Inertial manifolds for Navier-Stokes equations in notions of Lie algebras

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Abstract: - Inertial manifolds of Navier-Stokes equations have been calculated approximately up to now. In this paper, drawing upon advanced ingredients of differential geometry and Lie groups a novel methodology is presented for finding the inertial manifolds of $(2+1)$-dimensional Navier-Stokes equation. It has been shown that the geometric notions about Lie groups and Lie algebras such as transformation groups, one-parameter groups, integral submanifolds, adjoint representations, group-invariant solutions and optimal systems not only cover all of the properties of inertial manifolds, but also result to the exact decomposition of the velocity field of the flow of Navier-Stokes equation by proposing the coordinate chart for it. In this way, the new procedure outperforms the numerical estimation methods by providing the analytic solution of the inertial manifolds. Also, the proposed methodology can be applied to the general problems by searching the optimal systems of them. Furthermore, this geometric approach results to the reduction theory which transforms these partial differential equations into a system of differential equations with fewer variables.

Key-Words: Navier-Stokes equation, inertial manifold, Lie algebra, optimal system, invariant solution, Frobenius' theorem

1 Introduction

An inertial manifold is a smooth finite dimensional manifold of the phase space which is positively invariant, attracts exponentially all orbits, and contains the global attractor. Finding this portion or surface of the phase space for Navier-Stokes equation (NSE), as the very starting point of all equations in fluid mechanics, is an open problem, [1-3]. The approximate inertial manifolds can be obtained by some numerical methods, [4-5]**.** The explanation of the problem and a review of its approximate solution have been proposed in Sections 2.1 and 2.2.

On the other hand, finding the group-invariant solutions of a complicated system of partial differential equations (PDEs) aroused from some physically important problem is a well-known technique in mathematics and physics. Consider Ξ is a system of partial differential equations which defined over an open subset $M \subset X \times U \cong \mathbf{R}^p \times \mathbf{R}^q$ of the space of independent and dependent variables and G be a local group of transformations acting on manifold M. Roughly, a solution $u = f(x)$ of the system is said to be G-invariant if it remains unchanged by all the group transformations in G, meaning that for each $g \in G$, the functions f and $g.f$ agree on their common domains of definition.

In fact, these solutions are large classes of special explicit solutions which are characterized by their invariance under some symmetry group of the system of PDEs. The machinery of Lie algebra theory provides a systematic method to search for these special group invariant solutions. For the globally attraction property of inertial manifolds, the examination of the optimal system of these group invariant solutions.

In effect, the exponential property inserted in the definition of inertial manifolds and the main problem about the decomposition of the velocity field of NSE can be solved by Frobenius' Theorem. For analytic calculation of inertial manifolds, some geometric notions has been organized in Section 3, step by step, which result to the exact presentation of them.

In Theorem 2 of Section 4, the main problem has been proved in notions of optimal systems of symmetry-group invariant solutions of NSE. This geometric approach not only translates the notion of inertial manifolds to the new concepts of optimal systems, but also results to the reduction theory.

In fact, for a system of differential equations, symmetry group can transform it into a system of differential equations with fewer variables which are more easily solved in principal than those PDEs, [6-7].

Also, as in Corollary 1 of Section 4, the proposed methodology results in the exact decomposition of the velocity field of NSE and the presentation of coordinate chart for it.

2 Preliminaries

2.1 Explanation of the problem about Navier-Stokes equation (NSE)

This fact that turbulent flows have a finite number of degrees of freedom strongly suggests that, for practical purposes, it might be possible to describe the evolution of a turbulent flow by a finite, reasonably sized set of parameters, [1].

Approximate inertial manifolds give (in an approximate sense) a practical answer to this question. For this, consider the decomposition of the velocity field of the flow $u = u(x,t)$, into two parts,

$$u = y+z \,, \tag{1}$$

where $y = y(x,t)$ is a variable with relatively few dimensions, while $z = z(x,t)$, the remaining part, is somehow enslaved by y. The relation between these two parts of the flow could be represented by a functional relation of the form

$$z = R(y). \tag{2}$$

Generally, such decomposition can be carried out exactly when there is an adequate separation of scales. That is not the case for turbulent flows, but an approximate decomposition is possible. For example, Fourier representation of a flow is a natural candidate to this.

An exact relation of the form (2) leads to a system of ordinary differential equations for the evolution of the flow. Indeed, the functional equation form of the Navier-Stokes equation is

$$\frac{du}{dt}+vAu+B(u) = f \,, \tag{3}$$

with the decomposition (1) in mind and with the representations of y and z, it can formally obtain the equations corresponding to the evolution of the low and high modes by applying to (3) the Galerkin projector P_m and its complement $Q_m = I\text{-}P_m$. Then,

$$\frac{dy}{dt}+vAy+P_mB(y+z) =P_m f \,, \frac{dz}{dt}+vAz+Q_mB(y+z) =Q_m f.$$

If an exact relation of the form (2) holds, then the high modes are given in terms of the low modes and hence only the evolution equation for the low modes in the form $\frac{dy}{dt}+vAy+P_mB(y+R(y)) =P_m f$ can be considered. This is a system of ordinary differential equations. However, the exact relation (2) is not known to exist. The existence of such a relation has been proved (in a mathematically rigorous way) for a number of partial differential equations modelling turbulent phenomena in mechanics, chemistry, and other fields.

Problem. The portion or surface of the phase space (the infinite-dimensional function space for the velocity field) defined by the relation (2) is known as an inertial manifold. It is still an open question whether the 2-dimensional or 3-dimensional Navier-Stokes equations possesses such an inertial manifold.

Approximate solution. More plausible is the existence of an approximate relation of the form (2), that is, $z \approx R(\tilde{y})$. More precisely, one could have that the flow $u(t)=y(t)+z(t)$ is close to $y(t)+R(\tilde{y}(t))$ in the sense that, for all times (or, at least, for large t),

$$|z(t)\text{-}R(\tilde{y}(t))| < \varepsilon \,, \tag{4}$$

in some suitable norm. When the relation (4) holds, the manifold $\tilde{z}=R(\tilde{y})$ is called an approximate inertial manifold. It provides one with an approximate law relating the high modes to the low modes. In contrast with inertial manifolds, a number of approximate inertial manifolds are known to exist and their explicit expressions have been derived. In this regard, the Galerkin approximation of the Navier-Stokes equations corresponds to the flat manifold $\tilde{z} \equiv 0$.

For a given approximate inertial manifold $\tilde{z}=R(\tilde{y})$, one can consider an approximation to the NSE by the finite-dimensional system

$$\frac{d\tilde{y}}{dt}+vA\tilde{y}+P_mB(\tilde{y}+R(\tilde{y})) =P_m f \tag{5}$$

Note that the solution $\tilde{y}=\tilde{y}(t)$ of this system does not coincide with the low-mode part of the exact flow, $y(t)=P_mu(t)$, because the enslaving is not exact. Nevertheless, one can expect that, the smaller the error ε in (4), the better the approximation (5) in the sense that $\tilde{y}(t)+R(\tilde{y}(t))$ would be closer to the exact solution $u(t)$.

Another practical aspect of the concept of an approximate inertial manifold is that, even when an exact inertial manifold is known to exist, it might be more useful (for computational purposes) to have on hand an explicit form for an approximate inertial manifold, since inertial manifolds are usually obtained implicitly, [4-5].

In two following sections, some necessary notions from inertial manifolds, differential geometry and Lie groups have been expressed to give the exact inertial manifolds for NSE.

2.1 Inertial manifolds

As mentioned before, an inertial manifold is finite-dimensional even if the original system is infinite-dimensional, and because most of the dynamics for the system takes place on the inertial manifold, studying the dynamics on an inertial manifold produces a considerable simplification in the study of the dynamics of the original system, [8].

Definition 1. An inertial manifold for a dynamical semigroup $S(t)$ is a smooth manifold M such that

1. M is of finite dimension,
2. $S(t)M \subseteq M$ for all times $t \geq 0$,
3. M attracts all solutions exponentially quickly, that is, for every initial value $u_0 \in H$ there exist constants $c_j > 0$ such that $dist(S(t)u_0, M) \leq c_1 \exp(-c_2 t)$.

The restriction of the differential equation $du/dt=F(u)$ to the inertial manifold M is therefore a well-defined finite-dimensional system called the inertial system.

Subtly, there is a difference between a manifold being attractive, and solutions on the manifold being attractive. Nonetheless, under appropriate conditions the inertial system possesses so-called asymptotic completeness; that is, every solution of the differential equation has a companion solution lying in M and producing the same behavior for large time means that for all u_0 there exists $u_0 \in M$ and possibly a time shift $\tau \geq 0$ such that $dist(S(t)u_0, S(t+\tau)u_0) \to 0$ as $t \to \infty$.

3 Theoretical Background on differential geometry and Lie groups

In this section, some main concepts of differential geometry and Lie groups will be reviewed, step by step to make the paper essentially self-contained, [9-11].

1) An m-dimensional manifold is a set M, together with a countable collection of subsets $U_\alpha \subset M$ called coordinate charts, and one-to-one functions $\chi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbf{R}^m$, called local coordinate maps, which satisfy the following properties: The coordinate charts cover M; $\bigcup_\alpha U_\alpha = M$. On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the map $\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \to \chi_\beta(U_\alpha \cap U_\beta)$ is a smooth (infinitely differentiable) function. If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M, then there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$, with $\chi_\alpha(x) \in W$, $\chi_\beta(x) \in \tilde{W}$, satisfying $\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(W)$ is an empty set. A submanifold of M is a subset $N \subset M$, together with a smooth, one-to-one map $\phi : \tilde{N} \to N \subset M$ satisfying the maximal rank condition and $N = \phi(\tilde{N})$ is the image of ϕ.

2) An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation $m : G \times G \to G$, $m(g,h) = g.h$, $g,h \in G$ and the inversion $i : G \to G$, $i(g) = g^{-1}$, $g \in G$, are smooth maps between manifolds. A Lie subgroup H of a Lie group G is given by a submanifold $\phi : \tilde{H} \to G$, where $\tilde{H}$ itself is a Lie group, $H = \phi(\tilde{H})$ is the image of ϕ, and ϕ is a Lie group homomorphism.

A local group of transformations acting on a manifold M is given by a (local) Lie group G, an open subset $\cup$, with $\{e\} \times M \subset \cup \subset G \times M$, which is the domain of definition of the group action, and a smooth map $\Psi : \cup \to M$ with this property that if $(h,x) \in \cup$, $(g, \Psi(h,x)) \in \cup$, and also $(g.h, x) \in \cup$, then $\Psi(g, \Psi(h,x)) = \Psi(g.h, x)$ or for brevity, $g.(h.x) = (g.h).x$, $x \in M$. Also, for all $x \in M$, $\Psi(e,x) = x$ or $e.x = x$, for all $x \in M$.

The subset $O \subset M$ is an orbit of a local transformation group G, provided it satisfies the following conditions: If $x \in O$, $g \in G$ and $g.x$ is defined, then $g.x \in O$.

3) Let C is a smooth curve on a manifold M, parametrized by $\phi : I \to M$, where I is a subinterval of $\mathbf{R}$. In local coordinates $x = (x^1, \ldots, x^m)$, C is given by m smooth functions $\phi(\varepsilon) = (\phi^1(\varepsilon), \ldots, \phi^m(\varepsilon))$ of the real variable ε. At each point $x = \phi(\varepsilon)$ of C the curve has a tangent vector, namely the derivative $\dot{\phi}(\varepsilon) = d\phi/d\varepsilon = (\dot{\phi}^1(\varepsilon), \ldots, \dot{\phi}^m(\varepsilon))$. The collection of all tangent vectors to all possible curves passing through a given point x in M is called the tangent space to M at x, and is denoted by $TM|_x$.

If M is an m-dimensional manifold, then $TM|_x$ is an m-dimensional vector space, with

$\{\partial/\partial x^1,\dots,\partial/\partial x^m\}$ providing a basis for $TM|_x$ in the given local coordinates. The collection of all tangent spaces corresponding to all points x in M is called the tangent bundle of M, denoted by $TM=\bigcup_{x\in M} TM|_x$. A vector field has the form $v|_x=\xi^1(x)\frac{\partial}{\partial x^1}+\xi^2(x)\frac{\partial}{\partial x^2}+\dots+\xi^m(x)\frac{\partial}{\partial x^m}$, in local coordinates $(x^1,\dots,x^m)$, where each $\xi^i(x)$ is a smooth function of x. An integral curve of a vector field v is a smooth parameterized curve $x=\phi(\varepsilon)$ whose tangent vector at any point coincides with the value of v at the same point: $\dot\phi(\varepsilon)=v|_{\phi(\varepsilon)}$ for all ε.

The flow generated by a vector field is the same as a local group action of the Lie group $\mathbf{R}$ on the manifold M, often called a one-parameter group of transformations. The vector field v is called the infinitesimal generator of the action. The orbits of the one-parameter group action are the maximal integral curves of the vector field v.

4) There is a one-to-one correspondence between local one-parameter groups of transformations and their infinitesimal generators. In terms of this exponential notation, $\exp[(\delta+\varepsilon)v]x=\exp(\delta v)\exp(\varepsilon v)x$ whenever defined, $\exp(0v)x=x$, and

$$\frac{d}{d\varepsilon}[\exp(\varepsilon v)x]=v|_{\exp(\varepsilon v)x}, \qquad (6)$$

for all $x\in M$. For a vector field $v=\sum\xi^i(x)\partial/\partial x^i$ on M and a smooth function $f:M\to\mathbf{R}$, using the chain rule and relation (6) , $\frac{d}{d\varepsilon}f(\exp(\varepsilon v)x)=\sum_{i=1}^{m}\xi^i(\exp(\varepsilon v)x)\frac{\partial f}{\partial x^i}(\exp(\varepsilon v)x)$.

If v and w are vector fields on M, then their Lie bracket $[v,w]$ is the unique vector field satisfying $[v,w](f)=v(w(f))-w(v(f))$ for all smooth functions $f:M\to\mathbf{R}$.

5) Let $v_1,\dots,v_r$ be vector fields on a smooth manifold M. An integral submanifold of $\{v_1,\dots,v_r\}$ is a submanifold $N\subset M$ whose tangent space $TN|_y$ is spanned by the vectors $\{v_1|_y,\dots,v_r|_y\}$ for each $y\in N$.

The system of vector fields $\{v_1,\dots,v_r\}$ is integrable if through every point $x_0\in M$ there passes an integral submanifold. An integrable system of vector fields $\{v_1,\dots,v_r\}$ is called semi-regular if the dimension of the subspace of $TM|_x$ spanned by $\{v_1|_x,\dots,v_r|_x\}$ does not vary from point to point.

An integrable system of vector fields is regular if it is semi-regular, and, in addition, each point x in M has arbitrarily small neighborhoods U with the property that each maximal integral submanifold intersects U in a pathwise connected subset. A system of vector fields $\{v_1,\dots,v_r\}$ on M is in involution if there exist smooth real-valued functions $c_{ij}^k(x)$, $x\in M$, $i,j,k=1,\dots,r$, such that for each $i,j=1,\dots,r$, $[v_i,v_j]=\sum_{k=1}^{r}c_{ij}^k.v_k$.

Frobenius' theorem, as generalized by Hermann to the case when the integral submanifolds have varying dimensions, states that this necessary condition is also sufficient.

Theorem 1 (Frobenius' theorem) ([9-10]) Let $\{v_1,\dots,v_r\}$ be smooth vector fields on M. Then the system $\{v_1,\dots,v_r\}$ is integrable if and only if it is in involution.

In this way, Frobenius' theorem gives necessary and sufficient conditions for finding a maximal set of independent solutions of an underdetermined system of first-order homogeneous linear partial differential equations.

6) For any group element g of a Lie group G, the right multiplication map $R_g:G\to G$ defined by $R_g(h)=h.g$ is a diffeomorphism, with inverse $R_{g^{-1}}=(R_g)^{-1}$. A vector field v on G is called right-invariant if $dR_g(v|_h)=v|_{R_g(h)}=v|_{hg}$ for all g and h in G. The set of all right-invariant vector fields forms a vector space.

A Lie algebra is a vector space $\mathbf{G}$ with a bilinear operation $[.,.]:\mathbf{G}\times\mathbf{G}\to\mathbf{G}$, called the Lie bracket for $\mathbf{G}$, satisfying the following axioms: $[cv+c'v',w]=c[v,w]+c'[v',w]$ and $[cv+c'v',w]=c[v,w]+c'[v',w]$, for all $c,c'\in\mathbf{R}$, $[v,w]=-[w,v]$ and $[u,[v,w]]+[w,[u,v]]+[v,[w,u]]=0$, for all u,v,v',w,w' in $\mathbf{G}$.

The flow generated by a right-invariant vector field $v\neq 0$ through the identity, namely $g_\varepsilon=\exp(\varepsilon v)e\equiv\exp(\varepsilon v)$ is defined for all $\varepsilon\in\mathbf{R}$ and forms a one-parameter subgroup of G. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

7) For each element g of a Lie group G, group conjugation $K_g(h)\equiv ghg^{-1}$, $h\in G$, determines a diffeomorphism on G. The differential

$dK_g : TG|_h \to TG|_{K_g(h)}$ is readily seen to preserve the right-invariance of vector fields, and hence determines a linear map on the Lie algebra of G, called the adjoint representation; $Ad\, g(v) \equiv dK_g(v)$, $v \in \mathbf{G}$.

Let H and $\tilde{H}$ be connected, s-dimensional Lie subgroups of the Lie group G with the corresponding Lie subalgebras $\mathbf{H}$ and $\tilde{\mathbf{H}}$ of the Lie algebra $\mathbf{G}$ of G. Then $\tilde{H}=gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathbf{H}}=Ad\,\mathrm{g}(\mathbf{H})$ are conjugate subalgebras.

If v generates the one-parameter subgroup $\{\exp(\varepsilon v)\}$, then $ad\,\mathrm{v}$ is the vector field on $\mathbf{G}$ generating the corresponding one-parameter group of adjoint transformations which $ad\,\mathrm{v}|_{\mathrm{w}} \equiv \frac{d}{d\varepsilon}|_{\varepsilon\text{-}0}\, Ad(\exp(\varepsilon v))w$, $w \in \mathbf{G}$. For each $v \in \mathbf{G}$, the adjoint vector $ad\,\mathrm{v}$ at $w \in \mathbf{G}$ is $ad\,\mathrm{v}|_{\mathrm{w}} = [w, v] = -[v, w]$.

8) For a local transformation group G, a subset $\Xi \subset M$ is called G-invariant and G is called a symmetry group Ξ, if whenever $x \in \Xi$, and $g \in G$ is such that $g.x$ is defined, then $g.x \in \Xi$. If Ξ be a system of differential equations then a symmetry group of the system Ξ is a local group of transformations G acting on an open subset M of the space of transformation independent and dependent variables for the system with the property that whenever $u=f(x)$ is a solution of Ξ, and whenever $g.f$ is defined for $g \in G$, then $u=g.f(x)$ is also a solution of the system.

Let $\mathbf{G}$ is an n-dimensional Lie algebra of a differential system with p independent variables $\{x_1, x_2 \ldots, x_p\}$ and q dependent variables $\{u_1, u_2, \ldots, u_q\}$, which is generated by n vector fields $\{v_1, v_2, \ldots, v_n\}$. The corresponding n-parameter symmetry group of $\mathbf{G}$ is denoted as G, which is the collections of transformations

$$\begin{aligned}&(\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{u}_1, \ldots, \tilde{u}_q) \\ &= \exp\left(\sum_{i=1}^{n} a_i v_i\right)(x_1, \ldots, x_p, u_1, \ldots, u_q)\end{aligned} \tag{7}$$

for all allowed values of the group parameters. For an s-parameter subgroup $H \subset G$, an H-invariant solution can be transformed into another one by the elements $g \in G$ not belonging the subgroup H. That is to say, two group invariant solutions are essentially different if it is impossible to connect them with any group transformation in (7).

9) An optimal system of a s-parameter subgroups of a Lie group G, as a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras forms an optimal system if every s-parameter subalgebra of $\mathbf{G}$ is equivalent to a unique member of the list under some element of the adjoint representation; $\tilde{\mathbf{H}}=Ad\,\mathrm{g}(\mathbf{H})$, $g \in G$.

4 Main Result

For the classification of the group-invariant solutions of NSE which satisfies in the first and second conditions of Definition 1, it is more convenient to work in the space of Lie groups.

On the other and, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so it will concentrate on this on the latter. Although some sophisticated techniques are available for Lie algebras with additional structure, in essence this problem is attacked by the naïve approach of taking a general element v in $\mathbf{G}$ and subjecting it to various adjoint transformations so as to simplify it as much as possible.

Theorem 2. Inertial manifolds for NSE can be found in the notion of optimal systems of invariant solutions of the symmetry group of it, together with the coordinate charts result in the decomposition (1) of the velocity field of the flow of NSE.

Proof. As in Definition 1, inertial manifolds are finite-dimensional, smooth, invariant manifolds that contain the global attractor and attract all solutions of dissipative dynamical systems exponentially quickly.

Then, from 9 steps of Section 3, the proof reduces to the finding an optimal system of subalgebras under the adjoin representation which it will be show that this attracts all of the solutions.

From the final step of Section 3, a family of r-dimensional subalgebras $\{g_\alpha\}_{\alpha\in\mathbf{R}}$ forms an r-parameter optimal system named as O_r means that any r-dimensional subalgebra is equivalent to some g_α, g_α and g_β are inequivalent for distinct α and β. Each member $g_\alpha \in O_r$ is a collection of r linear combinations of generators. Also, an optimal system of s-parameter group-invariant solutions to a system of differential equations is a collection of solutions $u=f(x)$ with the following properties:

Each solution in the list is invariant under some s-parameter symmetry group of the system of differential equations. If $u=\tilde{f}(x)$ is any other solution invariant under an s-parameter symmetry group, then there is a further symmetry g of the system which maps $\tilde{f}$ to a solution $f=g.\tilde{f}$ on the list. Let $\mathbf{G}(v,w)\equiv\mathbf{G}\left(\sum_{i=1}^{n}a_iv_i,\sum_{i=1}^{n}b_iv_i\right)$ be a general two-dimensional algebra, which remains closed under commutation.

In $\mathbf{G}(v,w)$, two subalgebras $\{w_1,w_2\}$ and $\{w_1',w_2'\}$ are called equivalent if one can find some transformation $g\in G$ and some constants $\{k_1,k_2,k_3,k_4\}$ so that

$$\begin{aligned} w_1'&=k_1Ad_g(w_1)+k_2Ad_g(w_2),\\ w_2'&=k_3Ad_g(w_1)+k_4Ad_g(w_2). \end{aligned} \tag{8}$$

Since w_1' and w_2' are linearly independent, it requires $k_1k_4-k_2k_3\neq 0$ in (8) or else $w_1'=cw_2'$. Hence, to find all the inequivalent elements in the optimal system O_2, it is necessary that each member $\{v,w\}\in O_2$ satisfy $[v,w]=0$ or $[v,w]=v$ which the latter case result in $k_2=0$ and $k_4=1$.

For $w_1=\sum_{i=1}^{n}a_iv_i$, its general adjoint transformation matrix A is the product of the matrices of the separate adjoint actions $A_1,A_2,...,A_n$, each corresponding to $Ad_{\exp(\varepsilon v_i)}(w_1)$, $i=1\cdots n$ which taken in any order $A\equiv A(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)=A_1A_2\cdots A_n$. Hence, the equivalence between $\{w_1',w_2'\}$ and $\{w_1,w_2\}$ shown in (8) can be rewritten as $2n$ algebraic equations with respect to $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ and k_1,k_2,k_3,k_4,

$$\begin{cases} (a_1',a_2',\ldots,a_n')=k_1(a_1,a_2,\ldots,a_n)A\\ +k_2(b_1,b_2,\ldots,b_n)A,\\ (b_1',b_2',\ldots,b_n')=k_3(a_1,a_2,\ldots,a_n)A\\ +k_4(b_1,b_2,\ldots,b_n)A. \end{cases} \tag{9}$$
$$(k_1k_4-k_2k_3\neq 0).$$

If the system (9) have the solution, then $\left\{w_1=\sum_{i=1}^{n}a_iv_i,\,w_2=\sum_{i=1}^{n}b_iv_i\right\}$ is equivalent to $\left\{w_1'=\sum_{i=1}^{n}a_i'v_i,\,w_2'=\sum_{i=1}^{n}b_i'v_i\right\}$.

In this way and from 9 steps of Section 3, the steps to reach the exact inertial manifolds will be as follows

1) presenting the commutator table,
2) giving the adjoint representation table of the generators $\{v_i\}_{i=1}^n$ for a given algebra,
3) giving the restrictions about $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$,
4) computing adjoint transformation matrix A,
5) determining the general equations about the invariants ϕ,
6) determining the respective invariants of two cases $[w_1,w_2]=0,[w_1,w_2]=w_1$,
7) selecting the corresponding eligible representative elements $\{w_1',w_2'\}$ of equations (8) are the steps of the construction of two dimensional optimal system.

As the notions of ([1]), $(2+1)$-dimensional Navier-Stokes equation can be expressed as the equation $u=\psi_{xx}+\psi_{yy},u_t+\psi_xu_y-\psi_yu_x-\gamma(u_{xx}+u_{yy})=0$ which equivalent to

$$\begin{aligned} &\psi_{xxt}+\psi_{yyt}+\psi_x\psi_{xxy}+\psi_x\psi_{yyy}-\psi_y\psi_{xxx}\\ &-\psi_y\psi_{xyy}-\gamma(\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy})=0 \end{aligned} \tag{10}$$

The commutator table of the NSE is

Table 1. Commutator table of the NSE, [6]

	v_1	v_2	v_3	v_4
v_1	0	$-v_2$	v_3	0
v_2	v_2	0	v_4	0
v_3	$-v_3$	$-v_4$	0	0
v_4	0	0	0	0

where the associated vector fields for the one-parameter Lie group of NSE (10) are given by

$v_1=\frac{x}{2}\partial_x+\frac{y}{2}\partial_y+t\partial_t, v_2=\partial_t, v_3=-yt\partial_x+xt\partial_y+\frac{x^2+y^2}{2}\partial_\psi, v_4=-y\partial_x+x\partial_y, v_5=f(t)\partial_x-f'(t)y\partial_\psi, v_6=g(t)\partial_y+g'(t)x\partial_\psi$ and $v_7=h(t)\partial_\psi$.

Table 2. The adjoint representation table of the NS equation, [6]

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$e^{\varepsilon}v_2$	$e^{-\varepsilon}v_3$	v_4
v_2	$v_1-\varepsilon v_2$	v_2	$v_3-\varepsilon v_4$	v_4
v_3	$v_1+\varepsilon v_3$	$v_2+\varepsilon v_4$	v_3	v_4
v_4	v_1	v_2	v_3	v_4

So

$$A=A_1A_2A_3A_4=\begin{pmatrix}1&0&0&0\\0&e^{\varepsilon_1}&0&0\\0&0&e^{-\varepsilon_1}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&-\varepsilon_2&0&0\\0&1&0&0\\0&0&1&-\varepsilon_2\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\varepsilon_3&0\\0&1&0&\varepsilon_3\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&-\varepsilon_2&\varepsilon_3&-\varepsilon_2\varepsilon_3\\0&e^{\varepsilon_1}&0&e^{\varepsilon_1}\varepsilon_3\\0&0&e^{-\varepsilon_1}&-e^{-\varepsilon_1}\varepsilon_2\\0&0&0&1\end{pmatrix}$$

is the general adjoint matrix. Then substituting $w_1=\sum_{i=1}^{4}a_iv_i$, $w_2=\sum_{i=1}^{4}a_iv_i$ into $[w_1,w_2]=\delta w_1$ result to the two-dimensional optimal system

Table 3. Two-dimensional optimal system of NSE, [6]

Case	Result
Not all a_1 and b_1 are zeros	$k_1'=a_1, k_2'=\frac{a_1a_4-a_2a_3}{a_1}, k_3'=b_1,$ $k_4'=\frac{b_4a_1^2-a_2a_3b_1}{a_1^2}, \varepsilon_2=\frac{e^{\varepsilon_1}a_2}{a_1},$ $\varepsilon_3=-\frac{a_3}{a_1e^{\varepsilon_1}}$
$a_1=b_1=0$	$a_2b_3-a_3b_2=0$
Not all a_1 and b_1 are zeroes and $a_2\neq 0$	If $a_3=0$ then $k_1'=e^{\varepsilon_1}a_2, k_2'=e^{\varepsilon_1}\varepsilon_3a_2+a_4,$ $k_3'=e^{\varepsilon_1}b_2, k_4'=e^{\varepsilon_1}\varepsilon_3b_2+b_4$ If $a_2a_3>0$ then $k_1'=\sqrt{\frac{a_3}{a_2}}a_2,$ $k_2'=\sqrt{\frac{a_3}{a_2}}(\varepsilon_3-\varepsilon_2)a_2+a_4,$ $k_3'=\sqrt{\frac{a_3}{a_2}}b_2, k_4'=\sqrt{\frac{a_3}{a_2}}(\varepsilon_3-\varepsilon_2)b_2+b_4$ If $a_2a_3<0$ then $k_1'=\sqrt{-\frac{a_3}{a_2}}a_2, k_2'=\sqrt{-\frac{a_3}{a_2}}(\varepsilon_3+\varepsilon_2)a_2+a_4,$ $k_3'=\sqrt{-\frac{a_3}{a_2}}b_2, k_4'=\sqrt{\frac{a_3}{a_2}}(\varepsilon_3+\varepsilon_2)b_2+b_4$
$a_2=b_2=0$	$k_1'=e^{-\varepsilon_1}a_3, k_2'=-e^{-\varepsilon_1}\varepsilon_2a_3+a_4,$ $k_3'=e^{-\varepsilon_1}b_3, k_4'=-e^{-\varepsilon_1}\varepsilon_2b_3+b_4$
$a_3=b_3=0$	Lie algebra $\{a_4v_4, b_4v_4\}$ is trivial.

On the other hand, these Lie algebra notions and commutator relations are suitable for the attraction property, the third property, of the inertial manifold. Finally, of Theorem 1, it can be conclude that if $\{v_1,\dots,v_n\}$ be an integrable system of vector fields such that the dimension of the span of $\{v_1|_x,\dots,v_n|_x\}$ in $TM|_x$ is a constant s, independent of $x\in M$, then for each $x_0\in M$ there exist flat local coordinates $y=(y^1,\dots,y^m)$ near x_0 such that the integral submanifolds intersect the given coordinate chart in the slices $\{y: y^1=c_1,\dots,y^{m-s}=c_{m-s}\}$. Also, $c_1,\dots,c_{m-s}$ are arbitrary constants which specially give one the representation (2).

Corollary 1. For a semi-regular (and therefore for a regular) system the decomposition (1) can be reduced to the direct sum.

If G act semi-regularly on the m-dimensional M with s-dimensional orbits and $x_0 \in M$, then by Frobenius' theorem, there exist precisely m-s functionally independent local invariants $\zeta^1(x),\ldots,\zeta^{m\text{-}s}(x)$ defined in a neighbourhood of x_0.

Moreover, any other local invariant of the group action defined near x_0 is of the form $\zeta(x)=F\left(\zeta^1(x),\ldots,\zeta^{m\text{-}s}(x)\right)$ for some smooth function F. On the other hand, any semi-regular system can be made regular, by restriction to a suitably small open subset of M. If the action of G is regular, then the invariants can be taken to be globally invariant in a neighbourhood of x_0.

Then the coordinate chart can be chosen so that each integral submanifold intersects it in at most one such slice. Then as structure of the proof of Theorem 2, it can be obtained the direct sum of relation (1) which specially give one the more exact coordination for the linearization of NSE.

5 Conclusion

In this paper, a novel methodology was presented to find the inertial manifolds of the Navier-Stokes equation (NSE) by developing a reformulation based on differential and Lie groups. For this purpose, some geometric notions about group-invariant solutions, commutator relation, adjoint representation, two-dimensional optimal systems of NSE studied.

The machinery of Lie algebra theory was applied to provide a systematic method to search for these special group invariant solutions. This geometric approach gives one more concepts of NSE and specially, covered all of the properties of inertial manifolds of positively invariant, exponentially attraction of all orbits.

The main advantage of the proposed methodology is that it outperforms the numerical estimation approach for approximate solution of inertial manifolds since it provides the exact solution of the problem and, thus, it yields better results than those obtained through numerical estimation.

Also, in general case, our Lie algebraic structure results to the reduction theory for simplification of a system of differential equations with fewer variables which are more easily solved in principal than those PDEs.

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