

Robust Admissibilization of Descriptor LPV Systems based on SOF

ADDISON RIOS-BOLIVAR

Universidad de Los Andes
Departamento de Sistemas de Control
Núcleo La Hechicera, Mérida 5101
VENEZUELA
ilich@ula.ve

FRANCKLIN RIVAS

Universidad Técnica "Federico Santa María"
Departamento de Informática
Valparaíso, Valparaíso, 2340000
CHILE
firivas@inf.utfsm.cl

Abstract: The contribution of this work is focused on the analysis and synthesis of controllers based on static output feedback (SOF), for a class of descriptor linear parameter variable (LPV) systems. Descriptors systems, called also: differential-algebraic systems, singular systems, semi-state systems or generalized state-space systems; are considered to possess disturbances and parametric uncertainties of polytopic type, as are described by the following equation:

$$\mathbb{E}\dot{z}(t) = \mathcal{F}(\rho)z(t) + \mathcal{B}(\rho)\omega(t) + Bu(t), \quad h(t) = \mathcal{C}(\rho)z(t),$$

where ρ is a parametric variation. From a condition of existence of a linear injective application, representing the generalized inverse matrix of \mathbb{E} , the original descriptor system is transformed to a LPV system. Then, the condition for the static output feedback on the LPV system is analyzed. Synthesis of the SOF-based controller is obtained considering performance indices in \mathcal{H}_2 and \mathcal{H}_∞ , described as linear matrix inequalities, LMIs, as criteria in order to obtain the gain of SOF, in the presence of uncertainties and disturbances.

Key-Words: Descriptor systems. LPV systems. Static output feedback (SOF). \mathcal{H}_2 - \mathcal{H}_∞ norms. Robust control.

1 Introduction

Since its introduction in 1977 [15], descriptor systems (DS), also called singular systems, semi-state systems, differential-algebraic systems or generalized state-space systems; have been one of the main research fields within control theory, since they are a natural and general representation of dynamic systems. Unlike their regular counterparts in state space, a DS allows a representation that incorporates algebraic constraints in their physical variables. Over the past two decades, descriptor systems have attracted much attention because of the comprehensive uses in many real world systems, such as in the economy (Leontief dynamic model), social models, electrical systems, chemical processes, and mechanical models (robotics). Considerable progress has since been made in the investigation of such systems. A problem that has been well studied is the admissibility of DS, being a research line that still remains open.

On the other hand, the context of linear parameter variable (LPV) systems refers to linear dynamical systems whose state-space representations depend on exogenous non-stationary parameters [24]. LPV systems are a generalization of LTV systems, establishing an intermediate model between linear and nonlinear dynamics, so they can be constituted in a representative model for the control of non-linear processes,

allowing the use of all machinery of control of linear systems to the case of particular nonlinear processes control [5]. In addition, if the nonlinear model is formulated as a parameterized linear system, where parameterization is state dependent, it allows an LPV description to represent a non-local nonlinear system, taking advantage of the consequences of a global stabilization [11, 5]. Thus, the LPV representation of a nonlinear system describes a class of systems larger than the original nonlinear system.

When there are combined the modeling of physical systems with uncertain parameters, there arise dynamic systems that represent uncertain DS. As is well known, for modeling many applications and technical processes, only approximate models are available, so that the analysis of DS subject to uncertainties has been a very active research line. For example, numerous analysis and synthesis problems have been addressed in the literature: the analysis of robust stability (admissibility), stabilization, analysis of the robust controllability and observability, robust control under the characterization of the \mathcal{H}_∞ - \mathcal{H}_2 norms, robust filtering, analysis and positive real control, among other lines of work, [25, 27, 10]. The main results in the analysis and synthesis of DS-dependent parameters are based on parametric Lyapunov functions, which allow to minimize the conservatism of classic Lyapunov functions, when searching numerical solutions

through LMIs, representing a formulation that allows the resolution of complicated control problems very efficiently, and with a remarkable degree of simplicity [7].

In this context, this paper addresses the analysis of robust admissibility and control for an DS class of continuous time and with polytopic type uncertainties in the dependent parameters, by using the characterizations of the \mathcal{H}_2 - \mathcal{H}_∞ norms as LMIs, which arise from parameter dependent Lyapunov functions. The DS class is the one where there is a linear injection application that allows to transform the parameter dependent DS to a regular LPV system. The existence of linear transformation ensures that the analysis of the properties results of the transformed LPV system are transferred to the finite modes of the original parameter dependent DS. Likewise, the robust control design, for the transformed LPV system, is a guarantee of satisfying the admissibility and robust performance for the original DS system. Thus, the condition for the static output feedback (SOF) on the transformed LPV system is analyzed. The SOF controller synthesis is obtained by considering performance indexes in \mathcal{H}_2 and \mathcal{H}_∞ , described as LMIs, as criteria to obtain the extended SOF gain, which considers a feedback gain for the output, and a feedback gain for its derivative.

Notation. \Re is the set of real numbers. For a matrix A , A^T denotes its transpose. $\text{tr}(A)$ defines the trace of the matrix A . In symmetric matrices partitions \star denotes each of its symmetric blocks. \mathbb{I} defines the identity matrix of appropriate dimension. L_2 is the Hilbert space of vectorial signals defined on $(-\infty, \infty)$, with scalar product $\langle x | y \rangle = \int_{-\infty}^{\infty} x(\tau)^* y(\tau) d\tau$ and such that $\|x\|_2 \triangleq \langle x | x \rangle^{1/2} < \infty, \forall x \in L_2$.

1.1 Preliminaries

Important results that must be taken into account, since they will be used in the development of the proposed technique, correspond to the extended characterizations as linear matrix inequalities (LMI)s of the \mathcal{H}_∞ and \mathcal{H}_2 norms for linear systems [26, 19].

Consider the LTI system defined by

$$\begin{aligned} \dot{x} &= Ax + B\omega \\ y &= Cx + D\omega \end{aligned} \quad (1)$$

Lemma 1 (Relaxed \mathcal{H}_2 performance) Consider the system (1), where $D = 0$. For $P = P^T > 0$, the following statements are equivalent:

i) A is stable and $\|C(s\mathbb{I} - A)^{-1}B\|_2^2 < \mu$.

ii) There exist P and Z , such that: $\text{tr}(Z) < 1$ and

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\mu\mathbb{I} \end{bmatrix} < 0, \quad \begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0 \quad (2)$$

iii) There exist P, Z and G , such that: $\text{tr}(Z) < 1$ and

$$\begin{bmatrix} -(G + G^T) & G^T A + P + G^T & G^T B \\ A^T G + P + G & -2P & 0 \\ B^T G & 0 & -\mu\mathbb{I} \end{bmatrix} < 0, \quad (3)$$

$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0$$

Proof: See [26]. ■

Similarly for the \mathcal{H}_2 case, there are some results to improve performance in \mathcal{H}_∞ , from improved versions of Bounded Real Lemma, as shown below.

Lemma 2 (Relaxed \mathcal{H}_∞ performance) Consider the system (1). For $P = P^T > 0$ and the matrix G , the following statements are equivalent:

i) A is stable and $\|C(s\mathbb{I} - A)^{-1}B + D\|_\infty < \gamma$

ii) There exist P , such that

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma^2\mathbb{I} & D^T \\ C & D & -\mathbb{I} \end{bmatrix} < 0 \quad (4)$$

iii) There exist P and G such that, for $\tau \gg 1$

$$\begin{bmatrix} -(G + G^T) & G^T A + P + \tau G^T & 0 & G^T B \\ A^T G + P + \tau G & -2\tau P & C^T & 0 \\ 0 & C & -\mathbb{I} & D \\ B^T G & 0 & D^T & -\gamma^2\mathbb{I} \end{bmatrix} < 0 \quad (5)$$

Proof: Conditions i) and ii) are the well known Bounded Real Lemma. Equivalence between ii) and iii) can be seen in [26]. ■

2 Descriptor and LPV systems

2.1 Descriptor systems

The DS systems, also called singular systems, semi-state systems, differential-algebraic systems or generalized state-space systems; have been one of the main fields of control theory research since its introduction in [15]. Over the last two decades, the DSs have attracted much attention due to the comprehensive uses in the economy, such as the Leontief dynamic model,

in electrical systems, chemical processes, and mechanical models. Since then, considerable progress has been made in the investigation of such systems [10].

An DS is dynamically defined by

$$\begin{aligned}\mathbb{E}\dot{z}(t) &= \mathcal{F}z(t) + Bu(t), \\ h(t) &= \mathcal{C}z(t)\end{aligned}\quad (6)$$

where $z(t) \in \mathbb{R}^n$ is the vector of descriptor variable (instead of state vector), $\mathbb{E} \in \mathbb{R}^{m \times n}$, with $m \leq n$ and $\text{rank}(\mathbb{E}) = r \leq n$ and which is called the descriptor matrix; and $\mathcal{F} \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times l}$, $\mathcal{C} \in \mathbb{R}^{p \times n}$; the control function u belonging to $L_2(0, \tau; \mathbb{R}^l)$.

If $m = n$ and if for all $t \in [0, \tau]$, the polynomial $p(s) = \det(s\mathbb{E} - \mathcal{F})$ satisfies that $p(s) \neq 0$, it is said that the pair $(\mathbb{E}, \mathcal{F})$ is *regular*. Otherwise, it is called *singular*.

The solution and many of the properties of a free DS ($u = 0$) can be characterized in terms of the Weierstraß canonical form [14, 10], which allows to transform the matrix \mathbb{E} into a Jordan canonical form, with a finite number of eigenvalues (finite dynamic mode), plus a nilpotent matrix, also in Jordan canonical form, representing a number of infinite eigenvalues (impulsive mode). The *nilpotency index* of the nilpotent matrix is called *system index*. If \mathbb{E} is non-singular, the system is said to have zero index.

Definition 3 Consider the system (6), and be $\kappa = \deg(\det(s\mathbb{E} - \mathcal{F}))$. If $\kappa = r$ is said that the DS is of free impulse.

Thus, the DS (6) has κ finite dynamic modes, $r - \kappa$ impulsive modes, and $n - r$ non-dynamic modes.

Definition 4 Let the DS given by (6), with $\mathbb{E}, \mathcal{F} \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$ and $\mathcal{C} \in \mathbb{R}^{p \times n}$. In addition, be the matrices: \mathbb{T} and \mathbb{S} , with $\text{img } \mathbb{T} = \ker \mathbb{E}^T$, $\text{img } \mathbb{S} = \ker \mathbb{E}$.

- i) For the triplet $(\mathbb{E}, \mathcal{F}, B)$ is said that the system is of stabilizable finite dynamics if $\text{rank}[\lambda\mathbb{E} - \mathcal{F}, B] = n \forall \lambda \in C^+$.
- ii) For the triplet $(\mathbb{E}, \mathcal{F}, B)$ is said that the system is impulse controllable if $\text{rank}[\lambda\mathbb{E}, \mathcal{F}\mathbb{S}, B] = n$.
- iii) For the triplet $(\mathbb{E}, \mathcal{F}, B)$ the system is said to be strongly stabilizable if i) and ii) are satisfied.
- iv) For the triplet $(\mathbb{E}, \mathcal{F}, \mathcal{C})$ is said that the system has detectable finite dynamics if $\text{rank}[\lambda\mathbb{E}^T - \mathcal{F}^T, \mathcal{C}^T] = n \forall \lambda \in C^+$.
- v) For the triplet $(\mathbb{E}, \mathcal{F}, \mathcal{C})$ is said that the system is impulse observable if $\text{rank}[\lambda\mathbb{E}^T, \mathcal{F}^T\mathbb{T}, \mathcal{C}^T] = n$.

- vi) For the triplet $(\mathbb{E}, \mathcal{F}, \mathcal{C})$ the system is said to be strongly detectable if iv) and v) are satisfied.

A controllability analysis for DS is presented in [3, 12]. In that order of ideas, in [13] a study of the controllability condition for a semilinear non-autonomous DS, by transforming the system from a linear injective application, is presented.

Theorem 5 Let the system (6), with the pair $(\mathbb{E}, \mathcal{F})$ regular; and let $u = 0$.

1. The trivial solution $z = 0$ of the system is stable if and only if all the finite eigenvalues of $\lambda\mathbb{E} - \mathcal{F}$ are in the closed left half-plane and the eigenvalues on imaginary axis are simple.
2. The trivial solution $z = 0$ of the system is asymptotically stable if and only if all the finite eigenvalues of $\lambda\mathbb{E} - \mathcal{F}$ are in the open left half-plane. This means that finite dynamic modes are asymptotically stable.

Proof: See [18, 27]. ■

Definition 6 Consider the system (6). It is said that the DS is admissible if it is regular, free impulse and stable.

Definition 6 allows to establish conditions for the control of DS in the sense of its stabilization [7]. Indeed:

1. For the triplet $(\mathbb{E}, \mathcal{F}, B)$ is said that the system has stabilizable finite dynamics and impulse controllable if a matrix \mathbb{K} exists such that the pair $(\mathbb{E}, \mathcal{F} + B\mathbb{K})$ is admissible.
2. For the triplet $(\mathbb{E}, \mathcal{F}, \mathcal{C})$ is said that the system is of finite dynamics detectable and impulse observable if a matrix \mathbb{L} exists such that the pair $(\mathbb{E}, \mathcal{F} + \mathbb{L}\mathcal{C})$ is admissible.

On the other hand, if the system (6) is regular and free impulse, through the algebraic-differential manipulation of the non-dynamic modes, it is possible to obtain a system descriptor of the form

$$\begin{aligned}\mathbf{E}\dot{z}(t) &= \mathbf{F}z(t) + \mathbf{B}u(t), \\ h(t) &= \mathbf{C}z(t)\end{aligned}\quad (7)$$

where $\mathbf{E} \in \mathbb{R}^{r \times n}$, $\mathbf{F} \in \mathbb{R}^{r \times n}$ and $\mathbf{B} \in \mathbb{R}^{r \times m}$, representing the dynamic modes of the original system. Thus, the admissibility problem of the system (6) is equivalent to the admissibility of the system (7).

2.2 LPV systems

Definition 7 An LPV is a dynamic system in which matrices contain functions that depend on a vector of known variant parameters.

According to Definition 7, a representative LPV model is of the form:

$$\begin{aligned} \dot{x}(t) &= A(\alpha(t))x(t) + B(\alpha(t))u(t); & x(t_0) &= x_0 \\ y(t) &= C(\alpha(t))x(t) \end{aligned} \quad (8)$$

where $x(t) \in \mathbb{R}^n$ are the states, $u(t) \in \mathbb{R}^p$ are the controls and $y(t) \in \mathbb{R}^q$ are the measured output. $\alpha(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^l$. If $\alpha(t) = t$, $l = 1$, the LPV model describes an LTV system.

The typical constraints on exogenous parameters are limits on magnitudes and their indexes of variations, that is, $\forall t \geq 0$

$$\underline{\rho} \leq \alpha(t) \leq \bar{\rho}, \quad \underline{\mu} \leq \dot{\alpha}(t) \leq \bar{\mu} \quad (9)$$

2.2.1 Polytopical LPV systems

Consider the system (8). That system can be characterized as a polytope if it is defined

$$\mathcal{P} := \left(\begin{array}{cc} A(\alpha) & B(\alpha) \\ C(\alpha) & 0 \end{array} \right) \in \Omega. \quad (10)$$

where Ω is a polytopical set, which is defined as:

$$\Omega := \left\{ \mathcal{P} : \mathcal{P} = \sum_{i=1}^l \alpha_i \mathcal{P}_i; \quad \alpha_i \geq 0; \quad \sum_{i=1}^l \alpha_i = 1 \right\}; \quad (11)$$

so that any admissible matrix \mathcal{P} of the system can be written as an unknown convex combination of l vertex matrices given, such that

$$\mathcal{P}_i = \left(\begin{array}{cc} A_i & B_i \\ C_i & 0 \end{array} \right) \quad (12)$$

where $A_i, B_i, C_i, i = 1, \dots, l$, are given matrices, representing the polytope vertices. Thus, this system can be characterized by the convex hull of Ω considering the vertices of the polytope, ie

$$\mathcal{C}_o \Omega = \left\{ \left(\begin{array}{cc} A_1 & B_1 \\ C_1 & 0 \end{array} \right), \dots, \left(\begin{array}{cc} A_l & B_l \\ C_l & 0 \end{array} \right) \right\}. \quad (13)$$

where these matrix vertices are known, provided that $\alpha_i \in \mathbb{R}, \alpha_i \geq 0, i = 1, \dots, l, \sum_{i=1}^l \alpha_i = 1$.

Consequently, from the dependence of the system matrices with respect to the α parameter, and from the

membership of those matrices to the polytope Ω , then, with $x(t_0) = x_0$:

$$\begin{aligned} \dot{x}(t) &= \left(\sum_{i=1}^l A_i \alpha_i \right) x(t) + \left(\sum_{i=1}^l B_i \alpha_i \right) u(t); \\ y(t) &= \left(\sum_{i=1}^l C_i \alpha_i \right) x(t) \end{aligned} \quad (14)$$

where $\alpha_i \in \mathbb{R}, \alpha_i \geq 0, i = 1, \dots, l, \sum_{i=1}^l \alpha_i = 1$.

The controllability and observability conditions of these systems can be analyzed in [1, 11] and [22]. The stability and robust stabilization of polytopical LPV systems can be studied in [6, 24], as well in [21, 22].

3 Problem formulation

Consider an DS as (6), but with parametric uncertainty and perturbations, that is:

$$\begin{aligned} \mathbb{E}\dot{z}(t) &= \mathcal{F}(\rho)z(t) + \mathcal{B}_1(\rho)\omega(t) + Bu(t) \\ h(t) &= \mathcal{C}(\rho)z(t) + \mathcal{D}_1(\rho)\omega(t) \\ y(t) &= C_2z(t) \end{aligned} \quad (15)$$

which constitutes an LPV descriptor system. There, $u(t) \in \mathbb{R}^q$ are the controls; $\omega(t) \in \mathbb{R}^d$ are disturbances; $h(t) \in \mathbb{R}^p$ are the controlled outputs and $y(t) \in \mathbb{R}^p$ are the measured outputs. The parametric variation ρ is assumed to meet the constraints defined in (9). $\mathbb{E} \in \mathbb{R}^{m \times n}$, and $\text{rank}(\mathbb{E}) = r < n$. It can be assumed that $r = m$, so that the system is of the form given by (7), with parametric uncertainties. The matrices $\mathcal{F}, \mathcal{B}_1, B, \mathcal{C}, \mathcal{D}$ and C_2 are of appropriate dimensions. In addition, for all ρ , it is assumed that:

1. For the triplet $(\mathbb{E}, \mathcal{F}, B)$, the finite dynamics of the system is stabilizable and impulse controllable.
2. For the triplet $(\mathbb{E}, \mathcal{F}, C_2)$, the finite dynamics of the system is detectable and impulse observable.

The above conditions lead to solutions to the stability and robust performance problem for the system (15). B and C_2 are matrices known for the fact that they characterize, from a practical point of view, the actuators and sensors, respectively, which are the suitably selected devices in the control systems.

The study of the robust stabilization of DS type LPV has been reported in [25, 14, 9, 4, 8]. In these contributions the controllers are state feedback type and the uncertainty is usually assumed only in the dynamic matrix. In [17, 16] and in [20] output feedback is applied, again in models with very particular uncertainties. Finally, in [7, 2] SOF is used for the robust stabilization of a polytopical type DS with the matrix \mathbb{E} certain and undisturbed.

3.1 Robust admissibility and performance problem

Consider the system (15) with $(\mathbb{E}, \mathcal{F}, B)$ defining an stabilizable and impulse controllable finite dynamics; and $(\mathbb{E}, \mathcal{F}, C_2)$ is such that the finite dynamics is detectable and impulse observable.

Problem 8 *Design a control $u(t)$ for the system (15) such that the corresponding closed loop system will be admissible.*

Problem 9 *Design a control $u(t)$ for the system (15) such that the corresponding closed loop system will be admissible and the effect of the perturbation $\omega(t)$ on the controlled output $h(t)$ will be minimum in the sense of the \mathcal{H}_2 - \mathcal{H}_∞ norms.*

For Problem 8, relative to robust stabilization, it is assumed that $\omega(t) = 0$. Next, the problem 9 demands, in addition to robust stabilization, to satisfy a robust performance index characterized under the \mathcal{H}_2 - \mathcal{H}_∞ norms.

4 Admissibilization of LPV descriptor systems

In this section we present the main results of the work, which consists in proposing an extended SOF control that depends on the output and its derivative. So, be the control of the form:

$$u(t) = \mathcal{K}_0 y(t) + \mathcal{K}_1 \dot{y}(t), \tag{16}$$

where \mathcal{K}_0 y \mathcal{K}_1 are the feedback gains, to be determined, for the output and its derivative. In this case, the derivative of the output is used in the context of the derivative action on PID controllers. Thus, the control will be given by

$$u(t) = \mathcal{K}_0 y(t) + \mathcal{K}_1 C_2 \dot{z}(t) \tag{17}$$

There are some aspects that determine the advantages of this type of controller [21]:

1. If $C_2 = \mathbb{I}$, the design is reduced to a typical state feedback control.
2. If $\mathcal{K}_1 = 0$, corresponds to a classic SOF control.
3. By using the \mathcal{K}_0 and \mathcal{K}_1 gains, many systems that can not be controlled by a classic SOF, can be stabilized by this way. In addition, it is easier to implement than a dynamic output feedback control.

As can be seen in (17), the control $u(t)$ depends on the dynamics of $z(t)$. In order to construct the control, a particular class of linear DSs to variant parameters is assumed, those in which the following condition is satisfied:

$$\det(\mathbb{E}\mathbb{E}^T) \neq 0 \tag{18}$$

This means that there is a linear injective application Γ , which represents the general inverse of \mathbb{E} , that is, $\mathbb{E}\Gamma\mathbb{E} = \mathbb{E}$.

Therefore, let the change of variable $z(t) = \Gamma x(t)$. Consequently, $\mathbb{E}\Gamma = \mathbb{I}$, then the system (15) is transformed into an LPV system:

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + \mathcal{B}_1(\rho)\omega(t) + Bu(t) \\ h(t) &= C_1(\rho)x(t) + \mathcal{D}_1(\rho)\omega(t) \\ y(t) &= C_\Gamma x(t) \end{aligned} \tag{19}$$

where $A(\rho) = \mathcal{F}(\rho)\Gamma$, $C_1(\rho) = \mathcal{C}(\rho)\Gamma$ and $C_\Gamma = C_2\Gamma$. The control design for the original system (15) can be constructed from the transformed system (19).

Proposition 10 *Consider the system (19). If for all ρ the pair $(A(\rho), B)$ is controllable and the pair $(C_\Gamma, A(\rho))$ is observable, then the system (15) is of finite dynamic stabilizable and impulse controllable, and of finite dynamic detectable and impulse observable.*

In fact, the existence of the linear transformation Γ implies that the system (15) is regular and impulse free: if there exists Γ , the regular system (19) is obtained, since the regularity of (15) depends on the pair $(s\mathbb{E}, \mathcal{F}(\rho))$ be regular, whose condition becomes the regularity of the pair $(s\mathbb{I}, A(\rho))$, which is always satisfied. In addition, (19) is characterized by the dynamic matrix $A(\rho)$ whose spectrum defines the finite modes of the system (15). Then, according to the results shown in [13], the controllability (observability) properties of the original system are transferred in the transformed system, that is, if for all ρ :

1. the triplet $(\mathbb{E}, \mathcal{F}(\rho), B)$ defines a system with finite dynamics stabilizable and impulse controllable, then the pair $(A(\rho), B)$ is controllable;
2. the triplet $(\mathbb{E}, \mathcal{F}(\rho), C_2)$ defines a system with detectable and impulse observable finite dynamics, then pair $(C_\Gamma, A(\rho))$ is observable.

Therefore, the problem of admissibility for LPV type DS becomes a problem of control of LPV systems. Consequently, let the system (19) and consider a control law given by (16), then

$$u(t) = (\mathbb{I} - \mathcal{K}_1 C_\Gamma B)^{-1} (\mathcal{K}_0 C_\Gamma + \mathcal{K}_1 C_\Gamma A(\rho)) x(t) \tag{20}$$

As can be observed, the existence of the control depends on the invertibility of the matrix $\mathbb{I} - \mathcal{K}_1 C_\Gamma B$, which is a more weak condition with respect to the conditions for the typical SOF control [21, 22]. In short, the admissibility problem of the system (15) corresponds to the synthesis of a control for the system (19), [23].

4.1 Robust Stabilization

Let the system (19) with the pair $(A(\rho), B)$ controllable and $\omega(t) = 0$. It is also assumed that the system supports a polytopic representation according to (12). Be a control of the form (20), then the closed loop dynamic matrix is:

$$A_c = A(\rho) + B\mathbb{M}^{-1}(\mathcal{K}_0 C_\Gamma + \mathcal{K}_1 C_\Gamma A(\rho)),$$

where $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C_\Gamma B$ and for which \mathbb{M} is nonsingular, so that \mathbb{M}^{-1} exists, which allows to calculate $u(t)$.

Theorem 11 *Let the system (19) with the pair $(A(\rho), B)$ controllable. There is an extended SOF control of the form (20) that stabilizes in closed-loop system, if there exists \mathbb{M} non-singular and the matrix $P = P^T \succ 0$, and matrices X, Y, Z such that the following LMI is satisfied*

$$\begin{aligned} PA_i + A_i^T P + BXC_\Gamma + C_\Gamma^T X^T B^T + \\ BYC_\Gamma A_i + A_i^T C_\Gamma^T Y^T B^T \prec 0, \end{aligned} \quad (21)$$

where $A_i, i = 1, \dots, l$, representing the polytope vertices, then the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{M}Z^{-1}X \quad (22)$$

$$\mathcal{K}_1 = \mathbb{M}Z^{-1}Y \quad (23)$$

with $PB = BZ$ and $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1}YC_\Gamma B$.

Proof: It is known that for closed-loop stability, there exists $P = P^T \succ 0$ such that $PA_c + A_c^T P \prec 0$, then substituting

$$\begin{aligned} PA_i + PB\mathbb{M}^{-1}\mathcal{K}_0 C_\Gamma + PB\mathbb{M}^{-1}\mathcal{K}_1 C_\Gamma A_i + A_i^T P + \\ C_\Gamma^T \mathcal{K}_0^T (\mathbb{M}^{-1})^T B^T P + A_i^T C_\Gamma^T \mathcal{K}_1^T (\mathbb{M}^{-1})^T B^T P \prec 0 \end{aligned}$$

For the linearization of the matrix inequality, if $PB = BZ$ and variable changes $X = Z\mathbb{M}^{-1}\mathcal{K}_0$, $Y = Z\mathbb{M}^{-1}\mathcal{K}_1$, the LMI given by (21) is obtained. Moreover, as $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C_\Gamma B$ and $Z^{-1}Y = \mathbb{M}^{-1}\mathcal{K}_1$, then the expression for \mathbb{M}^{-1} is obtained, which depends on the numerical solution of the LMI (21) and the known matrices of the system (15). ■

It may be noted that $Z^{-1} = (B^T B)^{-1} B^T P^{-1} B$. Thus, the admissibility problem of the system (15) is solved by robust stabilization of the system (19).

4.2 Robust stabilization and performance

Let the system (19) with the pair $(A(\rho), B)$ controllable. This system supports a polytopic representation according to (12). Be a control given by (20), then the closed loop system is:

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + \mathcal{B}_c \omega(t) \\ h(t) &= C_1(\rho)x(t) + \mathcal{D}_1(\rho)\omega(t) \end{aligned} \quad (24)$$

where

$$\begin{aligned} A_c &= A(\rho) + B\mathbb{M}^{-1}\mathcal{K}_0 C_\Gamma + B\mathbb{M}^{-1}\mathcal{K}_1 C_\Gamma A(\rho) \\ \mathcal{B}_c &= B_1(\rho) + B\mathbb{M}^{-1}\mathcal{K}_1 C_\Gamma B_1(\rho) \end{aligned}$$

As it has been proposed, it is required to design \mathcal{K}_0 and \mathcal{K}_1 such that A_c be robustly stable, and that the effect of the perturbation $\omega(t)$ on controlled output $h(t)$ be minimum according to performance indices in $\mathcal{H}_2 - \mathcal{H}_\infty$, which are characterized by LMIs according to the Lemma 1 and the Lemma 2.

4.2.1 Design in \mathcal{H}_2

Theorem 12 *Let the system (19) with the pair $(A(\rho), B)$ controllable and $\mathcal{D}_1(\rho) = 0$, which supports a polytopic representation whose vertices are defined by A_i, B_{1i}, C_Γ and C_{1i} . There is a control law of the form (20), which guarantees a suboptimal performance in \mathcal{H}_2 for the closed loop system (24), if there exist $G \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times p}$, $Y \in \mathbb{R}^{n \times p}$, $P_i = P_i^T \succ 0 \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{p \times n}$ such that $\text{tr}(W) < 1$ and the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & \Upsilon \\ \star & -2P_i & 0 \\ \star & \star & -\mu\mathbb{I} \end{bmatrix} \prec 0, \begin{bmatrix} P_i & (C_{1i})^T \\ C_{1i} & W \end{bmatrix} \succ 0, \quad (25)$$

for $i = 1, \dots, l$, where $\Phi = G^T A_i + BXC_\Gamma + BYC_\Gamma A_i + P_i + G^T$ and $\Upsilon = G^T B_{1i} + BYC_\Gamma B_{1i}$. The feedback gains are:

$$\mathcal{K}_0 = \mathbb{M}Z^{-1}X \quad (26)$$

$$\mathcal{K}_1 = \mathbb{M}Z^{-1}Y \quad (27)$$

with $G^T B = BZ$ and $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1}YC_\Gamma B$.

Proof: Applying clause iii) of the Lemma 1 to the closed loop system (24), matrix inequalities are obtained. After, for the matrix inequalities linearization are used the changes of variables $G^T B = BZ$ and $X = Z\mathbb{M}^{-1}\mathcal{K}_0$, $Y = Z\mathbb{M}^{-1}\mathcal{K}_1$, which generate, by substitution, the LMI (25). ■

In this case, $Z^{-1} = (B^T B)^{-1} B^T (G^T)^{-1} B$, so that the gains are obtained from the numerical solution of the LMI and known matrices of the original system (15). Consequently, the admissibility with robust performance of the system (15), has been solved in the transformed system as a robust control problem in \mathcal{H}_2 , using extended SOF.

4.3 Design in \mathcal{H}_∞

Theorem 13 *Let the system (19) with the pair $(A(\rho), B)$ controllable, which supports a polytopic representation whose vertices are defined by A_i, B_{1_i}, C_Γ and C_{1_i} . There is a control law of the form (20), which guarantees a suboptimal performance in \mathcal{H}_∞ for the closed loop system (24), if the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & 0 & \Upsilon \\ * & -2\tau P_i & (C_{1_i})^T & 0 \\ * & * & -\mathbb{I} & \mathcal{D}_{1_i} \\ * & * & * & -\gamma^2 \mathbb{I} \end{bmatrix} < 0, \quad (28)$$

for $i = 1, \dots, l$, where $\Phi = G^T A_i + B X C_\Gamma + B Y C_\Gamma A_i + P_i + \tau G^T$ and $\Upsilon = G^T B_{1_i} + B Y C_\Gamma B_{1_i}$; $G \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times p}$, $Y \in \mathbb{R}^{n \times p}$, $P_i = P_i^T \succ 0 \in \mathbb{R}^{n \times n}$ and $\tau \gg 1$. The feedback gains are:

$$\mathcal{K}_0 = \mathbb{M} Z^{-1} X \quad (29)$$

$$\mathcal{K}_1 = \mathbb{M} Z^{-1} Y \quad (30)$$

with $G^T B = B Z$ and $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1} Y C_\Gamma B$.

Proof: Considering the closed-loop system (24), the Lemma 2 is applied. Then, the procedure of linearization of matrix inequalities is followed by changes of variables, as has been done for the proof of the Theorem 12. ■

In order to reduce conservatism, in the characterization of the relaxed norms in \mathcal{H}_2 - \mathcal{H}_∞ as LMIs, the P matrix does not necessarily have to be unique, so that the matrices $P_i = P_i^T \succ 0$ can be used. On the other hand, mixed performance indices in \mathcal{H}_2 - \mathcal{H}_∞ can be imposed, so control synthesis, for robust admissibility and performance in closed loop, meet multiple objectives.

5 Concluding remarks

From the results obtained in this research, the contributions are focused on the analysis and synthesis of controllers for a class of linear descriptor systems dependent on parameters. First, a model of linear descriptor systems with variable parameters has been

considered. Then, an analysis of admissibility and robust control for a class of descriptor systems with polytopic parametric uncertainties has been presented. The class is defined by those processes where there is a linear injective application that allows to transform the parameter-dependent descriptor system to a regular LPV system. Thus, the properties and conditions of the original system are conserved in the transformed system, which guarantees the design of a control for the robust admissibility (stability) and the robust performance. The design of the control in the transformed system is a guarantee of satisfying the robust admissibility and performance for the original descriptor system. The synthesis of the control law has been done by means of the static feedback of the extended output, which is based on obtaining a feedback gain for the output and a feedback gain for its derivative. These gains are derived by robust stabilizing and robust performance of LPV systems, using the characterizations of the \mathcal{H}_2 - \mathcal{H}_∞ norms as LMIs, which arise from parameter dependent Lyapunov functions. The design technique also allows to impose multi-objective specifications. The theoretical results have been evaluated through simulations. The technique can be extended to systems with uncertainties in the descriptor matrix.

References:

- [1] Gary J. Balas, Jozsef Bokor, and Zoltán Szabó. Invariant subspaces for lpv systems and their applications. *IEEE Trans. Automat. Control*, 48:2065–2069, 2003.
- [2] Gabriela Iuliana Bara. Robust analysis and control of parameter-dependent uncertain descriptor systems. *Systems & Control Letters*, 60(5):356–364, 2011.
- [3] Thomas Berger and Timo Reis. Controllability of linear differential-algebraic systems - a survey. Technical report, Institut für Mathematik, Technische Universität Ilmenau, Germany, 2009.
- [4] Anis Bouali, Mohamed Yagoubi, and Philippe Chevrel. Gain scheduled observer state feedback controller for rational LPV systems. *IFAC Proceedings Volumes*, 41(2):4922–4927, 2008.
- [5] Corentin Briat. *Robust control and observation of LPV time-delay systems*. PhD thesis, Institut National Polytechnique de Grenoble-INPG, 2008.

- [6] F. Bruzelius. *Linear Parameter-Varying Systems: an approach to gain scheduling*. PhD thesis, Chalmers University of Technology, Göteborg, Sweden, 2004.
- [7] M Chaabane, F Tadeo, D Mehdi, and M Souissi. Robust admissibilization of descriptor systems by static output-feedback: an LMI approach. *Mathematical Problems in Engineering*, 2011(1):1–10, 2011.
- [8] Mohammed Chadli, Jamal Daafouz, and Mohamed Darouach. Stabilisation of singular LPV systems. *IFAC Proceedings Volumes*, 41(2):9999–10002, 2008.
- [9] Mohammed Chadli, Mohamed Darouach, and Jamal Daafouz. Static output stabilisation of singular LPV systems: LMI formulation. In *CDC*, pages 4793–4796, 2008.
- [10] Guang Ren Duan. *Analysis and Design of Descriptor Linear Systems*. Springer, New York, 2010.
- [11] K. M. Hangos, J. Bokor, and G. Szederkényi. *Analysis and Control of Nonlinear Process Systems*. Springer, London, 2004.
- [12] J Klamka. Controllability of dynamical systems. a survey. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, 61(2):335–342, 2013.
- [13] Hugo Levia, Addison Ríos-Bolívar, Ambrosio Tineo-Moya, and Miguel Narvaez. Análisis de la controlabilidad de sistemas descriptores semi-lineales (Analysis of the Controllability of Semi-linear Descriptor Systems). *Ciencia e Ingeniería*, 38(2):187–196, 2017.
- [14] Philip Losse, Volker Mehrmann, Lisa Katrin Poppe, and Timo Reis. Robust control of descriptor systems. Technical report, Institut für Mathematik, TU Berlin, 2007.
- [15] D. G. Luenberger. Dynamic equations in descriptor form. *IEEE Trans. Automat. Control*, 22(3):312–321, 1977.
- [16] Izumi Masubuchi. Output feedback controller synthesis for descriptor systems satisfying closed-loop dissipativity. *Automatica*, 43(2):339–345, 2007.
- [17] Izumi Masubuchi, Tomoaki Akiyama, and Masami Saeki. Synthesis of output feedback gain-scheduling controllers based on descriptor LPV system representation. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, volume 6, pages 6115–6120. IEEE, 2003.
- [18] Volker Mehrmann and Tatjana Stykel. Descriptor systems: A general mathematical framework for modelling, simulation and control. Technical report, Institut für Mathematik, Technische Universität Berlin, Germany, 2006.
- [19] Goele Pipeleersa, Bram Demeulenaere, Jan Swersers, and Lieven Vandenbergh. Extended LMI characterizations for stability and performance of linear systems. *Systems & Control Letters*, 58(7):510–518, July 2009.
- [20] İlhan Polat, Esref Eskinat, and İE Kose. Dynamic output feedback control of quasi-LPV mechanical systems. *IET Control Theory & Applications*, 1(4):1114–1121, 2007.
- [21] A. Ríos, F. Hidrobo, and J. Suárez. Control robusto por realimentación estática de la salida extendida. *Revista Técnica de Ingeniería*, 34(3):255–262, 2011.
- [22] Addison Ríos-Bolívar. *Control de Sistemas Lineales: Realimentando la salida*. Editorial Académica Española, Alemania, 2014.
- [23] Addison Ríos-Bolívar and Miguel Narvaez. On the admissibility of LPV descriptor systems. *Revista INGENIERÍA UC*, 24(3):327–340, 2017.
- [24] Jeff S. Shamma. *Control of Linear Parameter Varying Systems with Applications*, chapter An Overview of LPV Systems, pages 3–26. Springer, 2012.
- [25] H-S Wang, C-F Yung, and F-R Chang. Bounded real lemma and H_∞ control for descriptor systems. *IEE Proceedings-Control Theory and Applications*, 145(3):316–322, 1998.
- [26] C.P. Wei. On applications of the projection lemma to analysis and design of continuous-time systems. Master’s thesis, National Sun Yat-Set University, 2003.
- [27] Elizabeth Yip and Richard Sincovec. Solvability, controllability, and observability of continuous descriptor systems. *IEEE Transactions on Automatic Control*, 26(3):702–707, 1981.