LQ-Moments: An Alternative to Standard L- and TL-Moments

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Abstract: The method of LQ-moments represents one of many alternatives to established population parameter estimation techniques. It is used in applied research in such fields as construction, meteorology or hydrology. The present paper focuses on the use of LQ-moments in economics, specifically in wage distribution modelling. The aim of the study is to highlight the advantages of this approach over other methods of estimating the parameters of continuous probability distributions (i.e. those of L- and TL-moments), the theoretical probability distribution being represented by three-parameter lognormal curves. The calculation of sample LQ-moments and identification of the statistical characteristics (those of level, variability, skewness and kurtosis) of a continuous probability distribution are also an integral part of the paper.

Key-Words: LQ-moments, probability distribution, normal distribution, lognormal distribution, large sample theory, comparison with L-moments and TL-moments

1 Introduction
The point estimation of parameters remains a widely discussed issue in the statistical literature, linear quantile (LQ) moments representing a more robust alternative to well-established methods of linear (L) and trimmed linear (TL) moments. Mudholkar & Hutson (1998), for example, introduce LQ-moments as analogues of L-moments acquired by replacing the expectations by functionals inducing the median, Gastwirth estimator and trimean. The same estimators are dealt with by Shabri & Jemain (2006a, 2006b) who develop extended and improved class of LQ-moments that do not impose restrictions on the values of \( p \) and \( \alpha \), their combinations lying within the range 0–0.5. Respectively, they design a weighted kernel estimator for quantile function estimation and conduct Monte Carlo simulations to check the performance of the proposed estimators of the three-parameter lognormal distribution. Shabri & Jemain (2010) also adapt the method of LQ-moments for a four-parameter kappa distribution considered as a combination of generalized distributions. Šimková & Picek (2017) derive L-, LQ- and TL-moments of generalized Pareto and extreme-value distributions up to the fourth order, using the first three moments to obtain estimators of their parameters. Performing a simulation study, they compare high-quantile estimates based on L-, LQ-, and TL-moments with the maximum likelihood estimate in terms of their respective sample mean squared errors. Ashour, El-Sheik & Abu El-Magd (2015) derive both TL-and LQ-moments of the exponentiated Pareto distribution, applying them to estimate the unknown parameters. In addition to distribution classification and model selection criteria, Mudholkar & Natarajan (2002) deal with L and LQ measures of skewness and kurtosis, noting that the former measures occur only in the case of finite expectation distributions. David & Nagaraja (2003) also consider measurements of probability distributions and some quick parameter estimators. Abu El-Magd (2010) obtains TL- and LQ-moments of the exponentiated generalized extreme value distribution and utilizes them to estimate the unknown parameters, dealing with some specific cases such as L-, LH- and LL-moments. Deka, Borah & Kakaty, (2009) determine the best-fitting distribution to describe annual time series of maximum daily rainfall data from nine measuring stations in North-East India for the period 1966–2007. GEV, generalized logistic, generalized Pareto, lognormal and Pearson distributions are fitted for this purpose employing L- and LQ-moments. Zin,

2 LQ-Moments

The method of L-moments – linear functions of the expected values in order statistics – has been widely used in different areas of applied research such as construction, hydrology and meteorology. Its LQ variant is addressed in the present study. LQ-moments are obtained by substituting the expected values in order statistics – has been widely discussed in the statistical literature. Other applications such as probability density functionals generating common skewness and kurtosis measurements based on LQ-moments represent more appropriate and efficient alternatives to standard beta coefficients, the asymptotic distribution of LQ estimators proving their effective simplicity. Their application, particularly in hydrological analysis of extreme flood data values, is discussed in the statistical literature. Other potential uses are outlined in this paper.

2.1 LQ-Moments of Probability Distribution

Let \( X_1, X_2, \ldots, X_n \) be a random sample from a continuous distribution with distribution and quantile functions \( F_X(\cdot) \) and \( Q_X(u) = F_X^{-1}(u) \), respectively, \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) representing order statistics. Then the \( r \)-th L-moment \( \lambda_r \) is given as

\[
\lambda_r = r^{-1} \sum_{k=1}^{n} (-1)^{k} \binom{r-1}{k} E(X_{r-k:n}), \quad r = 1, 2, \ldots
\]  

(1)

Analogously, we define the \( r \)-th LQ-moment \( \xi_r \) as

\[
\xi_r = r^{-1} \sum_{k=1}^{n} (-1)^{k} \binom{r-1}{k} \tau_{\alpha}(X_{k:n}), \quad r = 1, 2, \ldots
\]  

(2)

where \( 0 \leq \alpha \leq \frac{1}{2}, 0 \leq p \leq \frac{1}{2} \) and

\[
\tau_{\alpha}(X_{k:n}) = pQ_{X_{k:n}}(\alpha) + (1 - 2p)Q_{X_{k:n}}(1/2) + pQ_{X_{k:n}}(1 - \alpha).
\]  

(3)

It is evident from equations (1) and (2) that the expected value \( E(\cdot) \) at point \( \tau_{\alpha}(\cdot) \) in the latter equation defines L-moments. Another generalization of L-moments, including the replacement of the expected value in equation (1), is possible by TL-moments.

The linear combination \( \tau_{\alpha} \), defined by equation (3) is a quick measure of the level of the random distribution of the order statistics \( X_{r-k:n} \). Candidates for \( \tau_{\alpha} \) include functionals generating common quick estimators –

median

\[
Q_{X_{r-1:n}} \left( \frac{1}{2} \right),
\]  

(4)

trimean

\[
Q_{X_{r-1:n}} \left( \frac{1}{4} \right) + Q_{X_{r-1:n}} \left( \frac{1}{2} \right) + Q_{X_{r-1:n}} \left( \frac{3}{4} \right),
\]  

(5)

Gastwirth

\[
0.3Q_{X_{r-2:n}} \left( \frac{1}{3} \right) + 0.4Q_{X_{r-2:n}} \left( \frac{1}{2} \right) + 0.3Q_{X_{r-2:n}} \left( \frac{2}{3} \right).
\]  

(6)

When sampling from a normal distribution, the LQ-Gastwirth estimator is the most efficient considering the possibilities given by equations (4)–(6). The following four LQ-moments of the random variable \( X \) are commonly used in practical applications such as probability density classification and parameter estimation

\[
\xi_1 = \tau_{\alpha}(X),
\]  

(7)

\[
\xi_2 = \frac{1}{2} [\tau_{\alpha}(X_{2:n}) - \tau_{\alpha}(X_{1:n})],
\]  

(8)

\[
\xi_3 = \frac{1}{3} [\tau_{\alpha}(X_{3:n}) - 2 \tau_{\alpha}(X_{2:n}) + \tau_{\alpha}(X_{1:n})],
\]  

(9)

\[
\xi_4 = \frac{1}{4} [\tau_{\alpha}(X_{4:n}) - 3 \tau_{\alpha}(X_{3:n}) + 3 \tau_{\alpha}(X_{2:n}) - \tau_{\alpha}(X_{1:n})].
\]  

(10)
It is obvious that location measures \( \tau_{p,a}(\cdot) \) exist for any random variable \( X \). Therefore, the \( r \)-th LQ-moment always exists, and it is unique if the distribution function \( F_X(\cdot) \) is continuous. Moreover, the evaluation of LQ-moments of any continuous distribution can be simplified if the following applies: \( Q_x(\cdot) = F_X^{-1}(\cdot) \) being the quantile function of a random variable \( X \), a quick measure of the level defined by equation (3) is equivalent to the equation

\[
\tau_{p,a}(X_{r,a}) = pQ_x[B_{r-k}^{-1}(\alpha)] + (1 - 2p)Q_x[B_{r-k}^{-1}(1/2)] + pQ_x[B_{r-k}^{-1}(1 - \alpha)],
\]

where \( B_{r-k}^{-1}(\alpha) \) denotes the corresponding \( \alpha \)-th quantile of a beta distributed random variable with parameters \( r - k \) and \( k + 1 \).

When constructing appropriate distribution models and estimating parameters, the coefficients of skewness and kurtosis \( \beta_1 \) and \( \beta_2 \), respectively, play an important role in terms of the classification of statistical distributions. Due to their drawbacks, however, alternative measures of skewness and kurtosis are used, including relatively recent ones such as

\[
\tau_3(F) = [F^{-1}(1 - u) + F^{-1}(u) - 2m_{p}][F^{-1}(1 - u) + F^{-1}(u)].
\]

Quantile-based measures of kurtosis for symmetric distributions include

\[
\frac{|Q(0,75 + u) + Q(0,75 - u) - 2Q(0,75)|}{|Q(0,75 + u) - Q(0,75 - u)|}, 0 \leq u < 1/4
\]

and

\[
\frac{|Q(0.5 + u) - Q(0.5 - u)|}{|Q(0.75) - Q(0.25)|}, 0 \leq u < 1/2.
\]

L-moment-based ratios \( \tau_3 \) and \( \tau_4 \) – L-skewness and L-kurtosis, respectively – are defined as

\[
\tau_3 = \frac{\lambda_3}{\lambda_2} \quad \text{and} \quad \tau_4 = \frac{\lambda_4}{\lambda_2},
\]

offering an alternative to \( \sqrt{\beta_1} \) and \( \beta_2 \). It is proved that \( \tau_3 \) meets the convex arrangement, \( \tau_4 \) maintaining van Zwet's symmetric ordering.

The skewness and kurtosis measures \( \eta_3 \) and \( \eta_4 \) based on LQ-moments – LQ-skewness and LQ-kurtosis – are defined as

\[
\eta_3 = \frac{\xi_3}{\xi_2} \quad \text{and} \quad \eta_4 = \frac{\xi_4}{\xi_2}.
\]

It is necessary to note that both LQ-skewness and LQ-kurtosis exist for all distributions and are invariant in terms of location and scale. However, other analogous properties of \( \tau_3 \) and \( \tau_4 \) mentioned above remain unexplored for LQ-skewness \( \eta_3 \) and LQ-kurtosis \( \eta_4 \), their behaviour being now more thoroughly analysed.

Another ratio measurement useful for comparing distributions with the usual origin and scale is an analogy of the coefficient of variation

\[
\gamma = \frac{s}{\bar{x}}
\]

where \( \xi_1 \) and \( \xi_2 \) are represented by equations (7) and (8). When modelling survival data, it is common practice to plot \( \sqrt{\beta_1} \) against the sample coefficient of variation

\[
\tilde{\gamma} = \frac{s}{\bar{x}}
\]

in the \((\sqrt{\beta_1}, \gamma)\) plane to verify the model selection.

### 2.2 Sample LQ-Moments

LQ-moments can be estimated directly by estimating the quantiles of order statistics in combination with equation (11). The simplest quantile estimator suitable for this purpose is the one based on linear interpolation, available in standard statistical software packages. However, alternative estimators of quantiles can be used as well.

Let \( X_{1,a} \leq X_{2,a} \leq \ldots \leq X_{a,a} \) be the sample order statistics. The quantile estimator \( Q(u) \) is then given by
\[ \hat{Q}_X(u) = (1 - u) X_{(n'u)_n} + u X_{(n'u)_{n+1}}, \]  
(16)

where \( u = n' u - \lfloor n' u \rfloor \) and \( n' = n + 1 \).

For random samples of sample size \( n \), the \( r \)-th sample LQ-moment is expressed by the relationship

\[ \hat{\xi}_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \frac{(r-1)_k}{k!} \hat{\tau}_{p,a}(X_{r-k}, r), \quad r = 1, 2, \ldots, \]  
(17)

where \( \hat{\tau}_{p,a}(X_{r-k}, r) \) is a quick estimator of the distribution of the order statistic \( X_{r-k} \) in a random sample of size \( r \).

Specifically, the first four sample LQ-moments from equation (17) are given as

\[ \hat{\xi}_1 = \hat{\tau}_{p,a}(X), \]  
(18)

\[ \hat{\xi}_2 = \frac{1}{2} [\hat{\tau}_{p,a}(X_{22}) - \hat{\tau}_{p,a}(X_{13})], \]  
(19)

\[ \hat{\xi}_3 = \frac{1}{3} [\hat{\tau}_{p,a}(X_{13}) - 2 \hat{\tau}_{p,a}(X_{23}) + \hat{\tau}_{p,a}(X_{13})], \]  
(20)

\[ \hat{\xi}_4 = \frac{1}{4} [\hat{\tau}_{p,a}(X_{24}) - 3 \hat{\tau}_{p,a}(X_{24}) + 3 \hat{\tau}_{p,a}(X_{24}) - \hat{\tau}_{p,a}(X_{13})], \]  
(21)

where the quick estimator \( \hat{\tau}_{p,a}(X_{r-k}, r) \) of the level of order statistics \( X_{r-k}, r \) is defined by the relationship

\[ \hat{\tau}_{p,a}(X_{r-k}, r) = p \hat{Q}_{\frac{r}{r-k+1}}(\alpha) + (1 - 2 p) \hat{Q}_{\frac{r}{r-k+1}} \left( \frac{1}{2} \right) + p \hat{Q}_{\frac{1}{r-k+1}} (1 - \alpha), \]  
(22)

where \( 0 \leq \alpha \leq \frac{1}{2}, 0 \leq p \leq \frac{1}{2} \), \( B_{r-k+1}^{-1}(\alpha) \) is the \( \alpha \)-th quantile of the random variable with a beta distribution with parameters \( r - k \) a \( k + 1 \), and \( \hat{Q}_X(\cdot) \) denotes an estimator using linear interpolation given by equation (16). The calculation of sample LQ-moment \( \hat{\xi}_r \) is as follows

\[ \hat{\xi}_r = \frac{1}{4} [\hat{\tau}_{p,a}(X_{24}) - 3 \hat{\tau}_{p,a}(X_{24}) + 3 \hat{\tau}_{p,a}(X_{24}) - \hat{\tau}_{p,a}(X_{13})], \]

being simplified using quantile \( B_{r-k+1}^{-1}(\alpha) \) that can be easily obtained from statistical spreadsheets.

Explicit schemes for the calculation of LQ-moments are presented, the three quick estimators – median \((p = 0, \alpha = \cdot)\), trimean \((p = 1/4, \alpha = 1/4)\) and Gastwirth \((p = 0.3, \alpha = 1/3)\) – are used for \( \hat{\tau}_{p,a}(X_{r-k}, r) \) given by equation (22). The calculation of the first four sample LQ-moments from equation (17) is simplified using pyramid schemes.

Sample LQ-skewness and LQ-kurtosis

\[ \hat{\eta}_1 = \frac{\hat{\xi}_1}{\hat{\xi}_2}, \quad \hat{\eta}_2 = \frac{\hat{\xi}_2}{\hat{\xi}_3}. \]  
(23)

can be used to identify \( \eta_3 \) and \( \eta_4 \) and to estimate parameters.

### 2.3 Large Sample Theory

Sample LQ-moments depend on the choice of quick and quantile estimators, their asymptotic normality, however, being consistent with the theory of linear order statistics of large samples. In order to develop expressions for the large sample mean and variance of sample LQ-moments, we shall limit ourselves to the Q class of quantile functions \( Q \) meeting the following conditions:

- the inverse function \( Q_X(u) = F_X^{-1}(u) \) is defined exclusively for \( 0 < u < 1 \);
- \( Q(\cdot) \) is twice differentiable on the interval \((0, 1)\) with a continuous second derivative \( Q''(\cdot) \) on the same interval;
- \( Q(\cdot) > 1 \) for \( 0 < u < 1 \).

Let us consider \( 0 < u_1 < u_2 < \ldots < u_k < 1 \), assuming the above conditions (1)–(3) are fulfilled. Then

\[ [\hat{Q}(u_1), \hat{Q}(u_2), \ldots, \hat{Q}(u_k)] \]

is asymptotically normal with a vector of expected values \([Q(u_1), Q(u_2), \ldots, Q(u_k)]\) as well as with covariances

\[ \sigma_{ij} = \text{Cov} \{\hat{Q}(U_i), \hat{Q}(U_j)\} = \]  
(24)

\[ = u_i (1 - u_j) \frac{Q'(u_i)Q'(u_j)}{n}, \quad i \leq j, \sigma_{ii} = \sigma_{jj}. \]
To create asymptotic expressions for covariances of LQ-moments, we will first obtain

\[
\text{Cov}[\hat{\xi}_{p,a}(X_{r+k}), \hat{\xi}_{p,a}(X_{s+l})],
\]

which is a function dependent on six specific percentiles \(u_1, u_2, \ldots, u_6\) used to derive equation (24) rewritten into a set of six percentiles

\[
B_{r-k}^{-1}(\alpha), B_{s-l}^{-1}(\alpha), B_{r-k}^{-1}(1/2), B_{s-l}^{-1}(1/2),
\]

\[
B_{r-k}^{-1}(1-\alpha), B_{s-l}^{-1}(1-\alpha),
\]

so that \(0 < u_1 < u_2 < \ldots < u_6 < 1\), where \(B_{r-k}^{-1}(\alpha)\) represents the \(\alpha\)-th quantile of a random variable with a beta distribution with parameters \(r-k\) and \(k+1\). Then, we can get \(\text{Cov}[\hat{Q}(U_r), \hat{Q}(U_s)]\).

The covariance between the estimated quick estimators of order statistics is defined as

\[
\text{Cov}[\hat{\xi}_{p,a}(X_{r+k}), \hat{\xi}_{p,a}(X_{s+l})] =
\]

\[
= p(p\text{Cov}[\hat{Q}(u_1), \hat{Q}(u_2)] + (1-2p)\text{Cov}[\hat{Q}(u_2), \hat{Q}(u_1)]) +
\]

\[
+ p\text{Cov}[\hat{Q}(u_3), \hat{Q}(u_1)] + p\text{Cov}[\hat{Q}(u_1), \hat{Q}(u_3)] +
\]

\[
+ (1-2p)\text{Cov}[\hat{Q}(u_1), \hat{Q}(u_3)] + p\text{Cov}[\hat{Q}(u_3), \hat{Q}(u_1)] +
\]

\[
+ (1-2p)\{p\text{Cov}[\hat{Q}(u_2), \hat{Q}(u_1)] + (1-2p)\text{Cov}[\hat{Q}(u_1), \hat{Q}(u_2)] +
\]

\[
+ p\text{Cov}[\hat{Q}(u_1), \hat{Q}(u_2)]\}.
\]

The \(r\)-th sample LQ-moment

\[
\hat{\xi}_r, \ r=1,2,\ldots,
\]

has an asymptotically normal distribution with expected value \(\hat{\xi}_r\). For \(r \leq s\), covariances of LQ-moments are given by equation

\[
\text{Cov}[\hat{\xi}_{p,a}(X_{r+k}), \hat{\xi}_{p,a}(X_{s+l})],
\]

where \(\text{Cov}[\hat{\xi}_{p,a}(X_{r+k}), \hat{\xi}_{p,a}(X_{s+l})]\) is described by equation (25) and \(u_1, u_2, \ldots, u_6\) are specified above. For \(r = s\), we obtain the variance of the \(r\)-th sample LQ-moment

\[
\hat{\xi}_r.
\]

As \(n \to \infty\), sample measures of LQ-skewness \(\hat{\eta}_b\) and LQ-kurtosis \(\hat{\eta}_4\)

have a two-dimensional normal distribution with the vector of expected values \((\eta_1, \eta_4)\) and

\[
\text{Var}(\hat{\eta}_b) = \text{Var}(\hat{\xi}_b) / \hat{\xi}_b^2,
\]

\[
\text{Cov}(\hat{\eta}_b, \hat{\eta}_4) = [\text{Cov}(\hat{\xi}_b, \hat{\xi}_4) - \hat{\xi}_b \text{Var}(\hat{\xi}_b)] / \hat{\xi}_4^2,
\]

\[
\text{Var}(\hat{\eta}_4) = \text{Var}(\hat{\xi}_4) / \hat{\xi}_4^2,
\]

where

\[
\text{Var}(\hat{\xi}_b) = \text{Cov}(\hat{\xi}_b, \hat{\xi}_b)
\]

and variances and covariances indicate the right side of the equation (26).

### 2.4 Application to Normal Distribution

We consider a random sample from a normal distribution and compare the use of the median, trimean and Gastwirth estimators when estimating LQ-skewness and LQ-kurtosis. Then the estimators

\[
\hat{\eta}_b
\]

and

\[
\hat{\eta}_4
\]

given by equation (23) have a common normal distribution with the corresponding expected value vectors

\[
(0; 0,116), (0; 0,118) \text{ and } (0; 0,117)
\]

and covariance matrices

\[
\begin{bmatrix}
0 & 0.116 \\
0.116 & 0
\end{bmatrix} 
\]
We can see from equations (31) that $$\hat{\eta}_3$$ and $$\hat{\eta}_4$$ are asymptotically uncorrelated for each of the above-mentioned quick estimators. It is also obvious that we prefer Gastwirth estimator to median and trimean ones in terms of skewness and kurtosis estimation in the case of large samples from an (almost) normal distribution.

2.5 Application to Lognormal Distribution

LQ estimators for the three-parameter lognormal distribution behave similarly to L-moment estimators. We get the following expressions for LQ-moments of the above distribution from equations (7)–(9) and (13)

$$\hat{\xi}_1 = \theta + \exp(\hat{\mu}) \tau_{p,a}(X_{11}),$$

$$\hat{\xi}_2 = \frac{1}{2} \exp(\hat{\mu}) \{ \tau_{p,a}(X_{22}) - \tau_{p,a}(X_{12}) \},$$

The LQ-skewness coefficient can be calculated using

$$\hat{\eta}_3 = \frac{1}{2} \left[ \tau_{p,a}(X_{33}) - 2 \tau_{p,a}(X_{23}) + \tau_{p,a}(X_{13}) \right] - \frac{1}{2} \left[ \tau_{p,a}(X_{22}) - \tau_{p,a}(X_{12}) \right].$$

LQ-parameter estimators $$\hat{\mu}, \hat{\sigma}$$ and $$\hat{\theta}$$ represent the solution of equations (7)–(9) in combination with equations (32)–(34) for $$\mu, \sigma$$ and $$\theta$$, where we replace $$\xi_r$$ with $$\hat{\xi}_r$$.

Conducting regression analysis, we obtain the following approximate relationship, allowing for estimation of $$\hat{\sigma}$$

$$\hat{\sigma} = 2.1684\hat{\eta}_1 + 0.3967\hat{\eta}_3 + 0.1744\hat{\eta}_4 - 0.1015\hat{\eta}_2.$$

Once we get the value $$\hat{\sigma}$$, we can also obtain estimates $$\hat{\mu}$$ and $$\hat{\theta}$$ using equations (33) and (32).

2.6 Appropriateness of the Model

It is also necessary to assess the suitability of the constructed model or choose another model from several alternatives, applying a criterion which can be the sum of absolute deviations of the observed and theoretical frequencies for all intervals

$$S = \sum_{i=1}^{k} |n_i - n \pi_i|,$$  \hspace{1cm} \text{(36)}

or criterion $$\chi^2$$

$$\chi^2 = \frac{\sum_{i=1}^{k} (n_i - n \pi_i)^2}{n \pi_i},$$  \hspace{1cm} \text{(37)}

where $$n_i$$ are the observed frequencies in individual intervals, $$\pi_i$$ are the theoretical probabilities of statistical unit membership in the $$i$$-th interval, $$n$$ is the total sample size of the corresponding statistical file, $$n \cdot \pi_i$$ are theoretical frequencies in individual intervals, $$i = 1, 2, ..., k$$, and $$k$$ is the number of intervals.

The appropriateness of the wage distribution curve is not a general mathematical-statistical issue for testing the null hypothesis $H_0$: the sample coming from the supposed theoretical distribution,

against the alternative hypothesis $H_1$: non $H_0$,

because in wage distribution goodness of fit testing we often work with large samples, tests usually leading to the null hypothesis rejection. This results
not only from the lower test power at a given significance level but also from the test construction itself. The smallest distribution deviations revealed by the test being of no practical significance, the general consistency between the model and reality proves sufficient for us to “borrow” the wage distribution curve, allowing for the tentative use of the $\chi^2$ test criterion. Relying on experience and logical insights, however, the model suitability assessment remains to a large extent subjective.

3 Results and Discussion

The research database – consisting of employees who were working in the Czech Republic over the period 2009–2016 – is broken down by various demographic and socio-economic factors. The research variable is the gross monthly (nominal) wage (in CZK). Data were drawn from the official website of the Czech Statistical Office. They are in the form of the interval frequency distribution (a total of 328 wage distributions) since the data on individual employees are not available.

Tables 1–3 present parameter estimates obtained using the three methods of point parameter estimation and the $S$-criterion. Generally, the method of LQ-moments yielded the best results, deviations occurring mainly at both ends of the wage distribution due to extreme open intervals. With respect to total wage distribution sets, LQ-moments always give the most accurate outcomes in terms of the $S$-criterion. In the research of all 328 wage distributions, the method of TL-moments produced the second most accurate results in more than half of the cases, deviations occurring again especially at both ends of the distribution. The above tables indicate that TL-moments brought the second most accurate results in terms of all total sets of wage distributions for the Czech Republic in the period 2009–2016, the method of L-moments yielding the third most accurate outcomes in most cases.

Table 1: Parameter estimates obtained using LQ-moments and $S$-criterion values for total wage distribution in the Czech Republic

<table>
<thead>
<tr>
<th>Year</th>
<th>Parameter estimation</th>
<th>$\mu$</th>
<th>$\sigma^2$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td></td>
<td>9.059 747</td>
<td>0.630 754</td>
<td>9.065 52</td>
</tr>
<tr>
<td>2010</td>
<td></td>
<td>9.215 324</td>
<td>0.581 251</td>
<td>8.552 10</td>
</tr>
<tr>
<td>2011</td>
<td></td>
<td>9.277 248</td>
<td>0.573 002</td>
<td>8.872 54</td>
</tr>
<tr>
<td>2012</td>
<td></td>
<td>9.313 803</td>
<td>0.577 726</td>
<td>9.382 66</td>
</tr>
<tr>
<td>2013</td>
<td></td>
<td>9.382 135</td>
<td>0.680 571</td>
<td>10.027 84</td>
</tr>
<tr>
<td>2014</td>
<td></td>
<td>9.438 936</td>
<td>0.688 668</td>
<td>10.898 39</td>
</tr>
<tr>
<td>2015</td>
<td></td>
<td>9.444 217</td>
<td>0.703 536</td>
<td>10.640 53</td>
</tr>
<tr>
<td>2016</td>
<td></td>
<td>9.482 060</td>
<td>0.681 258</td>
<td>10.616 80</td>
</tr>
</tbody>
</table>

Table 2: Parameter estimates obtained using TL-moments and $S$-criterion values for total wage distribution in the Czech Republic

<table>
<thead>
<tr>
<th>Year</th>
<th>Parameter estimation</th>
<th>$\mu$</th>
<th>$\sigma^2$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td></td>
<td>9.017 534</td>
<td>0.608 369</td>
<td>7.664 46</td>
</tr>
<tr>
<td>2010</td>
<td></td>
<td>9.241 235</td>
<td>0.507 676</td>
<td>6.541 16</td>
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<tr>
<td>2011</td>
<td></td>
<td>9.283 399</td>
<td>0.515 290</td>
<td>6.977 45</td>
</tr>
<tr>
<td>2012</td>
<td></td>
<td>9.283 883</td>
<td>0.543 225</td>
<td>7.868 21</td>
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<td>2013</td>
<td></td>
<td>9.387 739</td>
<td>0.601 135</td>
<td>7.902 64</td>
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<td></td>
<td>9.423 053</td>
<td>0.624 340</td>
<td>8.754 64</td>
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<tr>
<td>2015</td>
<td></td>
<td>9.431 478</td>
<td>0.631 013</td>
<td>8.684 51</td>
</tr>
<tr>
<td>2016</td>
<td></td>
<td>9.453 027</td>
<td>0.621 057</td>
<td>8.746 20</td>
</tr>
</tbody>
</table>

Source: Own research
Table 3: Parameter estimates obtained using L-moments and S-criterion values for total wage distribution in the Czech Republic

<table>
<thead>
<tr>
<th>Year</th>
<th>Parameter estimation</th>
<th>S-criterion</th>
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<tr>
<td></td>
<td>µ</td>
<td>σ²</td>
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<tr>
<td>2009</td>
<td>9.741305</td>
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<td>2010</td>
<td>9.780008</td>
<td>0.232406</td>
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<td>2012</td>
<td>9.890594</td>
<td>0.210672</td>
</tr>
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<td>2013</td>
<td>9.950263</td>
<td>0.268224</td>
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<td>2014</td>
<td>10.017433</td>
<td>0.264124</td>
</tr>
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<td>2015</td>
<td>10.019787</td>
<td>0.269047</td>
</tr>
<tr>
<td>2016</td>
<td>10.033810</td>
<td>0.269895</td>
</tr>
</tbody>
</table>

Figure 1: Development of sample and theoretical median of three-parameter lognormal curves with parameters estimated using different methods of estimation

Figure 2: Development of probability density function of three-parameter lognormal curves with parameters estimated using LQ-moments

Figure 3: Development of probability density function of three-parameter lognormal curves with parameters estimated using TL-moments
Figures 2–4 illustrate the development of the probability density function of three-parameter lognormal curves with parameters estimated by all three methods of LQ-, TL- and L-moments, again representing the course of model distributions of total wages earned by all employees in the Czech Republic between 2009 and 2016. We can see that the shapes of lognormal curves with parameters estimated using TL- and L-moments are similar to each other, while being markedly different from those whose parameters were estimated by the method of LQ-moments (cf. Figs. 3, 4 and 2, respectively).

4 Conclusion
The present paper deals with an alternative moment analysis of probability distributions. The method of LQ-moments is compared with those of L- and TL-moments, particularly in terms of their parameter estimation accuracy, using the sum of all absolute deviations of the observed and theoretical frequencies for all intervals as a criterion. The higher accuracy of LQ-moments approach was proved by examining the set of 328 wage distributions, advantages of TL-moments compared to L-moments being also confirmed. The values of the $\chi^2$ criterion having been calculated for each wage distribution, the test always led to the rejection of the zero hypothesis about the supposed shape of the wage distribution because of the typically large sample sizes.

References:
[3] Bhuyan, A., & Borah, M., LQ-Moments for Regional Flood Frequency Analysis: A Case Study for the North-Bank Region of the


**Acknowledgement:**

This paper was subsidized by the funds of institutional support of a long-term conceptual advancement of science and research (no. IP400040) at the Faculty of Informatics and Statistics, University of Economics, Prague, Czech Republic.