Properties of the heat energy allocation models in systems with partial distribution of heat allocators

ALEKSANDAR HATZIVELKOS
University of Aplied Sciences Velika Gorica
Department of Mathematics
Zagrebačka 5, 10 410 Velika Gorica
Croatia
hatzi@vvg.hr

Abstract: Systems with partial distribution of heat allocators present a problem in terms of energy allocation. Since heat allocators provide pure numerical, and not the consumed energy value, energy consumed in the apartments without heat allocators is unknown. Furthermore, energy loss within the building is also unknown. Therefore, models for heat energy allocation are used. With those models, based upon partial (numerical) consumption readings, allocation to all apartments is formed. This paper addresses the allocation methods used for allocation of the heat energy in systems with partial distribution of the heat allocators. Mathematical definition and analysis of three heat energy allocation methods is given. Two of them were officially legislated in Croatia, and in use since year 2008. Third one is proposed in works of Hatzivelkos. Properties of allocation methods are introduced: consistency, monotonicity and local consistency. While consistency, as a global property can be viewed as a necessary allocation property, special attention is given to later two properties, monotonicity and local consistency. Those properties describe allocation methods from the perspective of a consumer. Mathematical analysis of allocation methods behavior in worst case scenario is given, i.e. scenario that produces the greatest error for observed allocation model. Another high consumer visibility concept is analyzed: consumption reading point value. For a consumer, it is only natural to seek relation between consumption reading and energy allocation, which is described with concept of allocated energy value of consumption reading points. Criteria for consumption reading point value comparison is introduced and described by usage of simulations. Finally, notions of new areas of heat allocation model analysis are given.

Key–Words: Heat energy allocation model, heat allocators, non-linear algorithms

1 Introduction

More than 150 000 households in Croatia have centralized heating system with heated water. Most of the heat energy consumers are connected to one common heat meter located in the heating substation of the building. These are the buildings built before 2001 in which the piping did not provide for individual metering of heat energy for each apartment. To make cost allocation fairer for customers on a common heat meter, there is a possibility of installing heat cost allocators. This article addresses the allocation methods used for allocation of the heat energy in systems with partial distribution of the heat allocators.

Hopefully, results will stimulate use of methods which can guarantee desirable properties of the heat energy allocation. Such methods can reinforce public confidence in fairness of individual energy consumption metering, even in situations where not all apartments in building have heat allocators installed.

2 Heat allocation models

A models for heat energy allocation among apartments connected to common heat meter presented in this article are based upon two assumptions:

1. All of the electronic heat allocators placed on heat emitters are of the same type. Therefore, their numerical values are comparable.
2. Not all apartments need to have electronic heat allocators placed on heat emitters, but apartments that do, have to have them placed on each heat emitter.

In this article we will use following notification:

1. There are m apartments connected to common heat meter. First k of them (k being smaller than m) have electronic heat allocators placed on each heat emitter. For those k apartments, with $N_1, N_2, ..., N_k$ we will denote the sum of consumption readings on heat allocators on all heat

emitters in those apartments in a given time period (for instance, in one month).

2. With $A_1, A_2, ..., A_m$ we will denote apartment living areas.

3. With E_{total} we will denote total amount of heat energy read on the common heat meter in a given time period (for instance, in one month).

4. With $E_1, E_2, ..., E_m$ we will denote an amount of energy allocated to each apartment in a given time period (for instance, in one month). It is clear that $E_{total} = \sum_{i=1}^{m} E_i$.

5. Finally, with α_A we will denote a factor which is determining a portion of energy total which is considered to be a energy loss or a common consumption in a building (energy used for heating of common areas). That portion of energy should be allocated with respect of an area share of the apartments.

First we will define model for energy allocation which was officially used for a heat energy allocation in Croatia (with minor modifications) from year 2008. to year 2015. [8] [9] [10] [11] [12]. Due to its structure, we shall call this model "*Static allocation model*", and it is given in following definition:

Definition 1 (Static allocation model, SAM) *For all $m - k$ apartments without installed heat allocators, their allocated energy equals to*

$$E_i = E_{total} \cdot \alpha_w \cdot \frac{A_i}{\sum_{j=k+1}^{m} A_j} \tag{1}$$

for all $i \in \{k+1, ..., m\}$. For all k apartments with installed heat allocators, their allocated energy equals to

$$E_i = E_{total}(1 - \alpha_w) \left[\alpha_A \frac{A_i}{\sum\limits_{j=1}^{k} A_j} + (1 - \alpha_A) \frac{N_i}{\sum\limits_{j=1}^{k} N_j} \right] \tag{2}$$

for all $i \in \{1, ..., k\}$, where α_w is a factor which defines the share of energy total that is allocated to all apartments without installed heat allocators:

$$\alpha_w = l \cdot \frac{\sum_{i=k+1}^{m} A_i}{\sum_{i=1}^{m} A_i}, \tag{3}$$

factor l being equal to 2.

Factor l which appears in the equation (3) in Definition 1, as part of the coefficient α_w (so called *correction factor*) changed several times in last eight years (from $l = 1$ in year 2008., $l = 1.25$ in year 2011., $l = 1.5$ in year 2014., to $l = 2$ in year 2015.) [8] [10] [12] [14]. With factor α_w model actually tries to determine the fixed share of the total energy used in apartments without the heat allocators. Because of this approach, we call model "*static*". Of course, share of apartments without the heat allocators in total energy could not be fixed, and it changes from one month to another. Furthermore, in some circumstances, the share of energy used in apartments without the heat allocators can be far greater than α_w. We will discus this in detail later in article.

As we can see from equation (1), the share of the total energy which is allocated to all apartments without the heat allocators is further allocated to apartments according to their area share. On the other hand, from equation (2) we can see that remaining energy is allocated to apartments with heat allocators as a convex combination of their share in area, and their share in the total number of consumption readings.

In year 2015. model for heat energy allocation was changed [13]. Although model kept its basic structure, new threshold was introduced. Therefore, model is named as "*Static allocation model with threshold*". That model, which was in use from April to November 2015. is given in following definition:

Definition 2 (SAM-wT) *We define threshold value T:*

$$N_t = \frac{\sum\limits_{j=1}^{k} N_j}{k}, \qquad T = \frac{\sum\limits_{\substack{j=1 \\ N_j < N_t}}^{k} A_j}{\sum\limits_{j=1}^{k} A_j} \tag{4}$$

For all $m - k$ apartments without installed heat allocators, their allocated energy equals to

$$E_i = E_{total} \cdot \alpha_w \cdot \frac{A_i}{\sum_{j=k+1}^{m} A_j} \tag{5}$$

for all $i \in \{k+1, ..., m\}$. For all k apartments with installed heat allocators, their allocated energy equals to

$$E_i = E_{total}(1 - \alpha_w) \left[\beta_A \frac{A_i}{\sum\limits_{j=1}^{k} A_j} + (1 - \beta_A) \frac{N_i}{\sum\limits_{j=1}^{k} N_j} \right] \tag{6}$$

for all $i \in \{1, ..., k\}$, where α_w is a factor which defines the share of energy total that is allocated to all apartments without installed heat allocators

$$\alpha_w = 2 \cdot \frac{\sum_{i=k+1}^{m} A_i}{\sum_{i=1}^{m} A_i}$$

and β_A equals to

$$\beta_A = \begin{cases} \alpha_A & \text{if } T \le 0.3 \\ T & \text{if } T > 0.3 \end{cases} \tag{7}$$

As we can see from equation (4), threshold value T was defined as a share in area of the apartments with heat allocators, of the apartments which consumption readings are greater than average consumption reading, N_t.

The allocation for apartments without heat allocators, remained the same as in a SAM model, defined with factor α_w. Allocation for apartments with heat allocators changed, and now it depends on a value of T. If T is less than 0.3, energy for those apartments is allocated the same way it was allocated in SAM. But if T exceeds 0.3, then instead of α_A, we use T as the split ratio between the part of energy allocated according to area share, and the part of energy allocated according to consumption readings share. This means, that if T is big, say 0.8, then 80% of energy will be allocated according to apartments area share, while only smaller part (20%) will be allocated according to consumption readings share.

To conclude with legislated models, we should point out that in November 2015. model for the allocation of heat energy in Croatia yet again changed [14]. The latest change abandoned usage of SAM-wT model, and returned to SAM model, with one difference. Value $l = 2$ in Definition 1 is not legislated by government anymore, but now it can be freely formed.

Let us finally define third model for heat energy allocation, which was proposed in works of Hatzivelkos [3][4][5].

Definition 3 (Dynamic allocation model, DAM) *For all $m - k$ apartments without installed heat allocators, we define their number of impulses (consumption readings) as*

$$N_i = \alpha_N \cdot A_i \cdot \max_{j=1,\dots,k} \left(\frac{N_j}{A_j} \right) \tag{8}$$

for all $i \in \{k+1, ..., m\}$, where is $\alpha_N \ge 1$ predefined ("motivation" or "punishment") factor, and it represents factor for which allocation of energy for an apartment without heat allocators should be greater than the allocation of energy for an apartment with heat allocators (with respect to apartments areas). Now we have

$$E_i = E_{total} \left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^{m} A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^{m} N_j} \right] \tag{9}$$

for all $i \in \{1, ..., m\}$.

We will call this model "*dynamic*" because it does not prescribe fixed share of total energy consumed in apartments without the heat allocators. Rather than that, DAM model establishes relations between apartments itself, therefore acting on a micro level. Relation between an apartment with heat allocators and one without heat allocators is defined by equation (8). Motivation for that relationship, where an apartment without heat allocators is assigned (relative) consumption reading greater or equal to maximal consumption reading among apartments with heat allocators, comes from stand that since we can not know the level of consumption in apartments without heat allocators, it is only fair to assume that it is maximal.

Let us now see how all three models allocate energy in following example:

Example 4 *The areas and consumption readings of ten apartments are given in the following table. First eight apartments have the heat allocators, while last two do not. Let us say that total energy that should be allocated to those apartments equals to $E_{total} = 1000$ MWh, and that factor α_A (which determines common part of energy, that should be allocated according to areas of the apartments) equals to $\alpha_A = 0.1$.*

Apartment	1.	2. − 4.	5. − 7.	8.	9. − 10.
Area	50	50	50	50	50
Impulses	80	20	10	0	−

In this example all areas are the same, so that difference in allocation methods would be more visible. If we use SAM method (see Definition 1), we get $\alpha_w = 0.4$, and so allocation to apartments without heat allocators (see equation (1)) equals to $E_9 = E_{10} = 200$. We calculate the allocations for apartments with the heat allocators from equation (2). See Table 1.

In case of SAM-wT model, average number of consumption reading N_t equals to $N_t = 21.25$. Therefore, all but the first apartment have consumption readings less than N_t, and threshold value T

equals to $T = 0.875$. This means that value for β_A equals to $\beta_A = T = 0.875$. The allocations for apartments without the heat allocators remain same as in SAM model, but allocations for other apartments change, with lowest allocations significantly rising. See Table 1.

Finally, if we use DAM model (with $\alpha_N = 1.1$), consumption readings prescribed for apartments 9. and 10. equals to $N_9 = N_{10} = 88$. Usage of equation (9) gives following allocation:

No.	A_i	N_i	SAM E_i	SAM-wT E_i	DAM E_i
1	50	80	261.62	100.92	218.09
2	50	20	71.03	74.45	62.02
3	50	20	71.03	74.45	62.02
4	50	20	71.03	74.45	62.02
5	50	10	39.26	70.04	36.01
6	50	10	39.26	70.04	36.01
7	50	10	39.26	70.04	36.01
8	50	0	7.50	65.63	10.00
9	50	–	200.00	200.00	238.90
10	50	–	200.00	200.00	238.90

Table 1: Example of SAM, SAM-wT and DAM allocations

As we can see in Table 1, in SAM model apartment 1. (with heat allocators) is assigned greater share of energy than apartments 9. and 10. (without heat allocators). SAM-wT model in this example produces allocation in which apartment 1. is allocated just 40% more energy than apartments 5., 6. and 7., although its energy consumption in eight times greater. Those observations will be in the focus of Section 3, in which we will analyze properties of the defined models for energy allocation.

3 Properties of the energy allocation models

3.1 Consistency

First thing that we expect from every model of energy allocation is consistency. This means that algorithm which assigns values of energy allocation to apartments, must do it in a way that the sum of all allocated energy is equal to total energy consumption reading. We will formalize this in following definition.

Definition 5 *We say an energy allocation model is consistent if*

$$\sum_{i=1}^{m} E_i = E_{total}.$$

Theorem 6 (Consistency) *Models for energy allocation SAM, SAM-wT and DAM are consistent.*

Proof: Let us first take a look at DAM model. Allocation to all apartments is given by equation (9). Let us sum up those values over all apartments:

$$\sum_{i=1}^{m} E_i = \sum_{i=1}^{m} E_{total}\left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^{m} A_j} + (1-\alpha_A)\frac{N_i}{\sum_{j=1}^{m} N_j}\right]$$

$$= E_{total}\sum_{i=1}^{m}\left[\alpha_A \frac{A_i}{\sum_{j=1}^{m} A_j} + (1-\alpha_A)\frac{N_i}{\sum_{j=1}^{m} N_j}\right]$$

$$= E_{total}\left[\sum_{i=1}^{m}\alpha_A \frac{A_i}{\sum_{j=1}^{m} A_j} + \sum_{i=1}^{m}(1-\alpha_A)\frac{N_i}{\sum_{j=1}^{m} N_j}\right]$$

$$= E_{total}\left[\frac{\alpha_A}{\sum_{j=1}^{m} A_j}\sum_{i=1}^{m} A_i + \frac{1-\alpha_A}{\sum_{j=1}^{m} N_j}\sum_{i=1}^{m} N_i\right]$$

$$= E_{total}\left[\alpha_A + 1 - \alpha_A\right] \quad = \quad E_{total}.$$

For SAM model, if we sum all allocations for apartments without heat allocators, given in equation (1) in Definition 1 we will get

$$\sum_{j=k+1}^{m} E_j = E_{total} \cdot \alpha_w.$$

On the other hand, if we sum all allocations for apartments with heat allocators, given in equation (2) we will get

$$\sum_{j=1}^{k} E_j = E_{total} \cdot (1-\alpha_w).$$

Therefore, if we sum all allocations, we have

$$\sum_{j=1}^{m} E_j = E_{total} \cdot (1-\alpha_w) + E_{total} \cdot \alpha_w = E_{total}.$$

Proof for SAM-wT model is carried out in the same way. If we sum allocations for apartments without heat allocators, given in equation (5) in Definition 2, and then allocations for all apartments with heat allocators given in equation (6) we would get the same conclusion as in the case of SAM model. Introduction of the threshold value T, and the factor β_A doesn't change argument. □

3.2 Monotonicity

Now it is time to analyze other properties of defined allocation models. One of the first, is concept of monotonicity, which is given in following definition.

Definition 7 *We say an energy allocation model is* monotone *if when all consumers maintain same level of energy consumption, except for one consumer who increases its energy consumption, then energy allocation to that consumer will increase.*

Monotonicity is important property for the heat energy allocation models. It ensures that energy allocation will follow most basic expectation: if you spend more, you will pay more. Unfortunately, not all of defined heat allocation models satisfy that property.

Theorem 8 (Monotonicity) *Models for energy allocation SAM and DAM are monotone, while SAM-wT is not.*

Proof: To prove that DAM model is monotone, suppose that consumption readings in one of the apartments with heat allocators increased from N_i to $N_i + \Delta N$, while consumption in all other apartments remained the same. This means that total energy consumption also increased from E_{total} to $E_{total} + k \cdot \Delta N$ for some factor $k > 0$. Let us compare energy allocations E_i before and after that increase. We want to prove that

$$(E_{total} + k \cdot \Delta N) \cdot \left[\alpha_A \cdot \frac{A_i}{\sum\limits_{j=1}^{m} A_j} + (1-\alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum\limits_{j=1}^{m} N_j} \right] >$$

$$> E_{total} \cdot \left[\alpha_A \cdot \frac{A_i}{\sum\limits_{j=1}^{m} A_j} + (1-\alpha_A) \frac{N_i}{\sum\limits_{j=1}^{m} N_j} \right]$$

This leads to

$$E_{total}(1-\alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum\limits_{j=1}^{m} N_j} + k \cdot \Delta N \cdot$$

$$\cdot \left(\alpha_A \cdot \frac{A_i}{\sum\limits_{j=1}^{m} A_j} + (1-\alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum\limits_{j=1}^{m} N_j} \right) > E_{total}(1-\alpha_A) \frac{N_i}{\sum\limits_{j=1}^{m} N_j}$$

We will prove that first summand on the left side of inequation is greater than right side of inequation:

$$E_{total}(1-\alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum\limits_{j=1}^{m} N_j} > E_{total}(1-\alpha_A) \frac{N_i}{\sum\limits_{j=1}^{m} N_j}$$

$$\Rightarrow \quad \frac{N_i + \Delta N}{\Delta N + \sum\limits_{j=1}^{m} N_j} > \frac{N_i}{\sum\limits_{j=1}^{m} N_j}$$

$$\Rightarrow \quad (N_i + \Delta N) \cdot \sum_{j=1}^{m} N_j > N_i \cdot \left(\sum_{j=1}^{m} N_j + \Delta N \right)$$

Last inequality holds when $\sum_{j=1}^{m} N_j > N_i$. The only time this inequality does not hold is when consumption in all other apartments (other than i-th one) equals to zero. But in that case, the original inequality holds because of factor $\alpha_A \cdot \frac{A_i}{\sum_{j=1}^{m} A_j}$ which does not depend on other apartments consumption readings.

On the other hand, if consumption is increased in an apartment without heat allocators, then all consumption readings in DAM model will remain the same. But increase of E_{total} then leads to increase of all allocations, including one in an apartment which increased consumption.

Proof of the monotonicity for the SAM model follows the similar argument as one for DAM model. But when we analyze SAM-wT model, we can see that same argument does not hold. Increase of energy consumption in one apartment with heat allocators, leads to increase of an average consumption reading N_t, and consequently possible increase of value T (see equation (4) in Definition 2). Increase of T can lead to increase of β_A (see equation (7) in Definition 2), which can significantly change outcome of energy allocation, putting much more weight on areas of the apartments instead of consumption readings. This can be seen in Example 9. □

Example 9 *Allocation is being made for a house with ten apartments. First eight apartments have the heat allocators, while last two do not. Let us say that total energy that should be allocated to those apartments equals to* $E_{total} = 1000$ *MWh, and that factor* α_A *(which determines common part of energy, that should be allocated according to areas of the apartments) equals to* $\alpha_A = 0.1$. *Let us suppose that one point in consumption readings represent* 7.5 *MWh of consumed energy (which is consistent with the consumption in last two apartments being similar to consumption in first apartment). SAM-wT model is allocating heat energy as shown in following table:*

No.	A_i	N_i	SAM-wT E_i	N'_i	SAM-wT E'_i
1	50	29	205.73	31	95.74
2	50	10	75.85	10	76.01
3	50	10	75.85	10	76.01
4	50	10	75.85	10	76.01
5	50	10	75.85	10	76.01
6	50	10	75.85	10	76.01
7	50	0	7.50	0	66.61
8	50	0	7.50	0	66.61
9	50	–	200.00	–	203.00
10	50	–	200.00	–	203.00

As we can see, increase of consumption reading for two points in first apartment resulted with decrease of its energy allocation. Before that consumption increase, average consumption reading was equal to $N_t = 9.88$*, and so we had* $T = 0.25$*, and* $\beta_A = 0.1$*. After increase of consumption reading in first apartment, average consumption reading rose to* $N'_t = 10.13$*, and consequently we have* $\beta_A = T = 0.875$*.*

From Example 9 we can see what distribution of consumption readings must look like, so that increase of reading for one consumer deceases its energy allocation. For that to happen, significant portion of apartments should have close consumption readings, slightly above average consumption reading N_t (see Definition 2). The greater the number of apartments whose readings are slightly above N_t, the greater will be impact of the threshold trigger.

Simulation provides us powerful tool for exploring such situations. Following simulation in Example 10 was computed in Wolfram Mathematica.

Example 10 *Allocation is being made for a house with fifty apartments. First forty of them have the heat allocators, while last ten do not. Energy consumption readings for those forty apartments is given in following vector:* $(i, 0, 0, 0, 0, 0, 10, 10, 10, 10, 10, 10, 10, 10, 10, 11, 11, 11, 11, 11, 11, 11, 11, 11, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12)$*. Furthermore, we will assume that those consumption readings represent consumption represented in percents of total available energy for each apartment. Therefore, consumption readings for the apartments without heat allocators (which is not measured, so for a modeling we should consider it as unknown) should equal to 100. Consumption for the first apartment is labeled with the unknown,* $i \in \{0, 100\}$*. While consumption of all other apartments is fixed, to analyze monotonicity (see Definition 7), consumption reading of the first apartment will rise from 0 to 100. Behaviour of all three allocation models (SAM, SAM-wT and DAM) is given in following figures.*

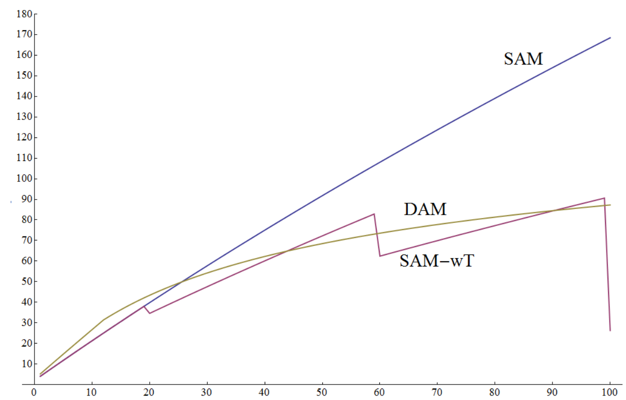

Figure 1: The energy allocation comparison of models with regard to monotonicity, for an apartment with energy consumption increase

On a graph in a Figure 1, on a x-axis we can read consumption reading (percentage of consumption, see Example 10) in one (first) apartment, while consumptions in other apartments are fixed. On the y-axis we can read level of energy allocation to the same apartment. Again, based on the construction of the example, those reading can be viewed as an percentage of an amount of heat energy available for consumption in that apartment.

As we can see in Figure 1, SAM and DAM models produce monotone allocation, while allocation of SAM-wT model fails demand of monotonicity in three occasions. There are significant allocation decreases for the apartment which is increasing its consumption. One other thing we can notice in this graph, is that the energy allocation for observed apartment under the SAM model goes over the value 100, which means that SAM model allocates greater amount of energy to that apartment than it is available for consumption. We will look deeper into this scenario in Chapter 4.

It is also interesting to see allocation to other apartments which, in this scenario, are consuming

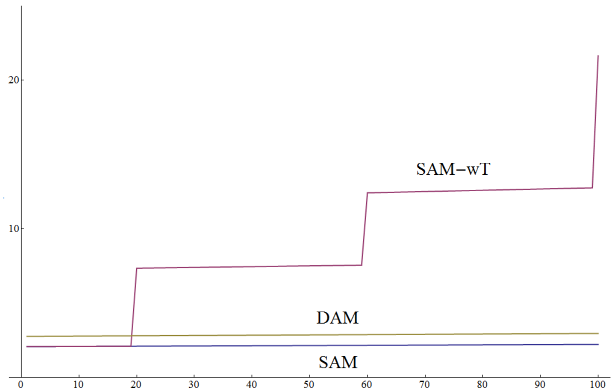

Figure 2: The energy allocation comparison of models with regard to monotonicity, for an apartment with no energy consumption

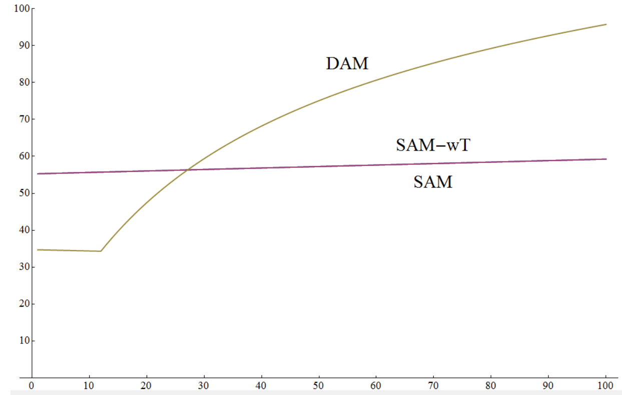

Figure 3: The energy allocation comparison of models with regard to monotonicity, for an apartment without heat allocators

same amount of energy all the time (while energy of first apartment ranges from 0 to 100). In Figure 2 we can see how SAM, SAM-wT and DAM model allocate energy to an apartment with heat allocators, and consumption reading equal to zero. In simulation described in Example 10 $\alpha_A = 0.1$ value is used (see Definitions of all three models). So it is expected that there would be some allocation in all of the models. But, while allocation to that apartment remains almost the same when using SAM and DAM model, energy allocation through SAM-wT model results with allocation jumps.

This way we can see that SAM-wT allocation to the apartment with no consumption significantly depends upon consumption in an another apartment. This rises an interesting question: how to measure (or define) resistance of the allocation model in a way that allocation to an apartment depends as less as possible of the consumption in another apartment. For now, this is an open question.

Finally, in Figure 3 we can see allocation to an apartment without heat allocators. On x-axis we can read (percentage of) consumption reading in first apartment with heat allocators, and on y-axis we again read (percentage of) consumption allocation for apartment without heat allocators. We can see that SAM and SAM-wT model allocates exactly the same amount of energy to the apartment, which stays the almost the same at allocation level 55. This is because of ”static” nature of those models, since they apriori determine total amount of energy for all apartments without heat allocators. On the other hand, DAM model allocation for that apartment ranges from 35 up to 95; so this time allocation of an apartment without heat allocators depend significately upon consumption in an another apartment. It is also good to point out that in this situation consumption of that apartment is supposed to be maximal, ie. at consumption level 100.

3.3 Local consistency

Although not being monotone is severe disadvantage for a energy allocation model, it is not easily open to manipulation. Consumers does not have information about current consumption and distribution of consummation readings, so it is very hard to manipulate own consumption in order to take advantage of lack of monotonicity. There is, however, another property which is very visible form the perspective of the consumer - a comparison with other consumers within the same building.

Definition 11 *We say an energy allocation model is* locally consistent *if its allocations to apartments, E_i, satisfy following:*

1. *For allocations E_i and E_j allocated to two apartments with same area and installed heat allocators, there is $E_i < E_j$ iff $N_i < N_j$.*

2. *For allocations E_i and E_j allocated to two apartments with installed heat allocators and same consumption readings, there is $E_i < E_j$ iff $A_i < A_j$.*

3. *For allocations E_i and E_j allocated to two apartments without installed heat allocators, there is $E_i < E_j$ iff $A_i < A_j$.*

4. *For allocations E_i and E_j allocated to two apartments with same area, where first one is assigned to an apartment with heat allocators, and second one is assigned to an apartment without heat allocators, there is $E_i \leq E_j$.*

Properties in Definition 11 establish consistency of an allocation model from the perspective of a consumer. For a consumer, it is natural to expect lower allocation than the one allocated to the same size apartment with higher consumption readings (1). On the other hand, if two apartments have the same consumption readings, it is also natural to expect lower allocation to one with smaller area, due to lower participation in common energy expenses (2). For consumers without heat allocators, apartment area should be deciding factor for allocated amount of energy (3). Finally, if we compare allocations for the same size apartments, where one of them has, and other has not heat allocators, it is only fair to assume (without any additional knowledge, such as an energy efficiency of apartments) that the apartment without allocators consumes at least the same amount of the energy as an apartment with heat allocators (4). Therefore, properties from Definition 11 play important role in establishing public confidence in *fairness* of an allocation model.

Theorem 12 *[Local consistency] Model for energy allocation DAM is locally consistent, while models SAM and SAM-wT are not.*

Proof: We will first prove that DAM is locally consistent. For some two apartments with heat allocators, properties (1) and (2) in Definition 11 follow from equation (9) in Definition 3. Property (3) is satisfied because of equation (8) in Definition 3; assigned consumption readings are formed as an area of an apartment multiplied with a fixed value. Finally, property (4) also follows from equation (8) in Definition 3, because consumption reading assigned to an apartment without heat allocator is always great or equal to consumption reading of an apartment with maximal (relative to its size) consumption reading among all apartments with the heat allocators.

For SAM model we can easily show that it satisfies properties (1), (2) and (3) from Definition 11, which follows from equations (1) and (2) in Definition 1. But SAM does not satisfy property (4) of local consistency. As we can see from Table 1 (Example 4), SAM allocated greater value to the 1st than to the 9th and 10th apartment. Because of static structure of that model, one can easily find a distribution of consumption readings which produces allocation for an apartment with heat allocators that is *several times* greater than allocation to an apartment without heat allocators. To do that, the actual share of consumption by apartments without heat allocators should be much greater than α_w, which is something that can happen during warm months, when energy savings among apartments with heat allocators are high, and so consumption share for apartments without heat allocators rise. We analyze that scenario in Chapter 4.

We can only assume that this result was motivation for introduction of SAM-wT model. Threshold defined in Definition 2 is activated in case when greater number of apartments saves energy. SAM-wT responds to described situation that violates property (4) of local consistency in a way that it rises share of an area part of allocation in process of determining allocations. This way, the model for energy allocation cancels the basic purpose of the heat allocators - to allocate energy (predominately) upon consumption. Even so, SAM-wT did not succeed to satisfy property (4) of local consistency, as can be seen in Example 9, where allocation to first apartment (with heat allocators) equals to $E_1 = 205.73$, while allocation to tenth apartment (without heat allocators) equals to $E_{10} = 200$. □

To show the extreme to which can go lack of local consistency for SAM model, we present following example:

Example 13 *Suppose there are 20 apartments in a building, each with area of $100\,m^2$. Two of them don't have heat allocators (we will denote their allocations with E_1 and E_2), while others have. Apartments without allocators consume 199 MWh each, one of the apartments with heat allocators consumes 2 MWh (we will denote its allocation with E_3), and all other apartments don't consume any energy. So, $E_{total} = 400$. Let us have $\alpha_A = 0.1$.*

If we look at SAM allocation, we will see that apartments without heat allocators have been allocated $E_1 = E_2 = 40$, apartment with minimal energy consumption and heat allocators has been allocated $E_3 = 289.78$, and all other apartments with heat allocators have been allocated $E_i = 1.78$. So, third apartment, with energy consumption 100 times smaller than first (or second apartment) has allocation seven times greater then theirs.

If we use SAM-wT model for allocation on same example, we will get $E_1 = E_2 = 40$, $E_3 = 34.568$ and $E_i = 16.79$. Now we don't have such huge anomaly, as with SAM, but side effect is that all allocations tend to be close to allocation solely upon area share of an apartment (which would be $E_i = 20$ in this example). This way, main purpose of heat allocators - to allocate energy on basis of consumption - is being negated. Finally, if we use DAM, allocations would be $E_1 = E_2 = 125.75$, $E_3 = 114.50$ and $E_i = 2.00$ for $i \in \{4, ..., 20\}$.

There is also a way to visualize lack of local consistency for a SAM and SAM-wT models. Let us look back again to the Example 10 and energy allocation dependance of energy consumption in one of the apartments. If we use data from that scenario to compare energy allocations to the equal sized apartments of the SAM model, one to the apartment with heat allocators (with its energy consumption reading ranging from 0 to 100 on a x-axis), and other to the apartment without heat allocators (and maximal energy consumption) we will get following graph:

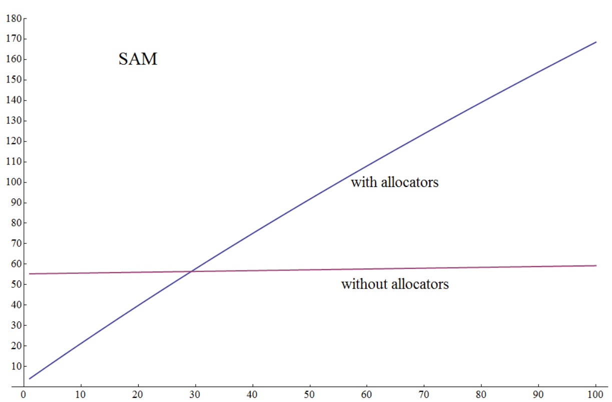

Figure 4: The energy allocation comparison for equal sized apartments with and without heat allocators, under SAM model allocation

From Figure 4 we can see that energy allocation to the apartment with heat allocators exceeds energy allocation to the apartment without heat allocators as soon as its energy consumption readings become greater than 30 (with 100 representing maximal consumption of the available heat energy). Comparing energy allocations of the SAM-wT model to that two apartments yields similar conclusion, as it is shown in Figure 5.

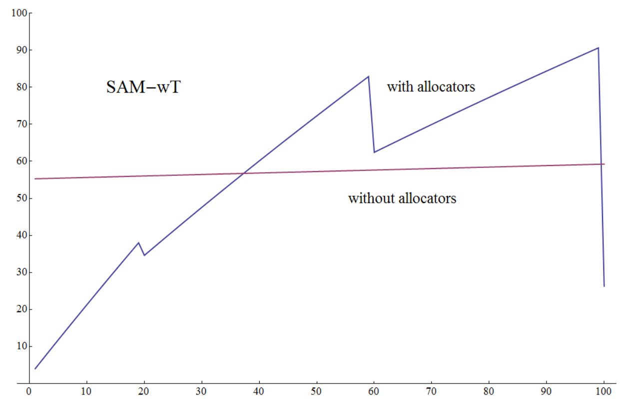

Figure 5: The energy allocation comparison for equal sized apartments with and without heat allocators, under SAM-wT model allocation

Here we can see that threshold introduction did not succeed to eliminate lack of local consistency. For instance, even if threshold did trigger at the level of consumption reading near 60, energy allocation to the apartment with heat allocators didn't decrease under the level of energy allocation to the apartment without heat allocators.

Figure 5 also clearly illustrates how lack of local consistency can be confusing for a consumer: as long as the consumption reading for a consumer with heat allocators ranged from 40 to 99, his / her energy allocation was greater than those to the equal sized apartment without heat allocators (and maximal consumption), but in the moment in which consumer with heat allocators maximize its consumption, his energy allocation becomes less if compared to allocation of the apartment without heat allocators!

Finally, in Figure 6 we can see allocations made by DAM model in Example 10. If we compare allocation of an apartment with heat allocators (and rising consumption, which is in percentage shown on a x-axis) and allocation to the apartment without heat allocators, we can see that graph confirms result of Theorem 12.

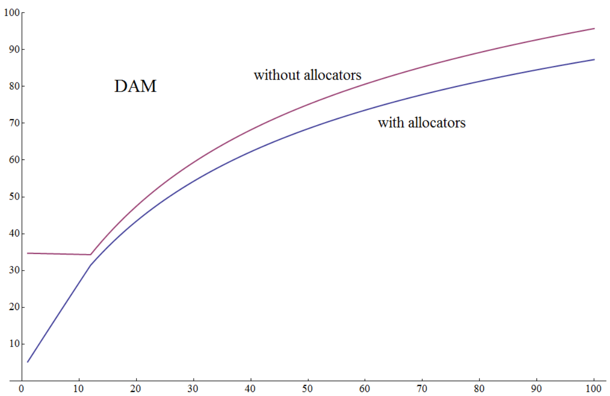

Figure 6: The energy allocation comparison for equal sized apartments with and without heat allocators, under DAM model allocation

4 The analysis of allocation models accuracy

4.1 SAM model

As we mentioned before, SAM model through factor α_w actually tries to predetermine share of energy consumption of the apartments without heat allocators. So what is the level of consumption (compared with the state prior to installation of heat allocators) for which SAM model accurately allocates heat energy?

To answer that question, we will assume that apartments without heat allocators maintain the same level of heat energy consumption as before heat allocators were installed in the building. Another assumption we are making, when talking about "accurate" allocation, is that we will assume that all apartments without heat allocators consume same amount of energy (relative to their size). This doesn't have to be so, because heat transfer between apartments changes after some of them consumes less energy. Also, different apartments have different energy consumption, due to different energy efficiency [7]. But if we want to stay within the same analysis framework, and if we don't want to introduce new class of information into the allocation models (such as database of energy efficiency factors for all apartments in building), we should accept those assumptions.

We will denote the consumption of the apartments with heat allocators before their installation with E_{with}, and consumption after allocators installation with E'_{with}. Similarly, energy consumption in the apartments without heat allocators we will denote with $E_{without}$ and $E'_{without}$.

Now, from Definition 1 we have:

$$E_{without} = E'_{without} \quad \Rightarrow$$

$$\frac{A_{without}}{A_{total}} \cdot E_{total} = l \cdot \frac{A_{without}}{A_{total}} \cdot E'_{total} \quad \Rightarrow$$

$$E_{total} = l \cdot E'_{total} \quad \Rightarrow$$

$$E_{without} + E_{with} = l \cdot (E'_{without} + E'_{with})$$

If we denote with α' share of energy apartments with heat allocators are consuming (relative to consumption prior to installation of allocators), we have:

$$E_{without} + E_{with} = l \cdot (E_{without} + \alpha' E_{with})$$

$$E_{without} \cdot (1 - l) = E_{with} \cdot (\alpha' l - 1)$$

$$\frac{A_{without}}{A_{total}} \cdot E_{total} \cdot (1 - l) = \frac{A_{with}}{A_{total}} \cdot E_{total} \cdot (\alpha' l - 1)$$

$$A_{without} \cdot (1 - l) = A_{with} \cdot (\alpha' l - 1)$$

Now, let us with α_{with} denote total area share of the apartments with installed heat allocators.

$$A_{total} \cdot (1 - \alpha_{with}) \cdot (1 - l) = A_{total} \cdot \alpha_{with} \cdot (\alpha' l - 1)$$

$$(1 - \alpha_{with}) \cdot (1 - l) = \alpha_{with} \cdot (\alpha' l - 1)$$

I we solve this equation for α', we get

$$\alpha' = \frac{1}{l} + \left(1 - \frac{1}{l}\right)\left(1 - \frac{1}{\alpha_{with}}\right) \tag{10}$$

Equation (10) gives us factor of change (reduction) of the consumption of apartments with heat allocators, at which SAM model accurately allocates heat energy, as a function of two parameters: factor l from Definition 1 and an area share of apartments with heat allocators.

For instance, if we use $l = 2$ (as in Definition 1) and $\alpha_{with} = 0.9$ (which means that total area of the apartments with heat allocators accounts for 90% of total buildings area), we get $\alpha' = 0.4444$. This means that SAM allocation will be accurate if apartments with heat allocators consume 44.44% of energy they consumed prior to installation of heat allocators. Dependance between α' and α_{with}, for a value $l = 2$, is shown in Figure 7.

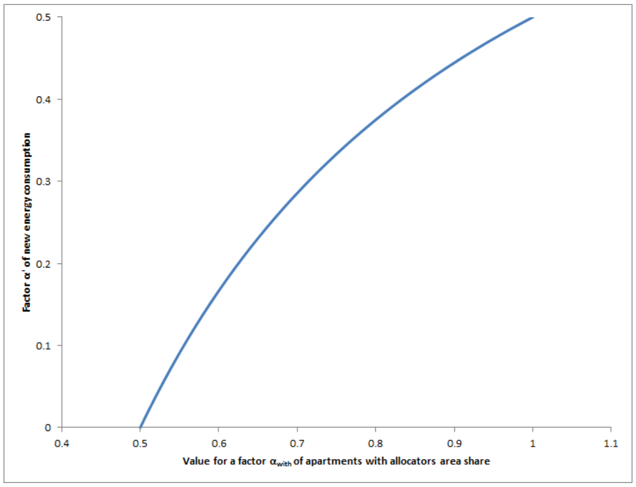

Figure 7: Dependance between α' and α_{with} for a value $l = 2$ (see equation (10))

Here we can see that, if we want SAM to provide accurate allocation, energy consumption (compared to the previous state without heat allocators) in the apartments with heat allocators must range from 0 (when there is only 50% of total building area with heat allocators) up to 0.5. Either way, if we want SAM to allocate energy accurately, savings in the apartments with heat allocators (for $l = 2$, as in Definition 1) must be significant.

Property described with equation (10) is something that requires information about share (in area) of apartments with installed heat allocators, and furthermore, the information that it provides is difficult to measure or estimate. Therefore, let us take look at total savings in building which provides accurate allocation for a SAM. Share of total energy consumption in building (compared to consumption prior to installation of heat allocators) we will denote β': $E'_{total} = \beta' \cdot E_{total}$. Now we have:

$$\begin{aligned} E'_{total} &= E'_{with} + E'_{without} \\ &= \alpha' \cdot \alpha_{with} \cdot E_{total} + (1 - \alpha_{with}) \cdot E_{total} \\ &= E_{total} \cdot (\alpha' \cdot \alpha_{with} + 1 - \alpha_{with}) \end{aligned}$$

For β' we have:

$$\begin{aligned} \beta' &= \frac{E'_{total}}{E_{total}} \\ &= \frac{E_{total} \cdot (\alpha' \cdot \alpha_{with} + 1 - \alpha_{with})}{E_{total}} \\ &= \left[\frac{1}{l} + \left(1 - \frac{1}{l}\right)\left(1 - \frac{1}{\alpha_{with}}\right)\right] \cdot \alpha_{with} + 1 - \alpha_{with} \\ &= \frac{1}{l} \end{aligned} \tag{11}$$

This result is global property of the model, which can easily be measured. If we want SAM model to accurately allocate energy within a building, total energy consumption in building must be at a level of $\beta' = \frac{1}{l}$ of energy consumed prior to allocators installation. Since in Definition 1 we have $l = 2$, building has to reduce its total heat energy consumption to one half of its prior consumption, for SAM model to be accurate.

This brings out the question: why is l set to 2? What was wrong with previous values for l? Especially when recent studies on behaviour change and energy use show that up to 20% of the currently consumed energy can be saved through changing behaviour [1]. Such savings would mean that statistically most appropriate value for l would be 1.25. Furthermore, HEP-Toplinarstvo d.o.o., the biggest heat energy provider in Croatia states that saving in heat energy for buildings, after installation of heat allocators vary from 15% up to 30% [2], which would yield a value for l between 1.17 and 1.42 (if we want a SAM model to be accurate in most of the situations).

To answer that questions, we have to look at a behavior of SAM model in the case when change of the total consumption factor, β', does not equal to $\frac{1}{l}$.

Let us first take a look at a SAM allocation for apartments without heat allocators at that level of consumption. SAM model allocates to all those apartments total amount of energy equal to $l \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}$, while allocation should be $\frac{1}{\beta'} \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}$, $\beta' \in [1 - \alpha_{with}, 1]$, for a model to be accurate. The ratio of those two values equals to

$$\frac{l \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}}{\frac{1}{\beta'} \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}} = l \cdot \beta', \tag{12}$$

where $\beta' \in [1 - \alpha_{with}, 1]$. Therefore, for $l = 2$, ratio of an energy allocated to apartments without heat allocators, and energy actually consumed in those apartments, equals to $2\beta'$. This means, that for $\beta' = 1$ (when no energy was saved compared to the consumption prior to installation of heat allocators, which can occur during very cold months, when all available heat energy is consumed), apartments without allocators are being allocated twice the consumed energy. On the level of consumption $\beta' = 0.75$ they are allocated 50% more energy than their consumption. On the other hand, if total savings are high (for instance during warmer weather period), and $\beta' = 0.25$, those apartments are being allocated only half the energy they spend. The extreme is when savings are maximal, and $\beta' = 1 - \alpha_{with}$. This means that all apartments with heat allocators are consuming no energy at all. But in this case, apartments without heat allocators are the one who benefit most from the situation. Although they consume all energy in the building, they will be allocated only $2 \cdot (1 - \alpha_{with})$ share of it.

Analysis for the apartments with heat allocators is more complex, since it depends on the distribution of the consumption readings. Because of that, we will analyze only "worst case scenario" for those apartments. This scenario occurs in situation when we maximize number (area) of the apartments with heat allocators and no energy consumption, because in that scenario, allocation for the rest of apartments with heat allocators will be greatest.

To make such analysis possible, we will work under assumption that maximal consumption in the apartments with heat allocators equals one in the apartment without them. Now, we have to denote (area) share of the apartments with heat allocators that consume maximal amount of heat energy (while others are consuming none). Let us denote that share with α_{mc} (as for "maximal consumption").

$$\alpha_{mc} = \frac{\text{Area of the apartments with heat allocators and maximal consumption}}{\text{Area of all apartments}}$$

Now we have

$$\alpha_{mc} = \beta' - \alpha_{without}$$

Why is this so? Because, in situation in which part of the apartments maintain full consumption, while other part consumes none of the available energy, β' equals to area share of the apartments with full consumption. When we subtract share of the area without heat allocators (for which we assume full consumption), we get α_{mc}.

To that part of the building (apartments with heat allocators and maximal consumption), SAM model allocates all the energy

$$(1 - \alpha_w) \cdot \beta' \cdot E_{total},$$

which equals to (see equation (3)):

$$(1 - l \cdot \alpha_{without}) \cdot \beta' \cdot E_{total}$$

Now, we calculate allocation relative to the area of the apartments with heat allocators and maximal consumption:

$$\begin{aligned} \frac{(1 - l \cdot \alpha_{without}) \cdot \beta' \cdot E_{total}}{\alpha_{mc} \cdot A_{total}} &= \\ \frac{(1 - l \cdot \alpha_{without}) \cdot \beta'}{\beta' - \alpha_{without}} \cdot \frac{E_{total}}{A_{total}} &= \\ \frac{1 - l \cdot \alpha_{without}}{1 - \frac{\alpha_{without}}{\beta'}} \cdot \frac{E_{total}}{A_{total}}. \end{aligned} \tag{13}$$

From (13) we can see that if $\beta' = \frac{1}{l}$, factor multiplying $\frac{E_{total}}{A_{total}}$ equals to 1. This means that allocation is accurate, because all (relative) consumptions (where there is some) are equal. If energy savings are low (if β' is equal or close to 1), we can easily see that factor in question is lower than 1, which means that apartments with heat allocators and maximal consumption are being allocated smaller amount of energy than they consume.

The most interesting scenario, however, is the one in which β' tends to $\alpha_{without}$, which means that energy consumed in all apartments with heat allocators tends to zero.

In that case value in denominator tends to 0, so factor in equation (13) doesn't have upper bond. This analysis shows that there are consumption reading distributions among apartments with heat allocators, that can easily lead to extremely high energy allocation to some of those apartments, using the SAM model.

Now, the unbounded expression in equation (13) tells us value of allocation relative to the area, and since heated area of the apartment is not continuous, but a discrete value, we can not achieve infinitesimal values in analyzed denominator. Nevertheless, values of allocation to that apartment can be really high, especially when compared with the allocations to the apartments without heat allocators. This is illustrated in the following example:

Example 14 *Allocation is being made for a house with fifty apartments. First forty of them have the heat allocators, while last ten do not. Energy consumption readings for those forty apartments is given in following vector:* $(i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Furthermore, we will assume that those consumption readings represent consumption represented in percents of total available energy for each apartment. Therefore, consumption readings for the apartments without heat allocators (which is not measured, so for a modeling we should consider it as unknown) should equal to 100.
Consumption for the first apartment is labeled with the unknown, $i \in \{0, 100\}$, while consumption of all other apartments is fixed to 0, which describes "worst case scenario" for the SAM allocation model.

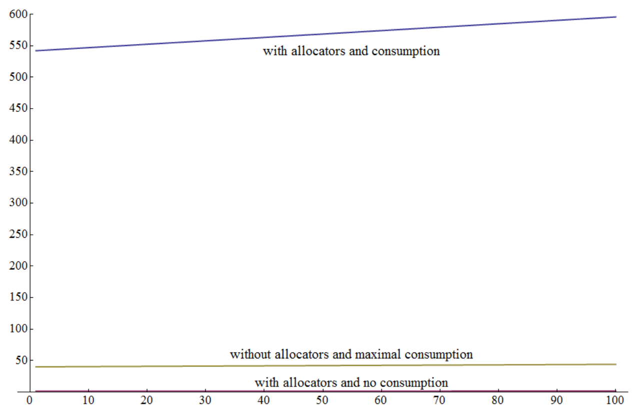

Figure 8: Example of the allocations under SAM model in "worst case scenario"

As we can see in Figure 8, allocation for the first apartment, the one with heat allocators and consumption reading that ranges from 0 to 100, SAM model energy allocation is high. In fact it is eleven to twelve times higher than the allocation to the apartment without heat allocators (and maximal heat energy consumption). Allocation for apartments with heat allocators and no heat consumption readings are low, because in convex combination defined with equation (2) (see Definition 1) part that depends upon consumption reading equals to zero.

Anomaly derived from equation (13) could have been the reason why SAM-wT model was introduced. Situation when β' tends to $\alpha_{without}$ means that α_{mc} tends to 0 and share of the apartments with no (or very low) consumption tends to α_{with}. In such scenario value T from equation (4) in Definition 2 tends to 1, and consequently value β_A in equation (7) tends to 1. This eliminates extremely high allocations, because allocation tends to be done according to area (rather than consumption). But, in the same time, SAM-wT model negates very purpose of heat allocators.

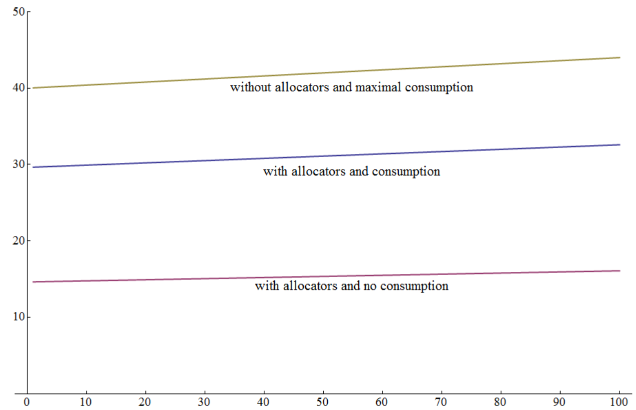

Figure 9: Example of the allocations under SAM-wT model in "worst case scenario"

This conclusion can be seen in the simulation described in Example 14, which describes "worst case scenario" for SAM allocation method. As we can see in Figure 9, SAM-wT allocation to all three types of apartment are almost constant when heat energy consumption reading of the first apartment ranges from 1 to 100. This is so, because value of convex combination index β_A (see Definition 2) equals to 0.78, which means that 78% of the allocation value is determined by area share of the apartment, while only 22% of the allocation value is determined by consumption reading share (which is in those type of scenario relatively low).

4.2 DAM model

Let us now analyze DAM method for energy allocation. This method produces accurate allocation when N_i,

$$N_i = \alpha_N \cdot A_i \cdot \max_{j=1,\ldots,k} \frac{N_j}{A_j}, \qquad i \in \{k+1, \ldots, m\}$$

$$\frac{N_i}{A_i} = \alpha_N \cdot \max_{j=1,\ldots,k} \frac{N_j}{A_j}, \qquad i \in \{k+1, \ldots, m\}$$

represents accurate consumption reading for energy consumed in the apartments without heat allocators. Let us denote consumption reading and area of an apartment which maximizes given fraction with N_{max} and A_{max}. Since we are assuming accurate allocation, for all consumption reading we have $N_i = \gamma E_i$ for some constant γ. Now we have

$$\frac{\gamma E_i}{A_i} = \alpha_N \cdot \frac{\gamma E_{max}}{A_{max}}, \qquad i \in \{k+1, \ldots, m\}.$$

So, if we want for an allocation to be accurate, there should be

$$E_{max} = \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i, \tag{14}$$

for all apartments without allocators, $i \in \{k+1, ..., m\}$.

What happens if we have $E_{max} > \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i$? In that case, the apartments without allocators are assigned consumption readings N_i':

$$\begin{aligned} N_i' &= \alpha_N \cdot A_i \cdot \tfrac{N_{max}}{A_{max}} \\ &= \alpha_N \cdot A_i \cdot \tfrac{1}{A_{max}} \cdot \gamma \cdot E_{max} \\ &> \alpha_N \cdot A_i \cdot \tfrac{1}{A_{max}} \cdot \gamma \cdot \tfrac{A_{max}}{A_i} \cdot \tfrac{1}{\alpha_N} \cdot E_i \end{aligned}$$

So, we have $N_i' > \gamma \cdot E_i = N_i$. Therefore, in the case of an apartment with maximal relative consumption among those with heat allocators, consumes more energy than stated in equation (14) (where i-th apartment is one of the apartments without heat allocators), apartments without heat allocators are allocated more energy than their actual consumption is. Consequently, apartments with heat allocators are allocated smaller amount of energy than their actual consumption.

Analogously, in case when we have $E_{max} < \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i$, we conclude that apartments with allocators are allocated greater amount of energy compared to their actual consumption, while apartments without heat allocators are allocated smaller amount of energy than their actual consumption.

Unfortunately, it is impossible to determine relation between E_{max} and E_i solely upon consumption readings on heat allocators, nor from the total energy consumption readings. No matter what are total saving in energy consumption, distribution of energy consumption within the building can produce both results, depending of actual consumption of just one apartment (the one that maximizes (relative) consumption).

Nevertheless, we can still give bound for maximal error of allocation model, through analyze of the worst case scenario. Based on the given analysis, worst case scenario for the DAM model of allocation occurs in case when $E_{max} \to 0$. Consequently, consumption in all apartments with heat allocators tends to zero. Therefore, we will analyze situation in which there is

$$\frac{E_{max}}{A_{max}} = \epsilon << \frac{E_i}{A_i}, \qquad i = k+1, ..., m \tag{15}$$

while energy consumption in all other apartments with heat allocators equals to zero. Furthermore, let us suppose (without loss of generality) that all apartments have equal area, so that we could have clearer readings of the results. So, we are working under assumption that $E_{max} = \epsilon << E_i$, where E_i is same amount of energy consumed in all apartments without heat allocators. In that case total energy consumption equals to:

$$E_{total} \;=\; \sum_{i=k+1}^{m} E_i + \epsilon \;=\; (m-k) \cdot E_i + \epsilon$$

Let us with N_0 denote energy consumption reading in the apartment with energy consumption $E_{max} = \epsilon$. All other apartments with heat allocators have consumption reading equal to 0, while according to DAM apartments without heat allocators are assigned consumption readings $N_i = \alpha_N \cdot N_0$. Furthermore, let us have $\alpha_C = 0$, so that complete allocation is done according to consumption readings (and none according to apartments areas) which emphasize energy allocation error. Now we have:

$$N_{total} = (m-k) \cdot \alpha_N \cdot N_0 + N_0$$

Now, energy allocated to the apartment with maximal energy consumption among apartments with heat allocators equals to (see Definition 3):

$$E_{i'} \;=\; E_{total} \cdot \frac{N_0}{N_{total}} \;=$$

$$\big((m-k) \cdot E_i + \epsilon\big) \cdot \frac{N_0}{(m-k) \cdot \alpha_N \cdot N_0 + N_0} \;=$$

$$\frac{(m-k) \cdot E_i + \epsilon}{(m-k) \cdot \alpha_N + 1} \;\approx\; \frac{(m-k) \cdot E_i}{(m-k) \cdot \alpha_N + 1}$$

$$E_{i'} \;=\; E_i \cdot \frac{1}{\alpha_N + \frac{1}{m-k}}. \tag{16}$$

Since $\alpha_N \geq 1$, from equation (16) we see that allocation to an apartment with heat allocators is smaller than allocation to apartments without heat allocators, which is conformation of Theorem 12. But, in this worst case scenario allocation to an apartment with heat allocators can be close to an allocation to the apartment without heat allocators, even if its actual consumption equals to ϵ which is very low compared to the consumption on the apartment without heat allocators.

For instance, if we use $\alpha_N = 1.1$ in a building in which 10 apartments does not have heat allocators, we have $E_{i'} = 0.833E_i$, even if i'-th apartment consumes few percent of energy consumed in i-th apartment.

We should point out that this worst case scenario for a DAM model, is in the same time a worst case scenario for a SAM model (see analysis following equation (13) on page 12). But, while worst case scenario produces unbounded share of energy allocation for an apartment with heat allocators in SAM model, the same scenario produces allocation for an apartment with heat allocators which is bounded by energy allocation share to the apartment without heat allocators.

4.3 Consumption reading point value

With use of the simulations, another area of analysis opens. Beside local consistency, there is one other aspect of heat energy allocation that is very visible to the consumers – energy value of consumption reading points.

Every consumer with heat allocators have first hand experience with consumption readings, and naturally question of the correlation between energy consumption readings points and energy allocation arises. Extensive research can be made in this area. At this point, however, we will use Example 14 just to illustrate the problem.

We should point out, that for this purpose value of (convex combination) coefficient α_A was set to zero. This way, allocation produced by models depends solely upon consumption.

In Figure 10 we can see how value of the consumption reading point changes for all three allocation models, as the consumption reading in one of the apartments rises from 1 to 100 (points, which is the same as percentage of maximal consumption in this example) which can be read on x-axis. On y-axis

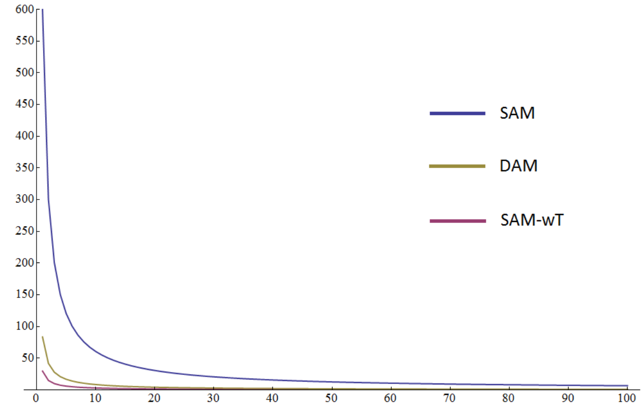

Figure 10: Energy consumption reading point value in the ”worst case scenario”

in this graph we can read value of the ratio of the allocated energy (in percentage of the maximal consumption in one of the apartments) and the number consumption reading points, again as a percentage of maximal consumption reading in that apartment. This means that accurate energy value of the consumption reading (in the case when energy allocation is equal to the actual energy consumption) should equal 1 (which can be seen as dashed line in Figure 11).

As explained in Example 14 this is the result of the ”worst case scenario” consumption reading distribution. Therefore, it is not so surprising to see that energy reading point value for a SAM method goes extremely high when consumption in only apartment with heat allocators that measures its consumption tends to zero. It is in fact expected result, which corresponds with analysis in the Chapter 4.1.

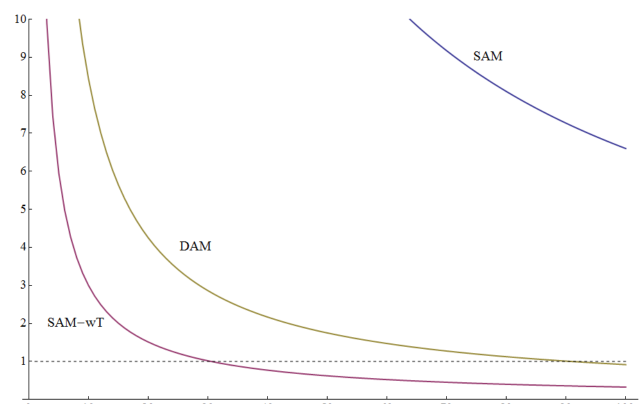

Figure 11: Energy consumption reading point value in the ”worst case scenario”

If we rescale y-axis, in order to have better look at lower consumption point values, as in Figure 11, we can see that consumption reading point value for SAM-wT model dives quickly under 1, ending at a value of 0.3. Although this looks like a good scenario, result should be viewed as a three times smaller than it should be. That is, of course, consequence of SAM-wT model putting the weight in allocation calculation on apartments area, rather than its consumption.

Energy reading point value in DAM model under the ”worst case scenario” does not fall from relatively high values to 1 so rapidly. But the question is, how to compare energy reading point values in DAM and SAM-wT models? What is the criteria for declaring one result better than other? Since accurate energy reading point value should equal to one, is it better for a model to value energy reading point 0.5 or 1.5?

We shall propose that criteria should be comparison between maximal ratios against accurate consumption reading point value. For instance, for consumption reading point value 0.5, ratios against accurate consumption reading point value are $\frac{0.5}{1} = 0.5$ and $\frac{1}{0.5} = 2$, and so we have $\max\left\{\frac{0.5}{1}, \frac{1}{0.5}\right\} = 2$. On the other hand, for consumption reading point value 1.5 we have $\max\left\{\frac{1.5}{1}, \frac{1}{1.5}\right\} = 1.5$. This result we interpret in following way: *consumption reading point value 0.5 is 2 times smaller than accurate one, while consumption reading point value 1.5 is 1.5 times greater than accurate one. Therefore, later consumption reading point value is more desirable result.*

In Figure 12 we can compare those maximal ratios for the DAM and the SAM-wT model, based upon simulation from Example 14.

Here we can see that SAM-wT model preforms better than DAM model (in terms of consumption reading point values analysis) for low energy readings (in the described worst case scenario). The DAM model provides much better results for higher energy reading. Of course graph in Figure 12 describes models behavior only in one specific energy reading distribution. But the concept of the consumption reading point value analysis opens the door for further reasearch in this area, based upon comprehensive usage of simulations. With that in mind, let us examine following example:

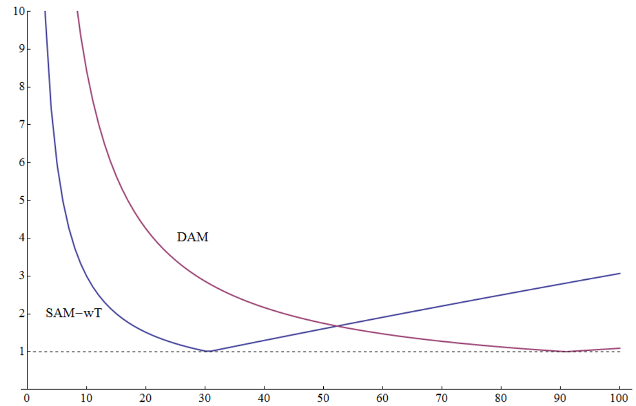

Figure 12: Maximal ratios of allocated consumption reading point values against accurate consumption reading values for SAM-wT and DAM model

Example 15 *Allocation is being made for a house with fifty apartments. First forty of them have the heat allocators, while last ten do not. Energy consumption readings for those forty apartments is given in following vector:* $(y, N_2, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x, x)$. *Furthermore, in this example consumption reading points represent (without loss of generality) percents of total available energy for each apartment. Therefore, consumption readings for the apartments without heat allocators (which is not measured, so for a modeling we should consider it as unknown) should equal to 100.*

Consumption reading for the first apartment is labeled with the variable, $y \in \{0, 100\}$*; for second apartment it equals to constant* $N_2 = 80$*; and in all other apartments with heat allocators, consumption reading is labeled with the variable* $x \in \{0, 100\}$. *With variable* y *we can (approximately) determine level of consumption (in percentage compared to the maximal available energy for an apartment). Total energy consumption level (which was in Chapter 4.1 labeled with* β'*) in this example equals to* $\beta' = 0.0076x + 0.0002y + 0.216$. *So for* $x = 37$ *we have* $\beta' \in [0.4972, 0.5172]$.

In this example, as in previous analysis, we will use value $\alpha_A = 0$ *(to put the emphasis on part of convex combination which allocates energy consumption on basis of consumption readings). For DAM model calculation* $\alpha_N = 1.2$ *value was used.*

After all parameters of simulations are set, let us calculate values of maximal ratios against accurate consumption reading point value. In Figures 13, 14 and 15 calculation of those maximal ratios are given for the first apartment under the SAM, SAM-wT and DAM model respectively.

Null	Null	Null	Null	Null	Null	Null	Null	Null	Null	Null
7.26667	1.8766	1.30588	1.0878	1.0281	1.10925	1.17211	1.22222	1.26312	1.29712	1.32584
6.6	1.85	1.29767	1.08387	1.03053	1.11111	1.17357	1.2234	1.26409	1.29794	1.32653
6.05455	1.82449	1.28966	1.08	1.03295	1.11296	1.17502	1.22458	1.26506	1.29875	1.32722
5.6	1.8	1.28182	1.07619	1.03535	1.11479	1.17647	1.22575	1.26603	1.29956	1.32791
5.21538	1.77647	1.27416	1.07244	1.03774	1.11661	1.17791	1.22691	1.26699	1.30037	1.3286
4.88571	1.75385	1.26667	1.06875	1.0401	1.11842	1.17934	1.22807	1.26794	1.30117	1.32928
4.6	1.73208	1.25934	1.06512	1.04245	1.12022	1.18076	1.22922	1.26889	1.30197	1.32997
4.35	1.71111	1.25217	1.06154	1.04478	1.122	1.18217	1.23037	1.26984	1.30277	1.33065
4.12941	1.69091	1.24516	1.05802	1.04709	1.12378	1.18357	1.23151	1.27078	1.30356	1.33132
3.93333	1.67143	1.2383	1.05455	1.04938	1.12554	1.18497	1.23264	1.27172	1.30435	1.33199

Figure 13: Maximal ratios against accurate reading point value for the first apartment in SAM model

Null	Null	Null	Null	Null	Null	Null	Null	Null	Null	Null
1.91658	2.19679	2.73827	3.28857	3.84144	1.10925	1.17211	1.22222	1.26312	1.29712	1.32584
1.11375	1.147	1.39257	1.6651	1.94013	1.11111	1.17357	1.2234	1.26409	1.29794	1.32653
1.20485	1.2516	1.04961	1.12388	1.30633	1.48984	1.17502	1.22458	1.26506	1.29875	1.32722
1.47275	1.6016	1.36269	1.16423	1.01072	1.12661	1.17647	1.22575	1.26603	1.29956	1.32791
1.7158	1.92614	1.66014	1.4274	1.24466	1.10052	1.01865	1.22691	1.26699	1.30037	1.3286
1.94161	2.2287	1.94332	1.68085	1.47097	1.30428	1.16983	1.22807	1.26794	1.30117	1.32928
2.15467	2.51213	2.21343	1.92517	1.69062	1.50257	1.35026	1.22496	1.26889	1.30197	1.32997
2.3578	2.7788	2.47152	2.16093	1.90392	1.69599	1.52661	1.38682	1.26984	1.30277	1.33065
2.55284	3.03065	2.71856	2.38863	2.11118	1.88474	1.69925	1.54555	1.42086	1.30356	1.33132
2.7411	3.26935	2.95539	2.60874	2.31269	2.06901	1.8683	1.70136	1.57027	1.30435	1.33199

Figure 14: Maximal ratios against accurate reading point value for the first apartment in SAM-wT model

Null	Null	Null	Null	Null	Null	Null	Null	Null	Null	Null
1.0381	1.02797	1.0221	1.01826	1.01556	1.01356	1.01201	1.01078	1.00978	1.01774	1.0409
1.03774	1.02778	1.02198	1.01818	1.0155	1.01351	1.01198	1.01075	1.00976	1.0177	1.04082
1.03738	1.02759	1.02186	1.0181	1.01544	1.01347	1.01194	1.01072	1.00973	1.01766	1.04073
1.03704	1.0274	1.02174	1.01802	1.01538	1.01342	1.0119	1.0107	1.00971	1.01762	1.04065
1.0367	1.02721	1.02162	1.01794	1.01533	1.01338	1.01187	1.01067	1.00969	1.01758	1.04057
1.03636	1.02703	1.02151	1.01786	1.01527	1.01333	1.01183	1.01064	1.00966	1.01754	1.04049
1.03604	1.02685	1.02139	1.01778	1.01521	1.01329	1.0118	1.01061	1.00964	1.01751	1.0404
1.03571	1.02667	1.02128	1.0177	1.01515	1.01325	1.01176	1.01058	1.00962	1.01747	1.04032
1.06838	1.05161	1.04145	1.03463	1.02974	1.02606	1.02319	1.02089	1.019	1.01743	1.04024
1.16949	1.12821	1.10309	1.08621	1.07407	1.06494	1.0578	1.05208	1.04739	1.04348	1.04016

Figure 15: Maximal ratios against accurate reading point value for the first apartment in DAM model

Rows in matrices in Figures 13, 14 and 15 represent maximal ratios for a fixed value of variable y, while columns represent maximal ratios for a fixed value of variable x. As we can see values of maximal ratios for the first apartment can not be calculated in a first row (i.e. when $y = 0$). Similar calculation can be made for all other apartments – in this case for a second and third apartment.

In following figures we will visualize calculated data. In Figure 16 is graph visualization of data from Figure 15. We can see that DAM model produces low values of maximal ratios against accurate reading point value for the first apartment (all values are lower than 1.2). If we compare data for a DAM model with data provided by the SAM model (from Figure 13),

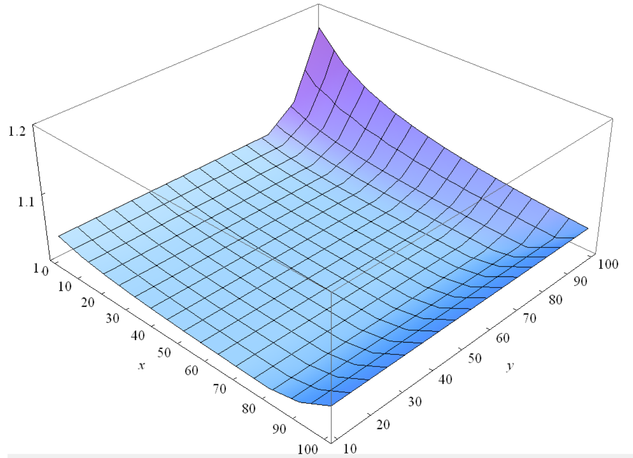

Figure 16: Maximal ratios against accurate reading point value for the first apartment in DAM model

we will get visualization presented in Figure 17.

Here we can see that DAM model in this case produces allocations with better (lower) maximal ratios against accurate reading point value in almost all situations. Only exception is situation in which value of x is close to 40, and value of y is maximal. In fact, SAM model produces lowest maximal ratios against accurate reading point value for $x = 37$ (with $\beta' \in [0.4972, 0.5172]$), as stated in Example 15. This result confirms analysis made in Chapter 4.1 about SAM model being accurate for $\beta' = 0.5$.

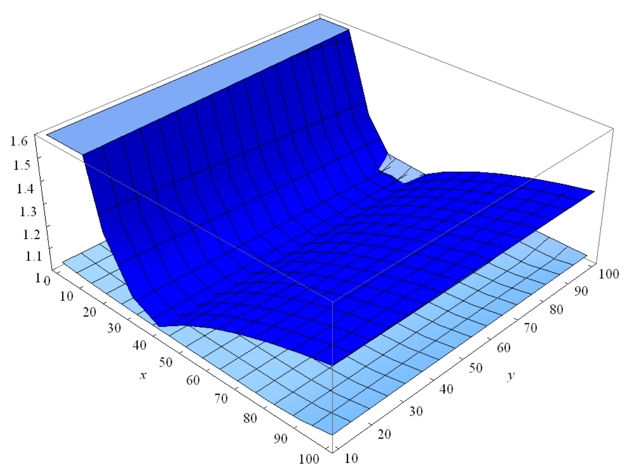

Figure 17: Comparison of the SAM (dark blue) and the DAM model maximal ratios against accurate reading point value for the first apartment

If we compare maximal ratios against accurate reading point value for a SAM and SAM-wT model (see Figure 18), we can see that SAM-wT model does not produce such high ratios against accurate reading point value when x is close to zero, as a SAM model. On the other hand, SAM model produces better result in case of (total building) energy consumption at level of $30 - 50\%$. In those situations, SAM-wT model triggers its threshold, producing more inaccurate energy allocation. For higher x values, both models produce same maximal ratios against accurate reading point value, because those models are the same in case when SAM-wT model threshold doesn't trigger.

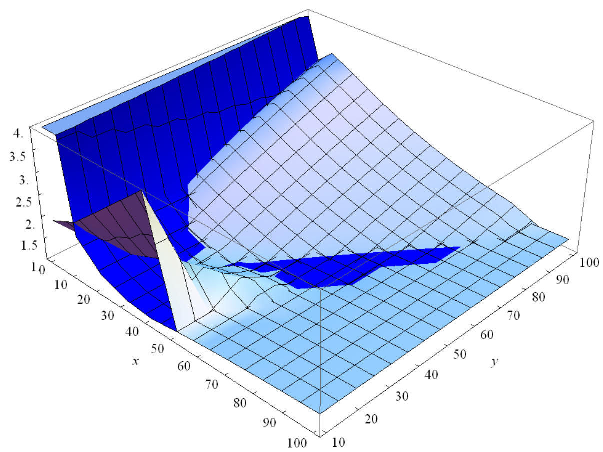

Figure 18: Comparison of the SAM (dark blue) and the SAM-wT model maximal ratios against accurate reading point value for the first apartment

However, we should point out that DAM model preformed so well in this example, partly due to selection of value for constant consumption reading $N_2 = 80$. Since DAM model produces allocation on basis of maximal (relative) consumption reading, such high isolated consumption reading ensures good results of the method even in case of low overall consumption (i.e. low x values). Even if such assumption is not unusual, let us look at a DAM model results in case when there is no such high isolated consumption reading.

In Figure 19 we can see how well DAM model preforms against SAM model, in case when value of N_2 is set to 20. In this case, DAM model still produces better maximal ratios against accurate reading point value, for x greater than 50 and for x less than 20. As mentioned before, SAM model

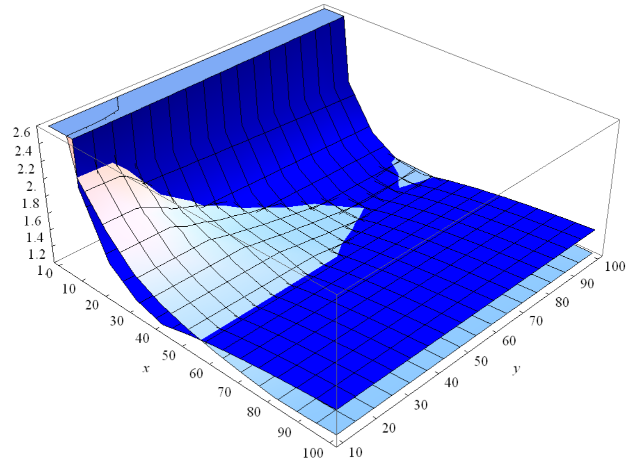

Figure 19: Comparison of the SAM (dark blue) and the DAM model maximal ratios against accurate reading point value for the first apartment (with $N_2 = 20$)

produces its best results for values of x from 30 to 40. Without high N_2 value, DAM model preforms slightly weaker than SAM model in that range.

Further comprehensive study of consumption reading point value can be made following method laid down by Example 15. Analysis could and should be complemented with statistical real life consumption reading data, which would provide more informations about consumption readings structure. With that additional knowledge, greater quality conclusions could be made. Let us conclude this chapter with notion that there also are differences between maximal ratios against accurate reading point value in some of the models (such as SAM-wT) when different apartments are compared. Analysis of such allocation model behavior is yet to be made.

5 Conclusion

In this article we presented and defined three methods for heat energy allocation with partial distribution of heat allocators. Two of them were legislated in Croatia during last seven years (SAM model defined in Definition 1 and SAM-wT model defined in Definition 2), while third method was proposed in works of Hatzivelkos (DAM model defined in Definiton 3).

Moreover, we defined several properties of heat energy allocation models: consistency, monotonicity and local consistency. Consistency of the allocation model is necessary property; the sum of all allocated energy must be equal to consumed energy total. As we showed, all three allocation models are consistent. Monotonicity is property that ensures that greater energy consumption will always lead to greater energy allocation to that consumer (see Definition 7). But, while SAM and DAM allocation models are monotone, SAM-wT is not.

Finally, local consistency is property that is most visible from the perspective of the consumer (see Definition 11). This property ensures that consumer sees allocation as ”*fair*”, when consumers compare their energy allocations. As we showed, only DAM model satisfy local consistency, while SAM and SAM-wT do not.

In the last part of article, we analyzed the behavior of SAM and DAM allocation models in ”worst case scenario”, that is, for distributions of consumption readings for which allocation distribution differs the most from actual energy consumption distribution. We find out that in those situations SAM model allocate share of energy to the consumer with heat allocators, that is far greater than its actual energy consumption. Even more, that ratio (of allocated and consumed heat energy) is unbounded from above. On the other hand, DAM model even in the worst case scenario, produced allocation for a consumer with heat allocators, that is bounded from above with allocation to the consumers without heat allocators.

Even if those worst case scenarios are rare, such allocation anomaly can take high media coverage. Such situations then lead to public distrust in very notion of individual metering through heat allocators. Problem then outgrows the individual injustice done to one consumer. In the long run, repetition of such anomalies, backed with allocation model properties which are seen as unfair (such as lack of local consistency) can lead to public disapprove of the very concept of individual heat energy metering through heat allocators.

Another open allocation models research area that deals with role of the coefficient α_A (see Chapter 2 and Definitions 1, 2 and 3). This coefficient should determine portion of the total energy consumption that should be allocated in respect to an apartments area share, as it represent the portion of the total heat energy loss within the building.

The SAM and DAM model treat this coefficient as a fixed, constant value. But this doesn't have to be so in real life. For instance, in warmer periods consumption could be much lower, and share of the heat energy loss could rise due to constant circulation of heated water within the building, which is not being used. On the other hand, the SAM-wT model intervenes into the value of coefficient α_A. What is the result in terms of accuracy of that intervention is an open question.

Additional study based upon real life energy consumption readings should be made to answer those questions. It could provide an answer to the question of dependance between consumption readings and energy loss share. There is also possibility that other factors, such ass value of average outside temperature in observed periods of time could play important role in modeling of the method or algorithm which would provide better estimation of the α_A value.

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