Proposal of a methodology for the estimation of slope, water depth and cross-sectional area by using the state function, ϕ (t): Testing with the measurements made by H. Fischer in M. Keck channel laboratory of CALTECH University

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Abstract: A basic topic in river studies, whether in hydrodynamics or water quality, is the accurate estimation of both geomorphological and geometric characteristics in cross sections in streams or channels. Many measurements or methodologies that are within the state of the art, are not direct or easy by several aspects. For this reason, this article analyses the application of a state function, Φ (t), which, acting as a thermodynamic potential, allows the magnitudes of the cross sections, depth of the water sheet, slope and longitudinal dispersion coefficient to be obtained directly, using NaCl as a tracer. In order to apply and validate this new method properly, an experiment conducted in 1966 by H.B. Fischer in the W.M. Keck Laboratory of Caltech in USA was studied on two points of the canal. It found average differences of 0.016 m^2 (with reference) in the area of the canal, 0.015 m of the height of the water sheet and an average difference of -0.00015 in the slope of the canal

Key-Words: Hydrodynamics, tracers, state function, thermodynamics, geomorphological and geometric characteristic

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1 Introduction

The measurement with traditional methods of physical variables in natural streams, such as the geometry of the cross sections of the flows (width and depth), and the parameters of the geomorphology, such as slope and roughness, in a number of cases is of great difficulty due to the very nature of the flows, which constitutes a serious problem for the successful completion of studies based on these data [1]–[5]. For this reason it is desirable to have new alternative, more general and easier methods to estimate these parameters and mainly in developing countries where the shortage of measurement stations and equipment with high pressures does not allow reliable data to be obtained.

Some authors have proposed the use of thermodynamics in river hydraulics, either for understanding fluid movement or for application in water bodies [6]–[8]. Accordingly, the use of state functions or thermodynamic potentials has been employed in river hydraulics and hydrodynamics [9], since these functions have the comparative advantage that they do not depend on the minimum details that make up physical processes, and therefore reflect their energy evolution in a general way [10].

The previous characteristic allows a better understanding and application through the use of tracers, since the physical phenomena of mass transport, such as Diffusion and Dispersion, are movements of large molecular populations that evolve in an irreversible way, the changes of degrees of freedom of their movements (and of the changes of the parameters linked to them), will be easily described by these thermodynamic potentials.

In accordance with the above, this article presents a method for estimating slopes, areas, depth of water sheets and dispersion coefficients, by developing equations that represent these characteristics using the state function, Φ (t) as the theoretical reference axis. For the verification, it is applied to an experiment carried out by H.B. Fischer in the W.M. Keck Laboratory Channel of Caltech in USA, since being a controlled experiment it has the advantage that the geometric parameters and the slope of this channel are known.

2 Theory

2.1 The State Function - Φ (t)

The pouring of a conservative solute into a turbulent flow is an irreversible process, naturally subject to the general laws of physics. Such an evolution, occurring close to thermodynamic equilibrium, is usually described by some thermodynamic potential corresponding to the constraints of the system. A usual potential is Gibbs' so-called "free enthalpy", suitable for systems with constant temperature and pressure, or appropriate for natural flows. However, the application of this thermodynamic potential requires knowledge of certain parameters that are difficult to evaluate, such as the electro-chemical potentials of the chemical species at play at the system boundaries.

It is then necessary to try to define some state function that will smooth out this analytical difficulty. Recently it has been found that such a function exists and that its calculation is extremely simple through time data in Gaussian tracer curves. Its definition is based on the concept of mutual spontaneous separation of the tracer particles, once these penetrate the water, as show in Figure 1.

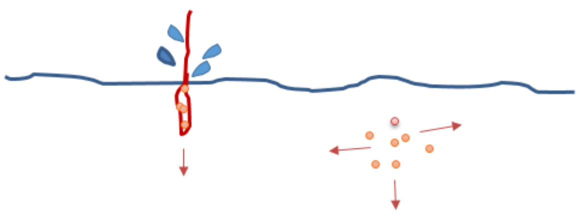


Figure 1. Injection and mutual spontaneous separation of tracer particles.

This happens because there is a natural reaction of the system to restore the lost balance (Le Chatelier-Braun principle). This separation reaction causes the concentration to decrease, which grew abruptly at the first instant of the injection [11].

A practical way to measure this effect is by a function that measures the separation velocity divided by the advective velocity. The separation velocity can be defined as the ratio between a characteristic Gaussian shift, Δ and the time it takes to perform, τ.

$$\Phi = \frac{Vsep}{U} = \frac{\left(\frac{\Delta}{\tau}\right)}{U} \qquad (1)$$

The characteristic one-dimensional Gaussian shift can be written as a completely random shift, i.e. Brownian.

$$\Phi = \frac{\left(\frac{\sqrt{2E\tau}}{\tau}\right)}{U} = \frac{\sqrt{\frac{2E}{\tau}}}{U} \qquad (2)$$

Now, you can show that:

$$\oint d\Phi = 0 \qquad (3)$$

So Φ is a state function, i.e. a thermodynamic potential close to equilibrium, which fully describes the evolution of the solute in turbulent flow. This function is decreasing with time, with an arbitrary initial value, Φo:

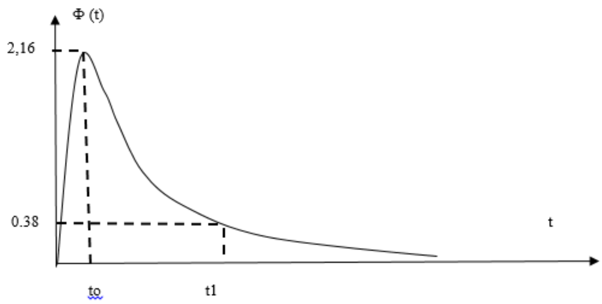


Figure 3. Absolute behavior of the curve Φ (t).

On the other hand, it should be noted that this state function has a very characteristic nature, since it is a dimensionless expression, which implies that it allows for the definition of a multitude of reasons within the set that is analyzed in the topic of interest. Remember that no other thermodynamic potential has this flexibility since they normally have the energy dimension and therefore their definitions are obliged to have this single dimension.

2.2 The state function applied to the Fick Function.

The one-dimensional Fick equation describes the evolution of a tracer pen in time and space, where *Co* is the base concentration of the flow.

$$C(x,t) = \frac{M}{A\sqrt{4\pi E\, t}} * e^{-\frac{(X-U*t)^2}{4E\, t}} + Co \qquad (5)$$

Now, if you use *E* and include it in equation (4) you have a new version of the Advection-Dispersion model:

$$E = \frac{\Phi^2 * U^2 * \beta * tp}{2} \qquad (6)$$

Here *β≈0.215* comes out of the application of a particle count by the Poisson statistics [12], *tp* is the peak time of the concentration distribution. Normally this definition of the Longitudinal Coefficient of Dispersion is compared with the classical Elder definition.

$$E \approx 5.93 * h * \sqrt[2]{h * g * S} \qquad (7)$$

Here h is the mean depth, g is the acceleration of gravity, and S is the slope of the power line. Now, by replacing in (5) the definition of E according to (6), we have the so-called modified Fick Equation.

$$C(x,t) = \frac{M}{Q*\Phi*t*1.16} * e^{-\frac{(X-U*t)^2}{2\beta\,(\Phi*U*t)^2}} + Co \qquad (8)$$

The advantage of this relationship is that it reflects fairly well the asymmetry (long tail) characteristics of the actual tracer experiments, which is not possible with state-of-the-art models.

2.3 Mass availability correction function

When a tracer is injected suddenly into a turbulent flow, the particles of the solute disperse gradually, making a fraction of the mass unavailable for dispersion very high at first, and decreasing non-linearly over time. To describe this process an associated function, $rq(\Phi)$, is defined as follows:

$$rq(\Phi) \approx \frac{Mo}{M} \qquad (9)$$

With the following specific expression:

$$rq(\Phi) \approx 1.0094\ \Phi^2 - 0.018\Phi + 0.9921 \qquad (10)$$

And with the curve shown in Figure 4.

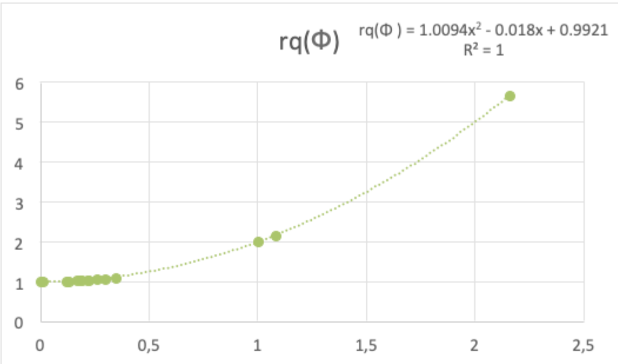


Figure 4. Behavior of the *rq* (Φ) curve.

At the beginning, when the tracer is very concentrated (and cohesive) in a small volume, and the state function has a high value, say greater than $\Phi \approx 0.38$ (situation for which solute transverse diffusion has been cancelled), then the nominal mass, *Mo*, is not all effectively available (only a smaller percentage, or effective mass *M*,), and therefore $Mo > M$.

This relationship between $rq(\Phi)$ and dispersion is shown in Figure 5.

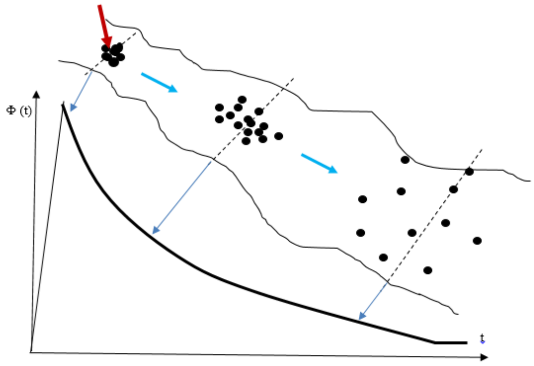


Figure 5. Correspondence between dispersion states and state function values

2.4 Analysis of the processes of dispersion and diffusion from the areas under the longitudinal distribution curve. Longitudinal Dispersion

In this case the concentration is a function of distance, X, and time, t, normally the observer is located at a given point downstream, therefore the only real variable is time. The corresponding integral is:

$$\int_{ta}^{tb} c(t)dt \qquad (11)$$

The dimensions of this function are $[M*L^{-3}*T]$ = Mass per unit of flow. It can be written, according to the principle of conservation of mass:

$$\int_{ta}^{tb} c(t)dt = \frac{M}{V} * t = \frac{M}{\left(\frac{V}{t}\right)} = \frac{M}{Q} \qquad (12)$$

2.5 The tracer distribution curve as a function of distance, X: Calculation of the cross section, *Ayz*, in the current tube.

If instead of expressing the tracer concentration as a function of time it is done as a function of distance (longitudinal coordinate, X), as see in Figure 6.

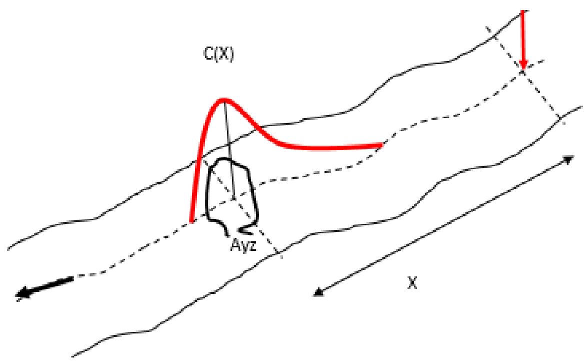


Figure 6. Definition of the current tube cross section

Then, with *Mo* the nominal mass injected this case can be written that:

$$\int_{Xa}^{Xb} c(X)dX = \frac{Mo}{V} * dX = \frac{Mo*dX}{dX*dY*dZ} = \frac{Mo}{dY*dZ} = \frac{Mo}{Ayzo} \quad (13)$$

Therefore, the nominal cross-sectional area remains:

$$Ayzo = \frac{Mo}{\int_{Xa}^{Xb} c(X)dX} \quad (14)$$

Finally, the effective definition of the cross-section, *Ayz*, remains.

$$Ayz = \frac{\left(\frac{Mo}{rq(\Phi)}\right)}{\int_{Xa}^{Xb} c(X)dX} \quad (15)$$

This is a very useful calculation, since in the dynamics of the tracers it is vital to know the cross-sectional area of the current tube through which the solute advances, as see in Figure 7.

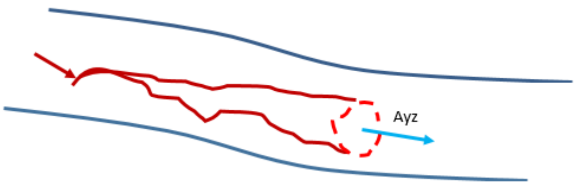


Figure 7. Cross-sectional area, *Ayz*, of the tracer advance.

As normally the RWT rhodamine concentration is given in PPB (µg/lts) and the flow rate in lts/s and the cross sectional area in m^2, the following dimensional adjustment must be made:

$$Ayz = \frac{M}{\int_{Xi}^{Xf} c(X)dX*(1000)} \quad (16)$$

To use this alternative for calculating the *Ayz* cross-section, you must change the time variable, *t*, in equation (7), and put in place the "distance" variable, *X*.

$$C(X) = \frac{M}{Q*\Phi*(tp)*1.16} * e^{-\frac{(Xo-X)^2}{2*\beta*(\Phi*X)^2}} + Co \quad (17)$$

Note that in the first factor of the member to the right of the previous expression, time is not a variable but a number, since in this case, an instant of time is fixed and the distance varies, unlike what is expressed in equation (7) where there is a fixed point (distance) and it is time that passes. For this reason the peak time (*tp*) is put in the denominator.

3 Practical application of the proposed formulas based on Φ. With data from H. Fisher (1966) on the CALTECH channel.

This controlled experiment allows verifying the formulas for the calculation of geometric and geomorphological parameters. H.B. Fischer conducted this experiment in the calibrated 40 meter channel with adjustable slope, at the W.M. Keck Laboratory at Caltech in 1966 [13], in order to verify Elder's formula for channel dispersion

This is a rectangular channel with plastic walls and a stainless steel bed. It has a uniform run from 0.0 m to 38.6 m. For the experiment in question, a salt mass of 38.6 grams of NaCl was injected, and a slope of 0.000257 adjusted by means of precision mechanical laboratory instrumentation was used. The experiment, named by its author as the "Series" 2700, consisted of eight runs. Four (2700 to 2704) made at a distance X1= 14.06 m and other four (2705-2708) made at a distance X2= 25.06 m. According to Figure 8, the flow in question has a cross-sectional area of *Ayz*=0.1408 m^2.

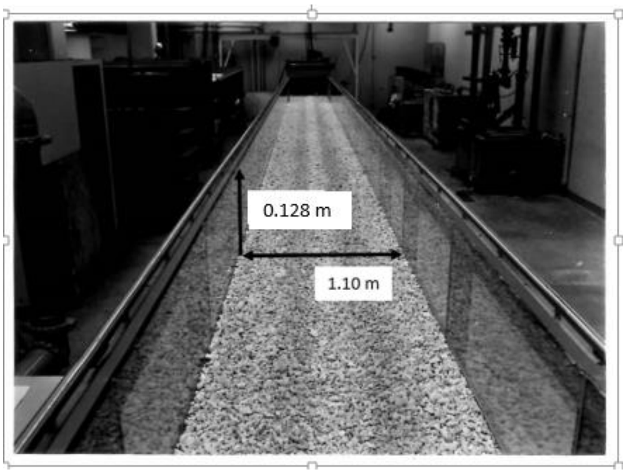


Figure 8. Appearance of the tracer measurement assembly in the Caltech channel, USA. Source: [13]

The data for the validation of this proposed model were obtained from publications given by H. Fisher, mainly his doctoral thesis and the article The Mechanics of Dispersion in Natural Streams, which describe the tests carried out and their results [13], [14].

Figure 9 shows the values reported by Fisher's 1966 experiment, in which the time of arrival (tpp) at point one was 32 s and a peak time (tp) of 7.8 s, for point two a tpp of 59 s and a tp of 64 s.

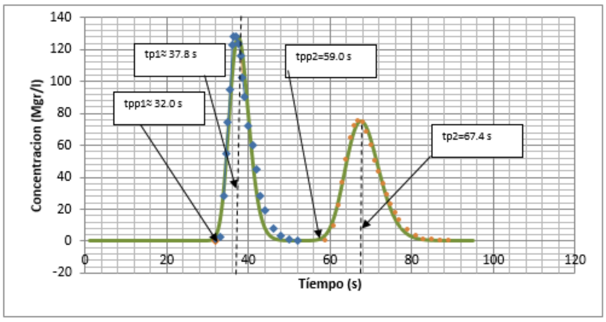


Figure 9. Sequential curves in Caltech channel for dispersion analysis.

According to these data Table 1 shows the basic data of the plotter applied to the experiment and the calculated values of the state function. While the two sequential curves are shown in Figures 9 (original) and 10 (Models), with the remarkable experimental values that are then compared with the calculated values. The models of both curves are made with the modified Fick Equation (7) using the notations of the experimental curves.

Table 1. Caltech channel salt tracer data (new methodology).

Distance (m)	Φ	Cp (Mgr/l)	tp (s)	U (m/s)
X1=14.06	0.137	135.0	37.8	0.372
X2=25.06	0.130	78.8	67.4	0.372

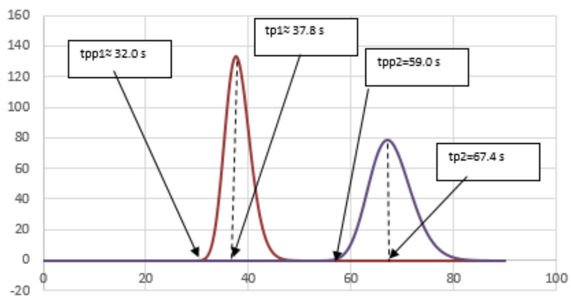
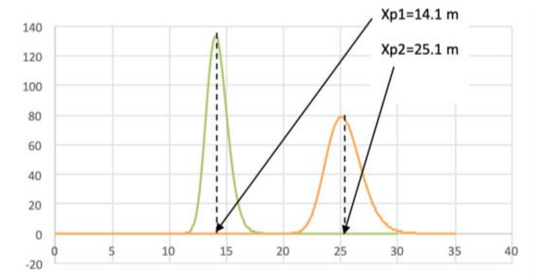


Figure 10. Curves of the models in time, according to the modified Fick Equation

3.1 Application of the formulas for the cross-sectional area in the experiment.

In this case it is necessary to apply Equation (13), with an additional correction for dimensional factors. The first thing to do is to set up the models of the two experimental curves, but not in the time domain, but in the distance domain. For this it is necessary to apply Equation (16) to the modeling curves shown in the previous Figure, which are already executed in the following Figure 11.



With an approximate flow of Q≈42.8 l/s, it is feasible to calculate the two areas under the plotter curves, roughly like this:

$$\int_{xi}^{xf} c1(X) * dX \approx 301.83\,(Mgr * m) \qquad (18)$$

And also:

$$\int_{xi'}^{xf'} c1(X) * dX \approx 301.34\,(Mgr * m) \qquad (19)$$

An additional step is to establish the relationship between the "nominal" mass, *Mo*, and the "effective" mass, *M*, by means of equation (9), for the two curves C1(X) and C2(X):

$$rq(0.137) \approx 1.0094 * 0.137^2 - 0.018 * 0.137 + 0.9921 = 1.009 \qquad (20)$$

And also:

$$rq(0.13) \approx 1.0094 * 0.13^2 - 0.018 * 0.13 + 0.9921 = 1.007 \qquad (21)$$

It is therefore evident that, thanks to the low values of Φ(t) for the two points under consideration, the two masses *M* and *Mo* are practically equal, indicating that the mass is all available. It is then possible to calculate the cross-sectional areas in the channel, at X1=14.06 m and X2=25.06 m, using Equation (15):

$$Ayz1 = \frac{38600\,mg}{301.83 * 1000(\frac{mgr}{m2})} \approx 0.1278\,m^2 \qquad (22)$$

And also:

$$Ayz1 = \frac{38600\,mg}{301.34 * 1000(\frac{mgr}{m2})} \approx 0.1215\,m^2 \qquad (23)$$

This theoretical area should be compared with the experimental area of the flow cross section of *Ayz≈0.141* m^2, a difference of 0.132 m^2 (9.36% error) was found for point 1 and for the second point a difference of 0.0195 m^2 (13.82% error), with an overall difference of 0.016 m^2 equivalent to an error of 11.59%.

Knowing this value of the area and the width of the channel, the effective depth at each measuring point can be calculated:

$$h1 \approx \frac{Ayz1}{W} \approx \frac{0.1278\ m2}{1.1\ m} \approx 0.1162\ m \qquad (24)$$

and

$$h2 \approx \frac{Ayz2}{W} \approx \frac{0.1215\ m2}{1.1\ m} \approx 0.1105\ m \qquad (25)$$

As for the height of the water sheet, for the first point the difference found was 0.0118 m (9.21% error), for the second point the difference is 0.0175 m (13.67%) an average error of 11.44% and an average difference of 0.01465 m.

The next step is to calculate the Longitudinal Dispersion Coefficient with equation (6), for the two measuring sites:

$$E1 = \frac{\Phi1^2 * U1^2 * \beta * tp1}{2} \approx \frac{0.137^2 * 0.372^2 * 0.215 * 37.8}{2} \approx 0.0106\ m^2/s \qquad (26)$$

and

$$E2 = \frac{\Phi2^2 * U2^2 * \beta * tp2}{2} \approx \frac{0.130^2 * 0.372^2 * 0.215 * 67.4}{2} \approx 0.0169\ m^2/s \qquad (27)$$

Using now the Elder equation (7), the corresponding slope, *S*, can be determined for each point, and verified with the value given in Fischer's thesis:

$$S \approx \frac{E^2}{g * (5.93 * h^{\frac{3}{2}})^2} \qquad (29)$$

Therefore, for each of the points you have:

$$S1 \approx \frac{E1^2}{g * \left(5.93 * h1^{\frac{3}{2}}\right)^2} \approx 2.05 * 10^{-4} \approx 0.000205 \ (30)$$

And also:

$$S2 \approx \frac{E2^2}{g * \left(5.93 * h2^{\frac{3}{2}}\right)^2} \approx 6.16 * 10^{-4} \approx 0.000616 \ (31)$$

Regarding this parameter, the slope for point 1 was found to be $5.11x10^{-5}$ (an error of 19.9%), however for point 2 the difference is -0.00035 (an overestimate of the slope) this is because the dispersion coefficient is high, however the average value of the calculation of the slope then is 0.00041, showing a value of the same order as that of the slope of the channel 0.000257.

4 Conclusion

1.This article proposes a thermodynamic interpretation of the phenomenon of Advection and Dispersion in natural turbulent flows. Based on this approach, a state function is found that describes in a general way this process, allowing to calculate both a new Fick distribution that incorporates this function and also to arrive at the values of the longitudinal coefficient of dispersion.
2. From this new distribution a version can be found that is based on distance instead of time, in this way the cross-sectional area of the flow through which the tracer circulates can be calculated, and from there with geometric data, the depth (or width) can be obtained.
3. Using the definition of the Elder Longitudinal Dispersion Coefficient that involves the Slope and the Depth.
4. This proposed methodology is easy to apply, since it only uses tracers, which is convenient in places where there are no mention or deficient systems.
5. It is necessary to extend the investigation, for this it is necessary to carry out measurements in rivers with different typologies.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Alfredo Constain, Carlos Peña-Guzmán and Gina Peña-Olarte developed the conceptual and mathematical models.

Alfredo Constain and Carlos Peña-Guzmán developed the analyses and experimental calculations.