function

$$W^j := w^{j+1}_0 - a^{j+1}x\ (j=0,\ldots,m-1),$$
$$W^m := x\ (ZW^m = 1)\ .$$

Theorem 15 may be applied if the normality condition

$$\sum\frac{\partial^2 f}{\partial w^i_1\partial w^j_1}(a^i-w^i_1)(a^j-w^j_1)\neq 0$$

is satisfied. We obtain the Routh variational integral

$$\int \tilde{\varphi} \quad (\tilde{\varphi} := (f-c)dx|_{G=c} = \tilde{F}dW^0)$$

on the space $\mathbb{M}(m,c)_{orb}$. Recall that functions

$$W^0 := w_0^1 - a^1 x,$$
$$W_r^i := \mathscr{D}^r W^i = \left(\frac{D}{w_1^1 - a^1}\right)^r (w_0^i - a^i x)$$
$$(i = 1, \dots, m-1)$$

may be taken as coordinates on $\mathbb{M}(m,c)_{orb}$.

For better clarity, let us directly verify that $\int \tilde{\varphi}$ is indeed defined on the orbitspace. Infinitesimal invariance of variational integral is ensured if and only if

$$f dx = A(W^0, \dots, W^{m-1}, w_1^1, \dots, w_1^m) dx$$

in terms of first integrals. Alternatively, this may be expressed as

$$f dx \sim$$
$$B(W^0, \dots, W^{m-1}, w_1^1 - a^1, \dots, w_1^m - a^m) \frac{dW^0}{DW^0}$$
$$(DW^0 = w_1^1 - a^1)$$

modulo contact forms, briefly

$$f dx \sim B(W^0, \dots, W^{m-1}, t^1, \dots, t^m) \frac{dW^0}{t^1}$$
$$(t^i = w_1^i - a^i)\ .$$

This is restricted to the level set $G = c$ or, explicitly, we suppose

$$f - c = \sum \frac{\partial f}{\partial w_1^i}(w_1^i - a^i) = \sum \frac{\partial B}{\partial t^i} t^i\ .$$

It follows that we have a first order homogeneous function on the level set:

$$f dx \sim C(W^0, \dots, W^{m-1}, \frac{t^2}{t^1}, \dots, \frac{t^m}{t^1}) dW^0\ .$$

However $t^i/t^1 = DW^i/DW^0 = dW^i/dW^0$ $(i = 2, \dots, m)$ may be interpreted as the derivative with respect to the (new) independent variable W^0. Altogether

$$\int f dx = \int \tilde{F}(W_0^0, \dots, W_0^{m-1}, W_1^1, \dots, W_1^m) dW^0$$
$$(\tilde{F} = C)$$

in the jet notation and we are done – this is integral on the orbit space.

In the particular case $a^1 = \cdots = a^m = 0$, $f = T - V$ with the kinetic energy T and the potential energy V, we obtain the classical Jacobi-Maupertuis theorem [1], [2] of reduction to the constant energy $G = H = c$ (the Hamiltonian function).

Example 9.3 (A non-abelian symmetry group). The variational integral

$$\int F(\tfrac{1}{2}(x^2+y^2+z^2), x\dot{x}+y\dot{y}+z\dot{z}, \tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)\, dt$$

admits the four-dimensional Lie algebra $\mathscr{G}$ of pointwise symmetries generated by vector fields

$$Z(1) = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + \cdots\ ,$$
$$Z(2) = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} + \cdots\ ,$$
$$Z(3) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \cdots\ ,$$
$$Z(4) = \frac{\partial}{\partial t},$$

namely the Lie algebra of the orthogonal group in the space x, y, z completed with time shifts. There are two-dimensional abelian Lie subalgebras of $\mathscr{G}$ with generators

$$AZ(1) + BZ(2) + CZ(3),$$
$$(\text{fixed } A, B, C \in \mathbb{R}; |A| + |B| + |C| \neq 0),\ Z(4)$$

that provide a Routh reduction (depending on parameters A,B,C) by applying Theorem 20

In order to avoid clumsy formulae, we will mention only the case of the function $F = F((\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2)$ and the particular case $B = C = 0$ of the abelian symmetry subalgebra with abbreviations

$$Z = Z(1) = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} - \dot{z}\frac{\partial}{\partial \dot{y}} + \dot{y}\frac{\partial}{\partial \dot{z}} + \cdots$$
$$T = Z(4) = \frac{\partial}{\partial t}\ .$$

Then

$$\breve{\varphi} = Fdt + F'(\dot{x}(dx - \dot{x}dt) + \dot{y}(dy - \dot{y}dt) + \dot{z}(dz - \dot{z}dt))$$

is the Poincaré-Cartan form and we obtain two conservation laws

$$\breve{\varphi}(Z) = (y\dot{z} - z\dot{y})F' = c(1),$$
$$\breve{\varphi}(T) = F - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)F = c(2) \qquad (41)$$

by applying the Noether theorem. All first integrals of vector fields Z, T are quite simple: the zeroth order functions x, $y^2 + z^2$ and their prolongations

$$\mathscr{D}^r x,\ \mathscr{D}^r(y^2+z^2) \quad (r = 0, 1, \dots;$$
$$\mathscr{D} = D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \cdots)$$

and moreover all functions

$$\mathscr{D}^r W \quad \left(W = \arctan\frac{z}{y};\ r = 1, 2, \dots \right)$$

as follows from $ZW = 1$, $TW = 0$ and the commutativity $[\mathscr{D}, Z] = [\mathscr{D}, T] = 0$. Recall that

$$\mathscr{D}^r \breve{\varphi}(Z),\ \mathscr{D}^r \breve{\varphi}(T) \quad (r = 0, 1, \dots)$$

are first integrals, too. If they may be included into coordinates on the orbit space, we have the normal case. We will not state the (rather clumsy) normality requirement (20) here. (Roughly speaking, it is satisfied on an open dense set for all nonconstant functions F.)

It follows that Theorem 20 with $K = 2$ may be applied. We may choose

$$W^{m-K+1} := \arctan\frac{z}{y},\ W^{m-K+2} := t$$
$$(m = 3, K = 2)$$

for the functions (39) and then the form

$$\tilde{\varphi} = (F - c(1)\mathscr{D}\arctan\frac{z}{y} - c(2)\mathscr{D}t)dt$$
$$= \left(F - c(1)\frac{y\dot{z} - z\dot{y}}{y^2+z^2} - c(2) \right) dt$$

determines the Routh integral (40). Recall that it is considered on the orbit space, i.e., under the restriction (41).

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