On the characterization of non-linear diffusion equations. An application in soil mechanics

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Abstract: - The search of dimensionless groups in engineering problems ruled by partial differential equations presents many problems whereby it is an untreated topic in the scientific literature. The main difficulties arise in the suitable choice of the reference quantities needed to define the dimensionless variables which must also be normalized, i.e., extended to the range of values (0,1). After setting the steps for a correct nondimensionalization protocol in this kind of problems, its application is illustrated by studying the soil consolidation problem, a process in which the constitutive dependences between the physical parameters and the dependent variables are strongly non-linear. Results are verified by numerical simulations.

Key-Words: - Nondimensionalization, dimensionless groups, soil consolidation, non-linear

1 Introduction

The nondimensionalization of the mathematical model (governing equations and boundary conditions) is a known method currently used for extracting the dimensionless groups that influence the solution of a large variety of complex problems in physics or engineering. This process rests on the properties of homogeneous functions, long time ago set by Buckingham in his paper of 1914 [1]. His famous pi-theorem, applied in the same manuscript to problems of different fields of physics, can be stated as follows: ‘any equation or system of equations that contains mathematical formulation of laws that determine a physical phenomenon can be represented as a r elation between dimensionless quantities’. From that date, a lot of papers have been published in the search for those relations or dimensionless numbers, a basic information for modelling [2].

Among the clearly differentiated protocols for the search of these numbers is, firstly, the dimensional analysis. This procedure, very extended in the literature despite its poor results in complex problems, does not part of the governing equations but a list of relevant variables from which a number of independent monomials of zero dimension can be set [3]. The difficulty of obtaining accurate results with this technique is the cause of its strong rejection by the majority of researchers. Secondly, the dimensionless groups may be obtained by writing, whether for theoretical reasoning – for which a deep understanding of the phenomena involved in the problem is required – or laboratory tests, the ratios between magnitudes that balance each other in the problem. Many famous numbers, such as R eynolds [4], were deduced in this way. Finally, the sought dimensionless numbers may be derived by relatively simple mathematical manipulation from the dimensionless-normalized form of the governing equations, for which a correct deduction of the same is required. This last procedure, called nondimensionalization process, leads to the most precise solution and is the one we will follow in this work.

However, the use of this technique in the scientific literature – both journals and specialized books – to obtain the dimensionless groups has been reduced to linear processes. Thus, lots of works devote to apply the method in various engineering problems ruled by partial differential equations – such as heat transfer [5], fluid dynamics [6], fluid flow and solute transport [7] and geothermic [8] – or by ordinary differential equations – such as mechanical engineering [9]. In all of them, the authors do not distinguish the potentially anisotropic character of the medium which causes a l ack of discrimination and leads inexorably to a less precise solution. Discrimination [10] assumes as different quantities those of the same nature but with an inherent vector character – for example, two perpendicular velocities in 2D problems are assumed to be dimensionally different quantities with discrimination – and joins classical non-discriminated dimensionless numbers in the new, more precise, independent groups that really rule the solution of the problem.

As regards non-linear problems, very few works can be found in which the nondimensionalization procedure is applied to derive dimensionless groups, whether in isotropic or anisotropic domains [7,8]. The reasons for this absence may be several but, undoubtedly, the most important is that non-linear problems generally involve non pr oportional dependences between one or more parameters and the dependent or, eventually, independent variables. The result is that the non-linear terms within the governing equations cannot turn into dimensionless terms in a direct way, doing so that this equation is very difficult to manage in order to derive the sought dimensionless groups.

From the point of view of its performance, nondimensionalization is a simple method that may – indeed sometimes should – precede or even supplant rigorous mathematical analysis. To set an equation in its dimensionless form we must firstly select the reference quantities, explicit or implicit in the statement, that normalize the dimensionless dependent and independent variables, for which a deep understanding of the physical processes involved is required. This normalization allows to average to an order of magnitude unity the changes of these variables and their derivatives. Once the governing equation is simplified the dimensionless numbers are the independent ratios between two addends of the equation.

After summarize the steps of the procedure in the next section, nondimensionalization is applied to the non-linear soil consolidation problem. The derived groups provided by the nondimensionalization are verified by a set of numerical simulations using the network method [11].

2 Procedure of nondimensionalization

The steps to apply the nondimensionalization procedure are summarized as follows:

i) To choice the references (explicit or not in the problem statement) to make dimensionless and normalized both the dependent and independent variables. References do not given in the statement are treated as unknowns of the problem.

ii) To define the dimensionless and normalized – dependent and independent – variables. According to an appropriate choice of references, the range of values of these variables should be extended to the interval (0,1) or very close to it.

iii) To yield the dimensionless governing equations. To do this, each term of the equation is averaged so that the factors of such term related to the dimensionless normalized variables and their changes are assume to be of order of magnitude unity in first approximation. Very sharp non-linearities in the problem may eventually move away from this assumption leading to dimensionless numbers of an order of magnitude larger or smaller than unity, accordingly. Also, the existence of complex mathematical expressions of the dependent variables that come from the constitutive relations should be studied in a particular way.

iv) To get the dimensionless numbers as the quotients formed by the groupings of the parameters of the problem. There are as many dimensionless groups as terms of the equation minus one. However, the number of terms may be less than expected since some numbers would appear in more than one equation. Again, non-linearities could make it advisable to separate a complex group into two for easier management for the engineer. In addition, the set of independent final dimensionless groups can be established in different forms by simple mathematical manipulation. For convenience, the final set can be chosen in such a way that each unknown appear in a single group.

v) If there are m+n dimensionless groups from which each of π_m contains a different unknown and the rest (π_n) does not contain unknowns, the solution for π_m is an arbitrary function (ψ) of the π_n groups:

$$\pi_{i,1\leq i\leq m} = \psi(\pi_{j,m+1\leq j\leq m+n}) \quad (1)$$

If all the groups are of the order of magnitude unity, it is evident that the arbitrary function also has this property. From the m-relations (1), the order of magnitude of each unknown can be obtained.

3 Nomenclature

$c_{v,1}$ initial coefficient of consolidation (m^2/s)
e void ratio (dimensionless)
e_o initial void ratio (dimensionless)
H_1 initial thickness (m)

H_2 final thickness (m)
k hydraulic conductivity (m/s)
k_1 initial hydraulic conductivity (m/s)
m_v coefficient of volumetric compressibility (m^2/N)
$m_{v,1}$ initial coefficient of volumetric compressibility (m^2/N)
q_o surface applied load (N/m^2)
t time (s)
u excess pore water pressure (N/m^2)
$\bar{U}_{\sigma'}$ average degree of pressure dissipation (dimensionless)
V volume (m^3)
z vertical spatial coordinate (m)
γ_k nonlinear coefficient of change of permeability with effective pressure (dimensionless)
γ_v non-linear coefficient of compressibility (dimensionless)
γ_w specific weight of water (N/m^3)
π_i dimensionless group or number (dimensionless)
σ' effective pressure (N/m^2)
σ_1' initial effective pressure (N/m^2)
σ_2' final effective pressure (N/m^2)
σ_m' mean value of the effective pressure (N/m^2)
$\tau_{o,\sigma'}$ characteristic time which takes the excess pore pressure to dissipate until approximately the value of zero (s)
Ψ arbitrary mathematical function

Subscripts
I, II… denote different numbers

Superscripts
$'$ denote dimensionless magnitude

4 The soil consolidation problem

A physical scheme of this problem is depicted in Fig.1. When a layer of clay soil saturated with water suddenly submits to loads on its surface, the initial excess pore water pressure gradually dissipates until all the load is supported by the soil skeleton. It is a typically asymptotic diffusion process whose duration (or characteristic time) depends on t he hydrological and mechanical soil properties.

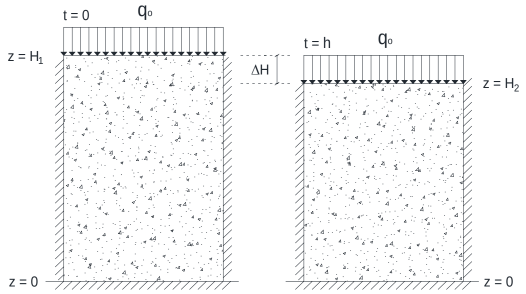

Figure 1. Physical scheme of 1D consolidation

The non-linearities of this civil engineering problem [12,13] relate to the constitutive dependences of the parameters void ratio (e) and hydraulic conductivity (k) on the effective pressure, generally of potential or logarithmic type. Since the formal procedure of nondimensionalization does not depend on the constitutive dependences, we will assume those of the model of Juárez-Badillo [14].

4.1 Mathematical model

Under the conditions imposed by the oedometer test

$$\frac{\partial u}{\partial z} = -\frac{\partial \sigma'}{\partial z}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \sigma'}{\partial t}$$

the constitutive dependences

$$\frac{dV}{V} = -\gamma_v \frac{d\sigma'}{\sigma'}, \quad \frac{dk}{k} = -\gamma_k \frac{d\sigma'}{\sigma'} \tag{2}$$

and some mathematical manipulation, the governing equation

$$\frac{d\sigma'}{dt} = -\frac{1}{\gamma_w m_v}\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right) \tag{3}$$

can be written in terms of an only dependent variable, the effective pressure σ'

$$\frac{\partial \sigma'}{\partial t} = \frac{k_1 \sigma_1'}{\gamma_w \gamma_v}\left(\frac{\sigma'}{\sigma'_1}\right)^{1-\gamma_k}\left[-\frac{\gamma_k}{\sigma'}\left(\frac{\partial \sigma'}{\partial z}\right)^2 + \frac{\partial^2 \sigma'}{\partial z^2}\right] \tag{4}$$

where

$$\frac{k_1 \sigma_1'}{\gamma_w \gamma_v} = \frac{k_1}{\gamma_w m_{v,1}} = c_{v,1} \tag{5}$$

is the initial coefficient of consolidation.

4.2 Dimensionless groups

To make dimensionless equation (4), the dimensionless variables $(\sigma')' = \frac{\sigma' - \sigma_1'}{\sigma_2' - \sigma_1'}$, $t' = \frac{t}{\tau_{o,\sigma'}}$ and $z' = \frac{z}{H_1}$ are defined, with $\tau_{o,\sigma'}$ the characteristic time – an unknown reference. Thus, equation (4) writes in its dimensionless form as

$$\frac{(\sigma_2'-\sigma_1')}{\sigma_m'\tau_{o,\sigma'}}\frac{\partial(\sigma')'}{\partial t'} = -\frac{c_{v,1}\gamma_k}{\sigma_1'^2 H_1^2}\left(\frac{\sigma_m'}{\sigma_1'}\right)^{-\gamma_k-1}(\sigma_2'-\sigma_1')^2\left(\frac{\partial(\sigma')'}{\partial z'}\right)^2 + \\ +\frac{c_{v,1}}{\sigma_1' H_1^2}\left(\frac{\sigma_m'}{\sigma_1'}\right)^{-\gamma_k}(\sigma_2'-\sigma_1')\frac{\partial^2(\sigma')'}{\partial z'^2} \quad (6)$$

with σ_m' a mean value for the effective pressure. By averaging the above equation and deleting the terms whose order of magnitude is unity – dimensionless variables and their derivative expressions –, the three emergent coefficients

$$\frac{1}{\tau_{o,\sigma'}},\quad \left(\frac{k_1\sigma_1'}{\gamma_w\gamma_v}\right)\frac{\gamma_k}{H_1^2}\left(\frac{\sigma_2'}{\sigma_1'}-1\right)\left(\frac{\sigma_m'}{\sigma_1'}\right)^{-\gamma_k}, \\ \left(\frac{k_1\sigma_1'}{\gamma_w\gamma_v}\right)\frac{1}{H_1^2}\left(\frac{\sigma_m'}{\sigma_1'}\right)^{1-\gamma_k} \quad (7)$$

give rise to two dimensionless independent groups:

$$\left.\begin{array}{l}\pi_I = \left(\frac{\gamma_w\gamma_v H_1^2}{\tau_{o,\sigma'}k_1\sigma_1'}\right)\left(\frac{\sigma_m'}{\sigma_1'}\right)^{\gamma_k-1} \\ \pi_{II} = \gamma_k\left(\frac{\sigma_2'-\sigma_1'}{\sigma_m'}\right)\end{array}\right\} \quad (8)$$

The complex expression of π_{II}, dependent on σ_m', makes understandable the partition of this group into two more simple groups, γ_k and $\frac{\sigma_2'}{\sigma_1'}$, giving to σ_m' the value of any of the ends of its range since they are of the same order of magnitude. Thus, the groups π_I and π_{II} disclose in three groups

$$\left.\begin{array}{l}\pi_1 = \left(\frac{\tau_{o,\sigma'}k_1\sigma_1'}{\gamma_w\gamma_v H_1^2}\right) \\ \pi_2 = (\gamma_k) \\ \pi_3 = \left(\frac{\sigma_2'}{\sigma_1'}\right)\end{array}\right\} \quad (9)$$

Doing this, from pi theorem $\pi_1 = \psi(\pi_2,\pi_3)$, the order of magnitude of $\tau_{o,\sigma'}$ is given by

$$\tau_{o,\sigma'} = \left(\frac{\gamma_w\gamma_v H_1^2}{k_1\sigma_1'}\right)\Psi_\tau\left(\gamma_k,\frac{\sigma_2'}{\sigma_1'}\right) \quad (10)$$

As regards $\bar{U}_{\sigma'}$, since it also depends on time, the solution is given by

$$\bar{U}_{\sigma'} = \Psi_U\left[\frac{t}{\tau_{o,\sigma'}},\gamma_k,\frac{\sigma_2'}{\sigma_1'}\right] \quad (11)$$

with Ψ_τ and Ψ_U unknown functions of their arguments.

5 Verification of the results

This section is devoted to check the solutions found in the above section, i.e., the dependences for $\tau_{o,\sigma'}$ and $\bar{U}_{\sigma'}$, expressions (10) and (11) respectively. Eight sets of simulations have been run; in each one, some of the particular parameters or initial values of the problem have been changed to give, as appropriate, the same or different values to the dimensionless groups in the search of the same or different solutions. Changes in the values of the individual parameters are enough to cover much of the real scenarios.

First, a reference set is established with all the rest referred to it or compared to each other. The physical and geometrical characteristics that change are: γ_v, γ_k, e_o, H_1 (m), σ_1' (N/m^2), σ_2' (N/m^2) and k_1 (m/year), Table 1. The values of π_2 and π_3 are derived from them while π_1 is obtained once the characteristic time $\tau_{o,\sigma'}$ is read from the simulation, Table 2. The criterion for the choice of $\tau_{o,\sigma'}$ is the time required by the soil to reach 90% of the total excess pore pressure dissipation.

set	γ_v	γ_k	e_o	H_1	σ_1'	σ_2'	k_1
1	0.10	0.5	1	1	30000	60000	0.020
2	0.10	0.5	1	1	60000	120000	0.020
3	0.10	0.5	3	2	30000	60000	0.040
4	0.20	0.5	1	1	30000	60000	0.020
5	0.40	0.5	1	1	30000	60000	0.020
6	0.10	1.5	1	1	30000	60000	0.020
7	0.10	0.5	1	1	30000	120000	0.020
8	0.16	0.2	1	1	30000	46265	0.032

Table 1. Physical and geometrical parameters

set	$\tau_{o,\sigma'}$	π_1	π_2	π_3
1	1.118	0.684	0.5	2.00
2	0.559	0.684	0.5	2.00
3	2.236	0.684	0.5	2.00
4	2.236	0.684	0.5	2.00
5	4.472	0.684	0.5	2.00
6	2.219	1.358	1.5	2.00
7	0.875	0.536	0.5	4.00
8	1.067	0.65	0.2	1.54

Table 2. Characteristic time and dimensionless groups

In view of the results, Tables 1 and 2, the characteristic time ($\tau_{o,\sigma'}$) is different for each problem, depending on the parameters and geometric characteristics of the same. However, the dimensionless form of the characteristic time (π_1) remains invariant as long as the values of π_2 and π_3 are kept constant, monomials on which we have already seen it depends (10).

As for $\bar{U}_{\sigma'}$ (11), two representations of this variable are given, Fig. 2 and Fig. 3. In the first, $\bar{U}_{\sigma'}$ is represented as a function of time, so that for each of the 8 sets studied a different curve of $\bar{U}_{\sigma'}$ is obtained. In the second graph, when we make its representation as a function of the dimensionless time, $t/\tau_{o,\sigma'}$, we see that the curves are arranged very close to each other. In addition, all cases with equal values of π_2 and π_3, and therefore of π_1, are represented by a single curve, as in our case for sets 1 to 5. This is a sign that the nondimensionalization process has been applied successfully, providing the most precise solutions to the non-linear consolidation problem.

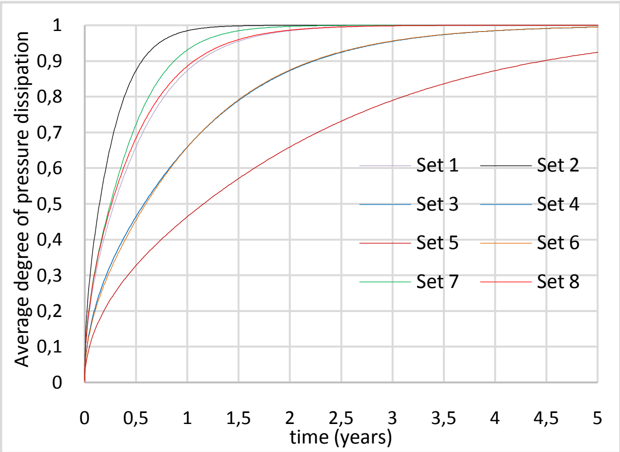

Figure 2. $\bar{U}_{\sigma'}$ as a function of time

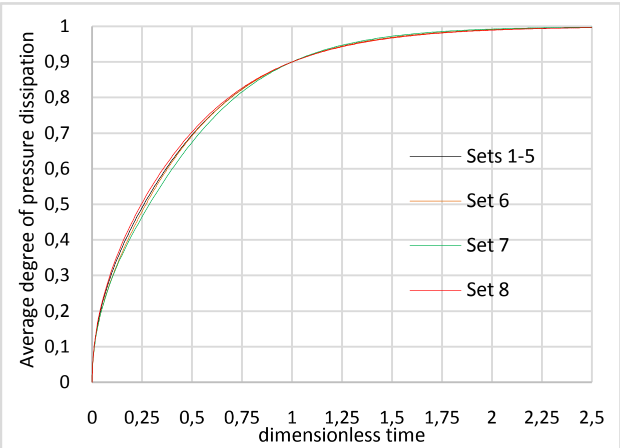

Figure 3. $\bar{U}_{\sigma'}$ as a function of dimensionless time

6 Conclusion

For the first time, as far as we know, the nondimensionalization procedure has been applied to a non-linear diffusion problem in order to determine the independent dimensionless groups that rule its solution. It has been demonstrated that the difficulties that emerge in the treatment of the nondimensionalization protocol can be surpassed what justifies the application of the method to such an important objective. The non-linearities inherent to soil consolidation are mainly related to the constitutive dependences between the parameters or coefficients of the governing equations and the dependent variable (they are potential type functions). These non-linearities are, perhaps, not very sharp, but undoubtedly complex in relation with the nondimensionalization protocol. In contrast to linear problems, attention should be paid to the choice of the references – to normalize the dependent and independent variables –, as well as the data averaging – needed to make unity the order of magnitude of the variables and their derivatives thus deleting them in the dimensionless governing equation –, allowing the coefficients of the averaged equation to be of the same order of magnitude.

Once the dimensionless number are derived, they can be organized in the best way for the management of the engineer and, as o ccurs in consolidation, separated in new groups (although it supposes a greater number), more familiar to the field in which the problem is involved.

To check the solutions provided by the nondimensionalization process applied to soil consolidation a set of simulations has been carried out by using a reliable numerical method. In these simulations, when the individual parameters are modified in such a way that the dimensionless groups do not change their value, the solutions remains unchanged.

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