Regression Analysis to Degrade the Highly Nonlinear Lumped Equation for Coupled Natural Convection and Radiation in Gases into a Bernoulli Equation

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Abstract: Within the framework of the lumped model, unsteady heat conduction takes place in a quasi-isothermal body whose mean temperature changes with time only. Fundamentally, the lumped model subscribes to the notion that the internal conductive resistance in a solid body is negligible with respect to the external convective resistance at the solid/fluid interface. The short technical paper seeks to establish an alternate basis for the utilization of the lumped model embodying heat interaction by coupled natural convection and radiation between a simple solid body and a quiescent gas. The governing lumped equation is highly nonlinear and needs to be solved by numerical methods, like the Runge-Kutta-Fehlberg algorithm. Utilizing regression analysis for the total heat transfer coefficient varying with the temperature excess, nonlinear lumped equation is conveniently transformed into a milder nonlinear Bernoulli equation. Despite that the latter equation is still nonlinear, it admits an exact analytic solution. The step-by-step computational procedure is developed in a case study centered in a horizontal solid cylinder cooled by air.

Keywords: natural convection, radiation, nonlinear lumped model, lumped Biot number criterion, Bernoulli equation.

Nomenclature

A surface area, m^2

Bi_l lumped Biot number, $\frac{h_T}{k_s}\left(\frac{V}{A}\right)$, dimensionless

c_v specific heat capacity at constant volume, J/kg K

c_p specific heat capacity at constant pressure, J/kg K

D diameter of long cylinder, m

g gravitational acceleration, m/s^2

h_C natural convection coefficient, W/m^2 K

h_R radiation coefficient, W/m^2 K

h_T total heat transfer coefficient, W/m^2 K

k thermal conductivity, W/m K

n exponent in Bernoulli equation (14)

Nu_D mean Nusselt number, $\frac{h_C D}{k_g}$, dimensionless

n exponent in Bernoulli equation (14)

p coefficient in Bernoulli equation (14)

Pr Prandtl number, $\frac{\mu c_p}{k_g}$, dimensionless

q coefficient in Bernoulli equation (14)

R radius of long cylinder, m

Ra_D standard Rayleigh number, $\frac{g\beta}{\nu\alpha}(T_s - T_g)D^3$, dimensionless

$Ra_{D,i}$ initial Rayleigh number for gas,

$\frac{g}{\nu\alpha T_g}(T_i - T_g)\, D^3$, dimensionless

t time, s

T temperature, K

T_i initial temperature, K

T_f film temperature, K

T_g gas temperature, K

T_s surface temperature, K

V volume, m^3

Greek symbols

α thermal diffusivity, m^2/s

β coefficient of volumetric thermal expansion, 1/K

ΔT temperature excess, $T - T_g$

ε total surface emissivity, dimensionless

μ dynamic viscosity, N s/m^2

ν kinematic viscosity, m^2/s

ρ density, kg/m^3

σ Stefan-Boltzmann constant, W/m^2 K^4

Subscripts

f refers to film

g refers to gas

s refers to solid

1 Introduction

When a solid body at a given temperature is immersed in an extensive fluid at a different temperature, heat conduction takes place inside the body and heat exchange between the body surface and the fluid usually occurs by convection and/or radiation (Mills [1]).

There are two resistances associated with the body/fluid ensemble: (1) an internal conductive resistance inside the body and (2) an external convective resistance or an external radiative resistance at the body/fluid interface. Whenever the external convective resistance dominates the internal conductive resistance, this translates into a small temperature difference between the center and the surface of the body and a large temperature difference between the body surface and the surrounding fluid. Putting this statement in perspective, it connotes that during the cooling or heating period, the solid body can be considered as a "lump" with nearly uniform temperature at any instant of time. In other words, unsteady heat conduction takes place in a spatially quasi-isothermal body so that the mean temperature changes only with time. This physical rationale sets the groundwork for the simple lumped model instead of the complex differential model [1]. Regardless whether the solid body is regular or irregular, the lumped model is oversight by the lumped Biot number $Bi_l < 0.1$, where h is the convection coefficient, k_s is the thermal conductivity, V and A are the volume and surface area of the body.

Setting aside forced convection, if natural convection cools or heats a solid body, the nonlinear mode of heat transfer is vulnerable to instantaneous changes in the body temperature. Therefore, the corresponding natural convective coefficient h_C does not stay constant, but varies with the mean temperature, which in turn varies with time. In addition, if the environment is a gas or a vapor, the magnitudes of natural convection and radiation may be comparable. In view of this, the governing lumped equation turns highly nonlinear necessitating numerical integration for its solution.

Focusing on combined heat transfer mechanisms, the total heat transfer coefficient h_T corresponds to the natural convection coefficient h_C plus the radiation coefficient h_R. In this work, linear regression analysis was applied to the tabulated total heat transfer coefficient h_T associated with the highly nonlinear lumped equation. As a direct result, the highly nonlinear lumped equation is degraded to a mildly nonlinear Bernoulli equation, giving way to an approximate analytic solution. As a case study, the lumped equation for coupled natural convection and radiation in the cooling of a horizontal solid cylinder will be solved in two ways: 1) by the numerical procedure using the Runge-Kutta-Fehlberg algorithm and 2) by the new approximate analytical procedure.

2 Lumped Equation for Coupled Natural Convection and Radiation

As sketched in Figure 1, a hot solid body at a uniform temperature T_i is immersed in a quiescent cold gas at a different temperature T_g. In general, the governing lumped equation for coupled natural convection and radiation along with the initial condition are

$$\rho_s c_{v,s} V \frac{dT}{dt} = -h_C A(T - T_g) - \varepsilon\sigma A\left(T^4 - T_g^{\,4}\right), \qquad T(0) = T_i \tag{1a}$$

This highly nonlinear equation may be rewritten in an alternate manner as

$$\rho_s c_{v,s} V \frac{dT}{dt} = -h_C A(T - T_g) - h_R A(T - T_g), \qquad T(0) = T_i \tag{1b}$$

where h_R stands for the radiation coefficient

$$h_R = \frac{\varepsilon\sigma\left(T^4 - T_g^4\right)}{T - T_g} \tag{2}$$

and T is in degrees Kelvin.

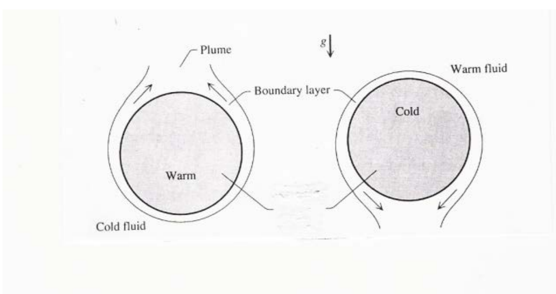

Figure 1. Horizontal solid cylinder immersed in a quiescent fluid

To deal with coupled natural convection and radiation, the lumped Biot number criterion

$$Bi_l = \frac{h_C}{k_s}\left(\frac{V}{A}\right) < 0.1 \tag{3a}$$

needs to be amplified to read

$$Bi_l = \frac{h_T}{k_s}\left(\frac{V}{A}\right) < 0.1 \tag{3b}$$

where the total heat transfer coefficient h_T equals the natural convection coefficient h_C plus the radiation coefficient h_R. That is,

$$h_T = h_C + h_R \tag{4}$$

Among the three simple solid bodies mostly used in engineering applications, the large plate, the long cylinder and the sphere, we chose the intermediate long cylinder whose volume–to–area ratio $\frac{V}{A} = \frac{D}{4}$. Accounting for this geometric characteristic, eq. (1a) reduces to

$$\rho_s c_{v,s} D \frac{dT}{dt} = -4h_C (T - T_g) - 4\varepsilon\sigma\left(T^4 - T_g^{\,4}\right), \qquad T(0) = T_i \tag{5a}$$

and its companion eq. (1b) reduces to

$$\rho_s c_{v,s} D \frac{dT}{dt} = -4\, h_C (T - T_g) - 4\, h_R (T - T_g), \qquad T(0) = T_i \tag{5b}$$

in terms of the radiation coefficient h_R given by eq. (2). In both equations (5a) and (5b), the enlarged lumped Bi number criterion becomes

$$Bi_l = \frac{h_T R}{k_s} < 0.2$$

where R is the radius. For laminar natural convection around a horizontal long cylinder, the mean Nusselt number correlation developed by Churchill and Chu [2] is

$$Nu_D = 0.36 + 0.518 \frac{Ra_D^{\;1/4}}{f(Pr)} \text{ for } Ra_D < 10^9 \tag{6}$$

where the standard Rayleigh number is

$$Ra_D = \frac{g\beta}{\nu\alpha}(T - T_g) D^3$$

$$= \frac{g}{\nu \alpha T_g} (T - T_g) D^3 \quad (7a)$$

because $\beta = \frac{1}{T_g}$ in K^{-1}, and $f\ (Pr)$ is the so–called "universal" Prandtl number function:

$$f\,(Pr) = \left[1 + \left(\frac{0.559}{Pr} \right)^{9/16} \right]^{4/9} \quad (7b)$$

Here, the intervening thermophysical properties of the gas are evaluated at the film temperature $T_f = \frac{T_s + T_g}{2}$. As stated by Holman [3], typical uncertainties in the determination of the natural convective coefficient h_C from most correlation equations, like eq. (6), lie within $\pm 10\%$ to $\pm 20\%$ margin.

Isolating the natural convective coefficient h_C for gases in eq. (6) knowing that $Pr = 0.71$ and $f\ (Pr) = 1.322$, the magnitude of h_C is expressed by the two–term expression in terms of primitive quantities:

$$h_C = 0.36 \frac{k_g}{D} + \left[\frac{0.39 k_g}{D^{1/4} f\,(Pr)} \left(\frac{g}{\alpha \nu T_g} \right)^{1/4} \left(T - T_g \right)^{1/4} \right] \quad (8)$$

Essentially here, h_C entails to a nonlinear single-value function of the temperature excess $T - T_g$.

For the case of cooling, eq. (8) turns over the largest $h_C = h_{C,max}$ happening at the initial time $t = 0$ where the temperature $T = T_i$. Correspondingly, $h_{C,max}$ is written as

$$h_{C,max} = 0.36 \frac{k_g}{D} + \left[\frac{0.39 k_g}{D^{1/4} f\,(Pr)} \left(\frac{g}{\alpha \nu T_g} \right)^{1/4} \left(T_i - T_g \right)^{1/4} \right] \quad (8a)$$

in the closed interval $[T_g, T_i]$, because $T_i > T_g$. Similarly, from eq. (2) in the case of cooling the largest radiation coefficient $h_R = h_{R,max}$ is

$$h_{R,max} = \frac{\epsilon \sigma \left(T_i^4 - T_g^4 \right)}{T_i - T_g} \quad (9)$$

In consequence, the lumped Biot number criterion for cooling a long cylinder is re-stated as

$$Bi_l = \frac{h_{T,\max} R}{k_s} < 0.2 \quad (10)$$

where $h_{T,max} = h_{C,max} + h_{R,max}$.

3 Case Study

Consider a long cylinder made of aluminum with diameter $D = 0.1$ m which is placed in a horizontal position. The long cylinder having initial temperature $T_i = 700$ K is immersed in stagnant air at a temperature $T_g = 300$ K. The objective is to determine the variation of the cylinder temperature with time under the influence of natural convection with radiation.

The thermophysical properties of aluminum at the film temperature $T_f = 500$ K are taken from References [1,3]: density $\rho_s = 8933$ kg/m^3, specific heat capacity at constant volume $c_{v,s} = 412$ J/kgK, thermal conductivity $k_s = 236$ W/mK, and total surface emissivity $\varepsilon = 0.80$.

As already stated, the nonlinear lumped equation for natural convection coupled with radiation is eq. (5),

$$\rho_s\, c_{v,s}\, D \frac{dT}{dt} = -4 h_C\ (T - T_g) - 4 \varepsilon \sigma \left(T^4 - T_g^{\,4} \right), \qquad T(0) = T_i \quad (5)$$

Further, evaluating the thermophysical properties of air at the film temperature $T_f = 500$ K [1,3], the proper expression for h_C turns out to be

$$h_C = \frac{0.014}{D} + 0.983 \frac{(T - T_g)^{1/4}}{D^{1/4}} \quad (11)$$

which is nonlinear. For the radiation part, the Stefan-Boltzmann constant is σ = 5.67 x 10^{-8} W/m^2K^4.

4 Presentation of Results

Eq. (5) was solved numerically using RKF45, a MATLAB library that implements the Runge-Kutta-Fehlberg algorithm [4]. Figure 2 shows the monotonic decreasing variation of temperature with time for the horizontal cylinder revealing that thermal equilibrium is reached at a time of 40,000 s. Additionally, Figure 3 contains the temperature-time variations for natural convection, radiation and natural convection combined with radiation, all within the umbrella of cooling.

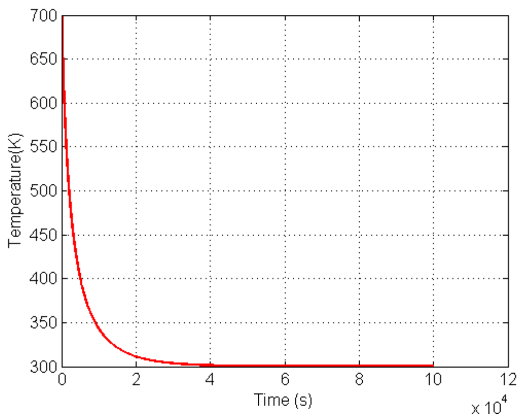

Figure 2. Temperature-time variation for coupled natural convection and radiation

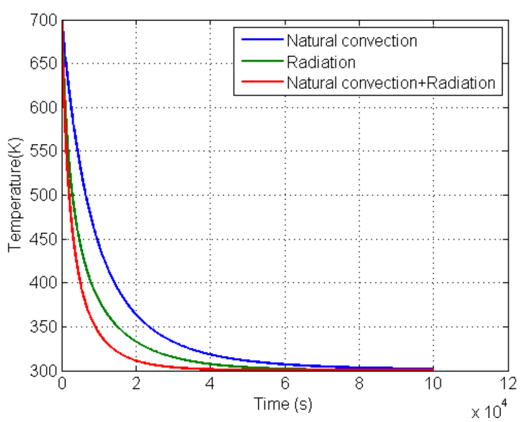

Figure 3. Comparison of the temperature-time variations for 1) natural convection, 2) radiation and 3) natural convection plus radiation

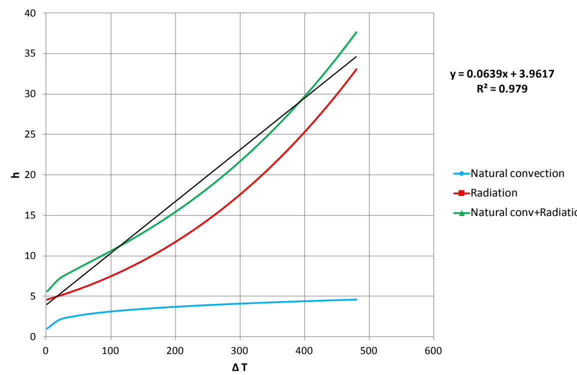

Figure 4. Variations of 1) the natural convection coefficient h_C, 2) the radiation coefficient h_R and 3) the total transfer coefficient h_T with the temperature excess ΔT

In Figure 4, the natural convection coefficient h_C given in eq. (11), the radiation coefficient h_R given in eq. (3) and the resultant total heat transfer coefficient $h_T = h_C + h_R$ are plotted on the ordinate and the temperature excess ΔT on the abscissa. Pausing here for a moment, it is observable in the figure that the uppermost curve for h_T varying with ΔT has a quasi linear shape. This behavior is exploited right away to perform a linear regression analysis of h_T versus ΔT. The MATLAB function polyfit [4] subsequently delivers the straight line

$$h_T = 3.9617 + 0.064\left(T - T_g\right) \quad (12)$$

with a high correlation coefficient R^2 = 0.979.

Next, substituting eq. (12) into eq. (5) provides a degraded lumped equation

$$\rho_s c_{v,s}\frac{dT}{dt} = -\frac{3.9617}{D}\left(T - T_g\right) - \frac{0.0639}{D}\left(T - T_g\right)^2 \quad (13)$$

which is still nonlinear because of the term $(T - T_g)^2$.

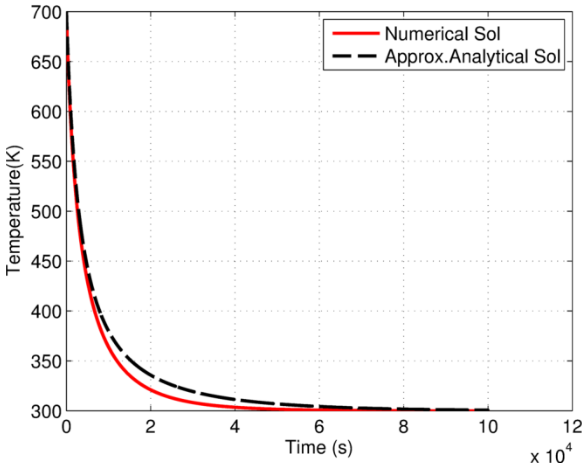

Figure 5. Comparison between the numerical solution T vs t of the highly nonlinear lumped equation (5) and the approximate analytical solution of the equivalent mildly nonlinear Bernoulli equation, eq. (15)

Within the classification of ordinary differential equations,

$$\frac{dy}{dx} + py = qy^{n}, \qquad n \neq 0, 1 \tag{14}$$

is named Bernoulli equation (Polyanin and Zaitsev [5]) where p and q are constant coefficients. Hence, the exact analytic solution of eq. (13) taken from [5] corresponds to

$$T(t) = T_g + \left\{ \left[\left(T_i - T_g\right)^{-1} + \frac{p}{q}\exp(-qt) - p \right] \right\}^{-1} \tag{15}$$

where the constants p and q are taken from the ratios

$$p = -\frac{0.2556}{\rho_s\, c_{v,s}\, D}$$

and

$$q = -\frac{1.5847}{\rho_s\, c_{v,s}\, D}$$

In consequence, eq. (15) supplies the approximate analytic temperature–time variation in the horizontal long cylinder, which is displayed in Figure 5 along with the temperature–time lumped variation obtained numerically from solving the highly nonlinear equation (5). Here, it is recognized that the approximate analytical solution of the equivalent Bernoulli equation is more conservative than the numerical solution of the original nonlinear equation using the Runge-Kutta-Fehlberg algorithm. From a qualitative standpoint, the largest temperature discrepancy between the two approximate solutions exposes a difference of around 15-20 K in the mid temperature region. Without question, this disagreement is considered small for engineering applications.

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