Equations of Nonlinear Waves in Thin Film Flows with Mass Sources and Surface Activity at the Moving Boundary

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Abstract: - The paper deals with the derivation of governing propagation equations of nonlinear waves in thin liquid films applying to two basic cases, namely for the perfect fluid flow with a weak mass source at the bottom and for the thin film of viscid liquid flow with a mass source and surface activity at the free moving boundary. The second case is considered on the example of a condensate film flow under the low heat transfer intensity. The conditions under which the model equation has the left-hand side of a type of the Korteweg-de-Vries equation with slowly evolved parameters, and perturbed right-hand side have been established for the both cases. The conditions under which the solitary wave solutions are possible have been defined too.

Key-Words: - nonlinear wave, thin film, condensate film, mass source, perfect liquid, viscid liquid

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1 Introduction

Despite a lot of works devoted to modeling the formation and propagation of nonlinear waves in liquid layers, almost all of them deal with constant-consumption flows, and the problem for the flows with non-constant consumption remains largely open [1, 2, 3]. This is explained not by the small importance of studies of flows with variable fluid flow rate or variable physical properties, but by the great variety of observed effects of nonlinearity and dispersion [4-8]. This is especially true for flows accompanied by heat and mass transfer processes, as well as by phase transitions [7-10].

A review of the scientific literature and analysis of the known results show that the mass gain in the condensate film can significantly affect the stability of the thin waveless liquid layer flows [6, 7, 11].

Special experimental studies confirmed the existence of solitary surface waves in falling condensate films [7, 12]. Formation of these waves can be quite convincingly explained by the combined influence of a variable consumption of fluid flow in the film and by a variable, due to the temperature dependence, condensate viscosity [4, 7].

Known analytical approaches to solve such problems are usually based on transformations of the evolution equations proposed by Taniuti and Wei [13], as well as on the classical studies of Grimshaw [14] and Witham [15]. At the same time, there are some factors that make it difficult to correctly use such a technique as applied to flows with mass sources. These difficulties are connected mainly with that noted methods rely on the existence of solutions-constants of the basic unperturbed transport equations in the stationary case [16, 17, 18].

The absence of constant solutions substantially complicates the analysis of the orders of smallness of the control parameters, since the constant value that is needed for comparison and parameters evaluation as a main scale disappears. In this case, the use of methods of perturbation theory and asymptotic analysis is fraught with difficulties in substantiating the correctness and estimating errors [19, 20, 21]. As shown in [2, 13] even in the case of a constant flow rate, but in the case of a variable bottom profile [2, 22], a correct analysis of the situation turns out to be complicated, and it leads to evolutionary equations that do not have a soliton type solution [2, 23].

At this in many cases a full and strictly reasoned assessment of the correctness of using asymptotic expansions while solving applied problems in the theory of nonlinear wave propagation is not carried out, because such an analysis often also encounters mathematical difficulties [24, 25, 26]. Therefore, a partially intuitive approach is used, and it is supposed relevant and acceptable from an applied point of view [27, 28].

A considerable amount of works carried out for a rather long time has been devoted to a numerical study of the formation and propagation of waves in film flows [5, 6, 29, 30]. In these works, special algorithms for numerically solving the equations of motion are debugged, or relevant problems are solved taking into account the specifics of particular processes [6, 31].

Numerical experiments using computer simulation devoted to wave propagation in condensate films have been conducted by Alekseenko with co-workers [5, 6] as well as by authors of works [4, 7].

The method of integral transforms and asymptotic expansions for deriving waves propagation equations in condensate films has been used in [4]. However, this work was not brought to deriving the equation of propagation of surface waves.

Without diminishing the significance of such studies, it should be noted that it is usually rather difficult to interpret the results obtained in this way in relation to more general statements of problems [18-21].

However, the basic equations for the propagation of nonlinear waves in thin films both for the case of shallow water in the presence of mass sources at the bottom, and for condensate films have not yet been proposed [4, 5].

It is also not known any general control equation for the propagation of nonlinear, in particular, solitary waves in condensate films, which could be considered as a basic model for studying and conducting numerical experiments aimed at obtaining reliable results [32].

So, today it is hard to consider the reliable type of model equations describing the propagation of nonlinear waves in film flows with variable flow rate or with mass sources of a different nature [33, 34, 35]. Therefore, only having established the possible types of propagation equations for various ratios of control parameters, it will be possible to draw certain reliable conclusions about the properties of the corresponding solutions and carry out a meaningful numerical experiment. It seems reasonable to give a more detailed analysis of the situation in order to correctly establish possible types of the evolution equations describing thin-layer flows in the presence of mass sources.

In this paper, it is proposed to discuss only the theoretical aspects of the problem, without touching a numerical study and a full-scale experiment.

The aim of this work is to derive the governing equations for the propagation of nonlinear surface waves based on an asymptotic analysis of the model parameters of moving thin liquid films.

The contribution and novelty of this work is that the equations of nonlinear wave propagation in moving thin films both of perfect and viscid liquid in the presence of weak sources of mass and possible surface activity have been derived and justified.

The preliminary analysis for both perfect fluid flows and viscous condensate flows with increasing consumption has been carried out. In both cases, an important issue is the allocation of small parameters of the problem.

2 Equations for a thin moving perfect fluid layer with a mass source at the bottom

2.1 Theoretical details. Derivation of the governing equation

Unlike the statement of the problem in [2, 36, 37], in the model proposed in this section, a source of liquid at the bottom will be taken into account. The goal of this section is to derive the governing propagation equation of nonlinear surface waves in moving thin films of perfect liquid in the presence of mass source of small intensity at the bottom.

Let us consider a potential flow of perfect liquid with free surface over a plate with slowly changing shape if there exists a weak mass source at the bottom (Fig. 1 (A)).

The approach used here on the whole follows the scheme submitted in [2, 8].

The equation of continuity reads [2]

$$\varphi_{xx} + \varphi_{yy} = 0 \ . \tag{1}$$

Unlike the known approach [2, 38] the boundary condition at the bottom with allowing for the bottom mass source here reads

$$\varphi_x h_x + \varphi_y = q; \quad y = -h(x). \tag{2}$$

The kinematical boundary condition at the free surface is

$$\eta_t + \varphi_x \eta_x - \varphi_y = 0 . \tag{3}$$

The dynamical boundary condition at the free surface is

$$\varphi_t + g\eta + \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2\right) = 0. \quad (4)$$

Here φ is the velocity potential, q is the density of the mass stream across the solid bottom, $y = -h(x)$ is the function for bottom shape, $\eta(x,t)$ is the perturbed free surface of the liquid layer, x is a longitudinal coordinate, y is a normal coordinate, t is time.

The density of mass stream q depends on the actual mechanism of physical processes in the neighbourhood of the bottom. Let us consider, for example, the simplest form of the appropriate dependence that can be obtained from the following condition

$$V = kU, \quad (5)$$

where V is the normal component of liquid velocity nearby the bottom and U is the tangent component.

Condition (5) can be interpreted as a linear dependence of the mass source intensity on the tangent component of liquid velocity nearby the bottom. The condition has physical meaning, since for an perfect fluid the adhesion condition is not set.

This can be written in a novel form that is different from the formulation of the problem without source [2]

$$\varphi_x h_x + \varphi_y = k\left(\varphi_x + \varphi_y h_x\right). \quad (6)$$

Let us consider only long-wave perturbations of the free surface supposing the wave length l much bigger than average thickness of the liquid layer h_0. It seems that such an assumption is quite correct for thin layers, when the characteristic longitudinal scale prevails the others [8, 39].

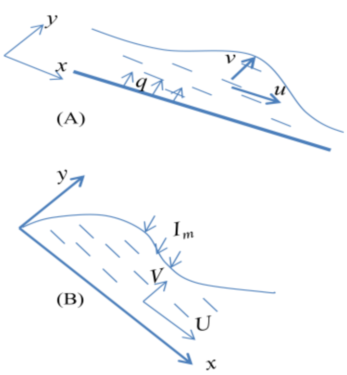

Fig. 1. Scheme of thin film flow: (A)- a film of perfect liquid with the mass source of intensity q at the bottom; (B)- condensate film flow.

Thus the small parameter μ is introduced as follows

$$h_0^2 / l^2 = \mu << 1. \quad (7)$$

In addition, let us suppose that the perturbation amplitude is also small that is necessary in order to stay in framework of weak nonlinearity approximation:

$$a/h_0 = \varepsilon << 1. \quad (8)$$

When solving the problem, it should bear in mind some uncertainty that creates problems for further asymptotic analysis. The fact is that the order of smallness, which characterizes the intensity of the source at the bottom, significantly affects the structure of the governing equation.

If both of small parameters introduced above have the same order ($\varepsilon \approx \mu$), but the coefficient of mass stream at the bottom k has a higher order, then following ratio is reasonable.

$$k = k_1 \varepsilon\mu \approx k_1 \varepsilon^2, \quad (9)$$

where $k_1 \sim O(1)$.

In framework of weak non-linearity the bottom shape is a function of the slow variable $X = \varepsilon x$. Let us use the dimensionless variables [2, 4]

$$x \to xl, \; \varphi \to \varphi \frac{a}{h_0}\sqrt{gh_0},$$

$$t \to t\frac{l}{\sqrt{gh_0}}, \; y \to yh_0, \; \eta \to \eta a, \; h \to hh_0. \quad (10)$$

Using dimensionless variables (10), system (1)-(4) takes the form

$$\varepsilon\varphi_{xx} + \varphi_{yy} = 0, \quad (11)$$

$$\varepsilon^2 \varphi_x h_X + \varphi_y = k_1 \varepsilon^2 \varphi_x, \quad (12)$$

$$\varphi_t + \eta + \frac{1}{2}\varepsilon\left(\varphi_x^2 + \varphi_y^2\right) = 0, \quad (13)$$

$$\eta_t + \varepsilon\varphi_x \eta_x - \frac{1}{\varepsilon}\varphi_y = 0. \quad (14)$$

The Taylor expansion of the velocity potential in a vicinity of the bottom gives [7]

$$\varphi = F(x,t) + \varphi_y (y+h) + \frac{1}{2}\varphi_{yy}(y+h)^2 + \frac{1}{6}\varphi_{yyy}(y+h)^3 + \frac{1}{24}\varphi_{yyyy}(y+h)^4 + ... \quad (15)$$

Using (7) and (8) the following estimations are obtained

$$\varphi_y = \varepsilon^2 (k_1 - h_X) F_x, \ \varphi_{yy} = -\varepsilon F_{xx}, \quad (16)$$

$$\varphi_{yyy} = O(\varepsilon^3), \ \varphi_{yyyy} = \varepsilon^2 F_{xxxx}. \quad (17)$$

After neglecting terms of higher than ε orders expression (13) takes the forms

$$\eta = -F_t + \varepsilon H F_{xxt} - \varepsilon F_x^2, \quad (18)$$

where $H = 1 + h$.

Substituting (18) into (14) the following expression is obtained

$$F_{tt} - HF_{xx} = \varepsilon\left[(H_x - k_1)F_x + H^2 F_{xxtt} - 2F_x F_{xt} - F_t F_{xx} - \frac{1}{6} F_{xxxx} H^3\right] \quad (19)$$

For more comfortable use of the methods of a secular perturbations theory [2, 32] it is reasonable to look for the function $F(x,t)$ in the form

$$F(x,t) = F\left(\frac{\theta}{\varepsilon}, X\right) + \varepsilon f\left(\frac{\theta}{\varepsilon}, X\right) + O(\varepsilon^2), \quad (20)$$

where θ is a special self-similar variable depending on slow coordinates $X = \varepsilon x$, $T = \varepsilon t$.

Thus the main equation for function $F(x,t)$ reads

$$F_{\theta\theta}(\theta_T^2 - H\theta_X^2) = \varepsilon\left[F_\theta(H\theta_{XX} - \theta_{TT} + H_X - k_1) - f_{\theta\theta}(\theta_T^2 - H\theta_x^2) - 2F_{\theta X}\theta_X + F_{\theta\theta\theta\theta}\left(H^2\theta_T^2\theta_X^2 - \frac{1}{6}\theta_X^4\right) - 3F_\theta F_{\theta\theta}\theta_X^2\theta_T\right] \quad (21)$$

In order to satisfy this equation in a zero order the following dispersion relation should be fulfilled

$$\theta_T^2 - H\theta_X^2 = 0. \quad (22)$$

Eliminating secular terms in the next order the following governing equation can be supposed

$$U_X - \frac{3}{2}\frac{\theta_X\theta_T}{H} UU_\theta + \frac{H\theta_X}{4}\left(\theta_T^2 - \frac{1}{3}H\theta_X^2\right)U_{\theta\theta\theta} = \left(\frac{\theta_T - H\theta_{XX} + \theta_X(k_1 - H_X)}{2H\theta_X}\right)U, \quad (23)$$

where $U = F_\theta$

Under choosing $\theta_T < 0$ and $\theta_X > 0$ this equation has a structure of the left side which bears a resemblance to the Korteweg-de-Vries equation (KdV) [1, 40].

As a result it can be concluded that for the accepted order of mass source intensity at the bottom corresponding to (9) its influence on non-linear waves propagation in thin liquid layer may be described by equation (23).

If a lower order for the weak mass source was accepted than the structure of evolution equation (23) would be destroyed.

Indeed let it be $k = k_1\varepsilon$. Thus the zero order instead of (22) reads

$$F_{\theta\theta}(\theta_T^2 - H\theta_X^2) + k_1\theta_X F_\theta = 0. \quad (24)$$

In next orders a system of linear recurrent equations which describe decrementing or incrementing perturbations would be obtained.

However in any case the complete structure of KdV equation is destroyed by the perturbation in the right-hand side, and that can be interpreted as damping influence of mass source [8, 41].

Else, if we consider the right-hand side of equation as a perturbing effect of the mass source, and assume, with a weak intensity of this source, a linear relationship between the source intensity and the flow rate U_θ, then the propagation equation can acquire the following structure

$$U_X + UU_\theta + \beta U_{\theta\theta\theta} = -cU_\theta. \quad (25)$$

Then equation (25) is reduced to the form known in the theory of solitary waves [16, 42]

$$U_X + (c+U)U_\theta + \beta U_{\theta\theta\theta} = 0. \quad (26)$$

Let us look for a solution of (26) in the form of a stationary wave

$$U = U(\xi) = U(\theta - cX), \quad (27)$$

where ξ is a phase variable and c is a phase velocity.

After twofold integration and some transformations, equation (26) in the case $\beta > 0$ transforms to the form [16, 19]

$$3\beta\left(\frac{dU}{d\xi}\right)_2 = (U_1 - U)(U_2 - U)(U_3 - U). \qquad (28)$$

Here U_1, U_2, U_3 are the constants that are expressed through control parameters of equation (25) and integration constants.

Finite solutions of equation (28) can be obtained under $U_1 \ge U \ge U_2 > U_3$ in the form

$$U(\xi) = (U_1 - U_3)\mathrm{dn}^2\left(\xi\sqrt{\frac{U_1 - U_3}{12\beta}}; \sqrt{\frac{U_1 - U_2}{U_1 - U_3}}\right) + U_3 . \qquad (29)$$

Here "**dn**" is the Jacobi elliptic function [16].

In the particular case $U_2 = U_3$ expression (27) is reduced to the form of classical soliton [16]

$$U(\xi) = \frac{U_1 - U_3}{\mathrm{ch}^2\left(\xi\sqrt{\frac{U_1 - U_3}{12\beta}}\right)} + U_3 . \qquad (30)$$

Structure of the propagation equation for non-linear waves may be also changed under the influence of phenomena taking place at the free surface [12, 30].

For example, if supplementing equation (13) with a term describing surface activity in the form $\varphi\varphi_x = \sigma\varepsilon\eta_t$, then instead of (13) the following equation can be obtained

$$\varphi_t + \eta + \frac{1}{2}\varepsilon\left(\varphi_x^2 + \varphi_y^2\right) + \varepsilon\sigma\varphi_{xx} = 0 , \qquad (31)$$

where σ is the coefficient of surface activity.

This form can be obtained, given in account the pressure gradient along the flow direction of a thin film, which arises due to the changing curvature of the film surface under the increasing consumption and influence of surface tension.

Namely, equation (25) after the rearrangements and transforms which pursue an aim to eliminate secular growth of perturbations [20, 21] can be expanded by the term describing a surface activity in the following form

$$U_X - \frac{3}{2}\frac{\theta_X\theta_T}{H}UU_\theta + \frac{H\theta_X}{4}\left(\theta_T^2 - \frac{1}{3}H\theta_X^2\right)U_{\theta\theta\theta} =$$

$$\left(\frac{\theta_T - H\theta_{XX} + \theta_X\left(k_1 - H_X\right)}{2H\theta_X}\right)U + \frac{\sigma\theta_T^3}{2H\theta_X}U_{\theta\theta} . \qquad (32)$$

2.2 Discussion and summary of Section 2

Equations (23) and (32) are offered in this work as contenders for the role of the governing equations for nonlinear waves in thin films of an inviscid fluid in the presence of a weak mass source at the bottom of the flow and surface activity. The derived equations are correct under the restriction that the intensity of a weak near-bottom source described by the small parameter k in relation (9) has a higher order of smallness than a small parameter ε characterizing the long-wave approximation.

The experience of a theoretical study of propagation equations for modulated nonlinear surface waves during fluid flow in channels with a varying bottom profile has shown the effectiveness of using conservation laws for these purposes [2, 43].

However, in our case, it is necessary to take into account the presence of mass sources, i.e. the role of relevant laws would play balance relations [29, 30].

Using the expansion of differential operators in the vicinity of critical values of the control parameters k_1, σ and excluding then the secular terms, the amplitude equations of the special type [19, 23] can be derived. Similar equations describe various nonlinear wave processes with dispersion. However, there are no soliton solutions in this case [23]. The stability of the similar wave flows depends on the order of the control parameter $\sigma\theta_T^3/2H\theta_X$ in equation (32).

In contrast to the perturbed KdV equation for the shallow water with a varying bottom profile [2, 22], in our case the right-hand side of equation (23) contains not only spatial derivatives, but also the derivative with respect to the time variable. This can be explained by the presence of a source of mass at the bottom.

Besides, the simultaneous presence of even and odd derivatives in (32) may be significantly manifested in the study of hydrodynamic stability [11]. The phenomenon of surface activity can significantly affect the flow pattern in the presence of contaminating surfactants in the liquid [30]. But this issue will need more detail analysis in further works.

3 Equations for nonlinear waves in condensate films

3.1 Theoretical details

This section deals with propagation equation for nonlinear waves in viscid condensate films with a slowly varying film thickness.

In this process, the situation is significantly complicated as a result of the presence of heat and mass sources, strong non-isothermality and, accordingly, due to these factors, the variability of the physical properties of the medium: viscosity, density, surface tension, etc. It was previously shown that for the moving condensate films, a situation may arise when the stationary Nusselt problem has no solution [4]. Therefore it can be assumed that nonlinear waves can be generated in the regions of significant temperature and viscosity gradients [7].

In order to carry out the analysis, let us obtain firstly the basic system of equations for the film thickness and flow rate during film condensation by the method of integral relations [6, 7].

Fig. 1.(B) depicts the scheme of the considered flow.

Equations of impulse and continuity in the long-wave approximation read [7]

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}\right)+g_{ef}+\frac{\sigma}{\rho}\frac{dK_S}{dx}, \quad (33)$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0. \quad (34)$$

Boundary conditions are

$$y=0\Rightarrow U=V=0,\; y=h\Rightarrow\frac{\partial U}{\partial y}=0. \quad (35)$$

Equation for material balance of condensate is

$$\frac{\partial h}{\partial t}+\frac{\partial j}{\partial x}=I_m, \quad (36)$$

The intensity of the mass source I_m in (36), i.e. increase in condensate flow rate, is related to the intensity of heat removal from the free surface of the film by the following relation

$$I_{m,(con)}=\frac{\lambda}{r\rho}\left.\frac{\partial T}{\partial y}\right|_{y=h}. \quad (37)$$

Integrating the equation of motion over the thickness it can be obtained

$$\frac{\partial j}{\partial t}+\frac{\partial}{\partial x}\int_0^h U^2dy-U_S\left(\frac{\partial h}{\partial t}+\frac{\partial j}{\partial x}\right)=-\nu_w\left.\frac{\partial U}{\partial y}\right|_{y=0}+$$

$$g_{ef}h+\frac{\sigma h}{\rho}\frac{dK_S}{dx}. \quad (38)$$

With using the self-similarity hypothesis, the velocity profile over the film thickness can be supposed in the form

$$U=U_S f(\eta),\eta=\frac{y}{h},\; f(1)=1. \quad (39)$$

Then the last equation can be written as

$$\frac{\partial j}{\partial t}+\frac{f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{j^2}{h}\right)-\frac{jI_m}{hf_1}=-\nu_w\frac{f_3}{f_1}\frac{j}{h^2}+g_{ef}h+\frac{\sigma h}{\rho}\frac{dK_S}{dx}, \quad (40)$$

The following notations are accepted here.

$$\int_0^1 fd\eta=f_1,\; \int_0^1 f^2d\eta=f_2,\; \left.\frac{\partial f}{\partial\eta}\right|_{\eta=0}=f_3. \quad (41)$$

The resulting equations form the basic system for film thickness and flow rate (condensate consumption).

However, the self-similarity hypothesis (39) [7] has some justification only for the case, when the temperature of the supporting surface over which the condensate film flows is constant.

If the temperature of the supporting surface is not constant, then it is necessary to include in the expression for the self-similar velocity profile a dependence on the film thickness as a parameter:

$$U=U_S f(\eta,x;h). \quad (42)$$

Then a more general form of the evolution equation reads

$$\frac{\partial j}{\partial t}+\frac{\partial}{\partial x}\left(\frac{f_2}{f_1^2}\frac{j^2}{h}\right)+\frac{j}{f_1h^2}\left(f_3\nu_w-I_mh\right)=g_{ef}h+\frac{\sigma h}{\rho}\frac{dK_S}{dx} \quad (43)$$

An equation of a similar type was derived in the work [4] but for a source I_m of a special and not general form.

3.2 Asymptotic analysis. Derivation of the governing equation

And again, as in Section 2, the problem of choosing the order of smallness of the control parameters arises, since the structure of the governing equation substantially depends on such a choice.

In our case, the problem is formulated as deriving the equation of propagation of nonlinear waves in a condensate film with a slowly varying consumption and, accordingly, with a slowly varying film thickness. In this case, it is necessary to evaluate the appropriate parameter of smallness.

At a sufficient distance from the starting point, the intensity of the mass source during film condensation is usually low. This makes it possible to introduce into consideration a small parameter

$$\varepsilon = \lambda \Delta T / r\rho \langle j \rangle, \tag{44}$$

where $\langle j \rangle$ is the averaged condensate consumption for the undisturbed film at the considered area [7].

This approach acquires the additional justification under the great phase transition heat [4, 6, 7].

Then the following balance ratio is correct

$$\frac{\partial h}{\partial t} + \frac{\partial j}{\partial x} = \varepsilon \frac{\langle j \rangle}{h}. \tag{45}$$

In addition, it becomes possible to introduce stretched slow variables $\tau = \varepsilon t$, $z = \varepsilon x$ and a fast phase variable $\eta = \theta(z,\tau)/\varepsilon$.

Then the system of basic equations is converted to the form:

$$\varepsilon \frac{\partial j}{\partial \tau} + \varepsilon A \frac{\partial}{\partial z}\left(\frac{j^2}{h}\right) + (B + \varepsilon B_1)\frac{j}{h^2} = g_{ef} h + \varepsilon^3 K_1 h \frac{\partial^3 h}{\partial z^3}, \tag{46}$$

$$\varepsilon \frac{\partial h}{\partial \tau} + \varepsilon \frac{\partial j}{\partial z} = \varepsilon \frac{\langle j_0 \rangle}{h}. \tag{47}$$

Because film flow consumption and the film thickness are functions of both slow and fast coordinates, derivatives in the basic equations are disclosed as follows

$$\frac{\partial}{\partial \tau} \equiv \frac{\partial}{\partial \tau} + \frac{1}{\varepsilon}\frac{\partial \theta}{\partial \tau}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial z} + \frac{1}{\varepsilon}\frac{\partial \theta}{\partial z}\frac{\partial}{\partial \eta}. \tag{48}$$

The coefficients in the resulting equations are disclosed as follows:

$$A = \frac{f_2}{f_1^2}, \; B = \frac{\nu_w f_3}{f_1}, \; B_1 = -\frac{\langle j \rangle}{f_1}, \; K_1 = \frac{\sigma}{\rho}. \tag{49}$$

As a result, it leads to the following system of evolution equations

$$\varepsilon\left(\frac{\partial j}{\partial \eta} + \frac{1}{\varepsilon}\frac{\partial j}{\partial \eta}\frac{\partial \theta}{\partial \tau}\right) + \varepsilon A\left[\frac{\partial}{\partial z}\left(\frac{j^2}{h}\right) + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\left(\frac{j^2}{h}\right)\frac{\partial \theta}{\partial z}\right] +$$

$$(B + \varepsilon B_1)\frac{j}{h^2} - g_{ef} h =$$

$$\varepsilon^3 K_1 h\left[\frac{\partial^3 h}{\partial z^3} + \frac{3}{\varepsilon}\frac{\partial^3 h}{\partial z^2 \partial \eta}\frac{\partial \theta}{\partial z} + \frac{3}{\varepsilon^2}\frac{\partial^3 h}{\partial z \partial \eta^2}\left(\frac{\partial \theta}{\partial z}\right)^2 + \right.$$

$$\left. + \frac{1}{\varepsilon^3}\frac{\partial^3 h}{\partial \eta^3}\left(\frac{\partial \theta}{\partial z}\right)^3 + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta}\frac{\partial^3 \theta}{\partial z^3}\right], \tag{50}$$

$$\varepsilon\left[\frac{\partial h}{\partial \tau} + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta}\frac{\partial \theta}{\partial \tau}\right] + \varepsilon\left[\frac{\partial j}{\partial z} + \frac{1}{\varepsilon}\frac{\partial j}{\partial \eta}\frac{\partial \theta}{\partial z}\right] = \varepsilon \frac{\langle j_0 \rangle}{h}. \tag{51}$$

Let us look for a solution to the basic system in the form of expansions in powers of a small parameter [2]

$$j = \sum_{i=0}^{N} \varepsilon^i J_i(\tau, z, \eta)\Big|_{\eta=\theta/\varepsilon} + \varepsilon^{N+1} R_{1N}(\tau, z, \eta, \varepsilon), \tag{52}$$

$$h = \sum_{i=0}^{N} \varepsilon^i H_i(\tau, z, \eta)\Big|_{\eta=\theta/\varepsilon} + \varepsilon^{N+1} R_{2N}(\tau, z, \eta, \varepsilon). \tag{53}$$

In the zero order, the system has the following form

$$\varepsilon^0 \rightarrow \begin{cases} \dfrac{\partial \theta}{\partial \tau}\dfrac{\partial J_0}{\partial \eta} + A\dfrac{\partial \theta}{\partial z}\dfrac{\partial}{\partial \eta}\left(\dfrac{J_0^2}{H_0}\right) + B\dfrac{J_0}{H_0^2} = \\ g_{ef} H_0 + K_1\left(\dfrac{\partial \theta}{\partial z}\right)^3 H_0 \dfrac{\partial^3 H_0}{\partial \eta^3}, \\ \dfrac{\partial \theta}{\partial \tau}\dfrac{\partial H_0}{\partial \eta} + \dfrac{\partial \theta}{\partial z}\dfrac{\partial J_0}{\partial \eta} = 0. \end{cases} \tag{54}$$

In the first order, the system is as follows:

$$\varepsilon^1 \to \begin{cases} \frac{\partial\theta}{\partial\tau}\frac{\partial J_1}{\partial\eta} + A\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\eta}\left(\frac{2J_0}{H_0} - J_1\right) - A\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\eta}\left(\frac{J_0^2}{H_0^2} - H_1\right) \\ + B\frac{J_1}{H_0^2} - 2B\frac{J_0}{H_0^3}H_1 - g_{ef}H_1 - K_1 H_0\frac{\partial^3 H_1}{\partial\eta^3}\left(\frac{\partial\theta}{\partial z}\right)^3 \\ + K_1 H_1\left(\frac{\partial\theta}{\partial z}\right)^3\frac{\partial^3 H_0}{\partial\eta^3} = -A\frac{\partial}{\partial z}\left(\frac{J_0^2}{H_0}\right) - B_1\frac{J_0}{H_0^2} \\ + 3K_1 H_0\frac{\partial^3 H_0}{\partial z\partial\eta^2}\left(\frac{\partial\theta}{\partial z}\right)^2 + 3K_1 H_0\frac{\partial^2 H_0}{\partial\eta^2}\frac{\partial\theta}{\partial z}\frac{\partial^2\theta}{\partial z^2}, \\ \frac{\partial\theta}{\partial\tau}\frac{\partial H_1}{\partial\eta} + \frac{\partial\theta}{\partial z}\frac{\partial J_1}{\partial\eta} = \frac{\langle j_0\rangle}{H_0} - \frac{\partial H_0}{\partial\tau} - \frac{\partial J_0}{\partial z} \end{cases} \quad (55)$$

The recurrence relations for the subsequent decomposition orders are obtained consequently. All systems except the first are linear with respect to the sought-for functions and are decoupled. At that, unlike [42, 43], the systems are disconnected without the additional assumption that the phase velocity is constant.

From relation (54) it follows

$$J_0 = -\frac{\partial\theta/\partial\tau}{\partial\theta/\partial z}H_0 + \Psi(\tau, z). \quad (56)$$

As from the physical meaning of the problem under consideration $\Psi(\tau, z) = 0$, the equation for H_0 reads

$$\frac{(\partial\theta/\partial\tau)^2}{\partial\theta/\partial z}(A-1)\frac{\partial H_0}{\partial\eta} - K_1\left(\frac{\partial\theta}{\partial z}\right)^3 H_0\frac{\partial^3 H_0}{\partial\eta^3} = g_{ef}H_0 + \frac{\partial\theta/\partial\tau}{\partial\theta/\partial z}\frac{B}{H_0}. \quad (57)$$

Similarly, systems become disjointed for subsequent approximations. At that, between consumptions and thicknesses as functions of fast and slow variables, there is a quasi-linear relationship

$$J_i = -\frac{\partial\theta/\partial\tau}{\partial\theta/\partial z}H_i + \mathrm{F}(H_{i-1}, J_{i-1}; \tau, z). \quad (58)$$

This shows that the contribution of the surface pressure gradient, due to the variable curvature of the film surface which is responsible for the dispersion of the waves, manifests itself as a dispersing factor only in the zero order. Waves of higher orders evolve, but are, strictly speaking, not dispersive.

The coefficient K_1 may be of the order ε or ε^2, depending on the nature of liquid. Therefore, to evaluate the effect of surface tension, we will present solution (57) in the form $H_0 = H_{01} + H_{02}$, where $H_{02} << H_{01}$ is the correction term taking into account the effect of surface tension. Then the following equation for H_{01} can be obtained

$$\frac{(\partial\theta/\partial\tau)^2}{\partial\theta/\partial z}(A-1)\frac{\partial H_{01}}{\partial\eta} = \frac{gH_{01}^2 + B(\partial\theta/\partial\tau)/(\partial\theta/\partial z)}{H_{01}}. \quad (59)$$

Equation (59) has an obvious traveling-wave type solution:

$$H_{01} = \sqrt{(H_{01}^2(\eta_0) + S_1)\exp(2gS_2(\eta - \eta_0)) - S_1}, \quad (60)$$

were

$$S_1 = \frac{B(\partial\theta/\partial\tau)}{g(\partial\theta/\partial z)}, S_2 = \frac{(\partial\theta/\partial z)}{(\partial\theta/\partial\tau)^2(A-1)}. \quad (61)$$

After discarding the terms of the second and higher orders of smallness with respect to the correction term H_{02} the following equation can be obtained

$$\frac{(\partial\theta/\partial\tau)^2}{\partial\theta/\partial z}(A-1)\frac{\partial H_{01}}{\partial\eta} = H_{02}\left[g - \frac{B(\partial\theta/\partial\tau)/(\partial\theta/\partial z)}{H_{01}^2}\right] + K\left(\frac{\partial\theta}{\partial z}\right)^3 H_{01}\frac{\partial^3 H_{01}}{\partial\eta^3} + K\left(\frac{\partial\theta}{\partial z}\right)^3 H_{01}\frac{\partial^3 H_{02}}{\partial\eta^3} \quad (62)$$

With discarding the first term on the right-hand side, as having a higher order of smallness than the others, homogeneous equation has been obtained. It is easy to verify that the Wronskian of this homogeneous equation is equal to zero, since there is no second derivative $\partial^2 H_{01}/\partial\eta^2$. From this it follows that the order of the obtained equation cannot be reduced to the first, and the equation has not monotonously growing solutions [19].

For the appearance of oscillating solutions in the form of a distortion of a wave profile such as ripples, it is necessary to fulfill the condition [4]

$$(A-1)\frac{\partial\theta}{\partial z} < 0 . \tag{63}$$

The last inequality is interesting in that it relates the integral characteristics of the velocity profile in the film, which depend on the temperature field over the thickness of the film, and the wave number of the carrier wave. If inequality (63) is not satisfied, then it can be expected that a small perturbation of the profile of the carrier wave will be smoothed out by capillary forces.

For positive wave numbers, the last inequality leads to the condition $A<1$.

Since the wave number and frequency in the film of variable flow rate vary with time, it is necessary to take into account at least two terms in their expansion in the Taylor series

$$\theta(z,\tau,\varepsilon)=\beta(\tau,\varepsilon)(z+\phi(\tau,\varepsilon))+\beta_1(\tau,\varepsilon)(z+\phi(\tau,\varepsilon))^2 . \tag{64}$$

Then the evolution equation follows

$$\beta\left(\frac{\partial\phi}{\partial\tau}\right)^2(A-1)\frac{\partial H_0}{\partial\eta}-K_1\beta^3 H_0\frac{\partial^3 H_0}{\partial\eta^3}=g_{ef}H_0+\frac{\partial\phi}{\partial\tau}\frac{B}{H_0} \tag{65}$$

Using the methods proposed and used in [19, 23], one would describe the evolution of functions β, β_1 and ϕ provided that special additional restrictions on the coefficients of the basic equations are satisfied. However, the fulfillment of such restrictions in our case is not obvious.

Therefore, in order to construct mathematical models capable of describing the evolution of wave perturbations of the condensate film profile, the methods of the secular perturbation theory [2, 32] have been applied. These methods are widely used in many works for studying nonlinear waves and solitons, such as models of nonlinear waves of modulation, perturbed solitons, slowly varying cnoidal waves, etc. [20, 21].

Let us suppose that near the stability boundary of the stationary flow regime, the process can be considered quasi-stationary. But then, provided that the functions j_0 and h_0 describing stationary solutions are slow, it can be assumed that some relation $j_1=L(h_1)$ is valid, where $j_1 << j_0$ and $h_1 << h_0$ are perturbations of the stationary solution of the film condensation problem, and L is also a slow function.

For condensation on a flat wall, it is possible to write a system of equations for perturbations of flow rate and film thickness, similar to that for a cylindrical surface [4]. However, unlike [4], to describe the evolution of the wave packet in the weakly nonlinear approximation, the second-order terms should be saved [2, 21].

As a result it leads to the following equations

$$\frac{\partial j_1}{\partial t}+\alpha_1\frac{\partial j_1}{\partial x}+\alpha_2\frac{\partial h_1}{\partial x}+\alpha_3\frac{\partial^3 h_1}{\partial x^3}+\alpha_4 j_1+\alpha_5 h_1=$$

$$\beta_1 j_1\frac{\partial j_1}{\partial x}+\beta_2 h_1\frac{\partial h_1}{\partial x}+\beta_3 j_1^2+\beta_4 h_1^2 , \tag{66}$$

$$\frac{\partial h_1}{\partial t}+\frac{\partial j_1}{\partial x}=zh_1+z_2 h_1^2 . \tag{67}$$

The coefficients of the resulting system of equations are described by the following expressions:

$$\alpha_1=\frac{2f_2}{f_1^2}\frac{j_0}{h_0};\ \alpha_2=-\frac{f_2}{f_1^2}\frac{j_0^2}{h_0^2};\ \alpha_3=-\frac{\sigma}{\rho}h_0;$$

$$\alpha_4=\frac{2f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{j_0}{h_0}\right)+\frac{1}{h_0}\left(\frac{\nu_w f_3}{f_1}-\frac{\lambda\Delta T}{r\rho f_1}\right);$$

$$\alpha_5=-\frac{f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{j_0^2}{h_0^2}\right)-\left(\frac{\nu_w f_3}{f_1}-\frac{\lambda\Delta T}{r\rho f_1}\right)\frac{2j_0}{h_0^3}-g-\frac{\sigma}{\rho}\frac{\partial^3 h_0}{\partial x^3}$$

$$\beta_1=\frac{2f_2}{f_1^2}\frac{1}{h_0};\qquad \beta_2=\frac{2f_2}{f_1^2}\frac{j_0^2}{h_0^3};\ \beta_3=-\frac{f_2}{f_1^2 h_0^2}\frac{\partial h_0}{\partial x};$$

$$\beta_4=\frac{f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{j_0^2}{h_0^3}\right)+\left(\frac{\nu_w f_3}{f_1}-\frac{\lambda\Delta T}{r\rho f_1}\right)\frac{3j_0}{h_0^4};$$

$$z_1=-\frac{\lambda\Delta T}{r\rho h_0^2};\ z_2=\frac{\lambda\Delta T}{r\rho h_0^3}.$$

If the temperature of the supporting surface of the flow can vary, then some coefficients of the system should be written differently:

$$\alpha_4=\frac{f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{2j_0}{h_0}\right)+\frac{\nu_w f_3}{f_1}\frac{1}{h_0^2}-\frac{1}{f_1}\left(\frac{j_0}{h_0^2}\frac{\partial I}{\partial j}\bigg|_0+\frac{I|_0}{h_0^2}\right);$$

$$\alpha_5=-\frac{\nu_w f_3}{f_1}\frac{2j_0}{h_0^3}+\left(\frac{j_0}{h_0^2}\frac{\partial I}{\partial h}\bigg|_0-\frac{2h_0 I|_0}{h_0^3}\right)-g-\frac{\sigma}{\rho}\frac{\partial^3 h_0}{\partial x^3}$$

$$\beta_3 = -\frac{f_2}{f_1^2 h_0^2}\frac{\partial h_0}{\partial x} + \frac{1}{2}\left(\frac{j_0}{h_0^2}\frac{\partial^2 I}{\partial j^2}\bigg|_0 + \frac{2}{h_0^2}\frac{\partial I}{\partial j}\bigg|_0\right);$$

$$\beta_4 = \frac{f_2}{f_1^2}\frac{\partial}{\partial x}\left(\frac{j_0^2}{h_0^3}\right) + \frac{\nu_w f_3}{f_1}\frac{3 j_0}{h_0^4} + \frac{1}{2}\left(\frac{j_0}{h_0^2}\frac{\partial^2 I}{\partial h^2}\bigg|_0 - \frac{4 j_0}{h_0^3}\frac{\partial I}{\partial h}\bigg|_0 + \frac{6 j_0}{h_0^4} I\big|_0\right)$$

Before analyzing system (66), (67), some of its features should be noted.

In addition to the usual convective nonlinearities, nonlinear terms appear in the equations due to the pumping of energy and mass due to an increase in the flow rate of the liquid in the film. The convective nonlinear terms of the type of $U(\partial U/\partial x)$ which are contained in the initial equations lead as a result of integral transformations to the terms $h(\partial h/\partial x)$ in the final form. The additional nonlinear terms in the equations appear when the intensity of the source and its derivatives are not be equal to zero: $I \neq 0$; $\partial I/\partial h \neq 0$; $\partial^2 I/\partial h^2 \neq 0$.

The pumping of energy into the system is also due to another source - gravitational forces. The resulting system, due to this and other reasons, cannot be disengaged within the framework of formal mathematical calculations. However, this can be done using the results of the analysis of an approximate linear problem and remaining within the framework of an adequate description of the qualitative behavior of small perturbations of the stationary solution [2, 20].

Let us introduce, as before, the stretched variables $X = \varepsilon x$, $T = \varepsilon t$ and the fast variable $\eta = \theta(X,T)/\varepsilon$ and will further search for a solution to the system (65), (66) in the form

$$h_1 = H\exp(\eta),\ j_1 = J\exp(\eta). \tag{68}$$

Then, after separating the terms of the equations by the powers of the small parameter and discarding the rapidly oscillating components of small amplitude of type $H^2\exp(2\eta)$ and $J^2\exp(2\eta)$, the approximate linear system for the amplitudes has been derived

$$J\left[\frac{\partial\theta}{\partial T} + \alpha_1(X)\frac{\partial\theta}{\partial X} + \alpha_4(X)\right] + H\left[\alpha_2(X)\frac{\partial\theta}{\partial X} + \alpha_3(X)\left(\frac{\partial\theta}{\partial X}\right)^3 + \alpha_5(X)\right] = 0 \tag{69}$$

$$J\frac{\partial\theta}{\partial X} + H\left(\frac{\partial\theta}{\partial T} - z_1(X)\right) = 0. \tag{70}$$

For the solvability of system (69), (70), the dispersion relation should be fulfilled

$$\begin{vmatrix} \dfrac{\partial\theta}{\partial T} + \alpha_1\dfrac{\partial\theta}{\partial X} + \alpha_4 & \alpha_2\dfrac{\partial\theta}{\partial X} + \alpha_3\left(\dfrac{\partial\theta}{\partial X}\right)^3 + \alpha_5 \\ \dfrac{\partial\theta}{\partial X} & \dfrac{\partial\theta}{\partial T} - z_1 \end{vmatrix} = 0. \tag{71}$$

From expressions (69), (70) the desired representation $J = L(X,T)H$ can be obtained, where

$$L(X,T) = \frac{(\partial\theta/\partial T - z_1)}{\partial\theta/\partial X}. \tag{72}$$

Given that $L(X,T)$ is the function of slow variables, and after substituting the last relation into the original system, the zero-order equation for the function h_1 has been derived:

$$\frac{\partial h_1}{\partial t} + \left(\alpha_1 + \frac{\alpha_2}{L}\right)\frac{\partial h_1}{\partial X} + \alpha_3\frac{\partial^3 h_1}{\partial X^3} + \left(\alpha_4 + \frac{\alpha_5}{L}\right)h_1 =$$

$$\left(\beta_1 L + \frac{\beta_2}{L}\right)h_1\frac{\partial h_1}{\partial X} + \left(\beta_3 L + \frac{\beta_4}{L}\right)h_1^2. \tag{73}$$

In the resulting equation, it is convenient to transfer to a moving coordinate system:

$$t;\quad \xi = t - \int\frac{dX}{\alpha_1 + \alpha_2/L}. \tag{74}$$

As a result equation (72) is converted to the form

$$\frac{\partial h_1}{\partial t} + \frac{\beta_1 L + \beta_2/L}{\alpha_1 + \alpha_2/L}h_1\frac{\partial h_1}{\partial \xi} - \frac{\alpha_3}{\alpha_1 + \alpha_2/L}\frac{\partial^3 h_1}{\partial \xi^3} =$$
$$-(\alpha_4 + \alpha_5/L)h_1 + (\beta_3 L + \beta_4/L)h_1^2. \tag{75}$$

Resulting equation (75) in structure is close to the Korteweg de Vries equation with non-linear perturbation on the right side and slowly varying coefficients [20]. The presence of such a perturbation leads to the fact that the dispersion relation of last equation (75) contains a nonzero imaginary part, and an undamping wave solution can exist only on the neutral line and in the growth region of amplitudes [19].

To simplify further analysis, it is convenient to rewrite equation (75) in the form

$$\frac{\partial h_1}{\partial t} + R_1 h_1 \frac{\partial h_1}{\partial \xi} + R_2 \frac{\partial^3 h_1}{\partial \xi^3} = R_3 h_1 + R_4 h_1^2 . \qquad (76)$$

It is important note that the problem has a control parameter in the form of the source intensity I .

3.3 Discussion and summary of Section 3

Equations (75), (76) with dispersion relation (71) are submitted here as the governing model for nonlinear waves in condensate films under the weak heat transfer intensity. The scale of smallness for control parameters is estimated by the parameter (44).

Further research would be interesting to focus on issues of the wave flow stability and on the evolution of waves characteristics. Computer simulation and numerical studies would also be carried out.

Below is a preliminary analysis of the developed model and a forecast of the possible behavior of wave solutions on the base of the known features of equations of similar types.

The solution of equation (76) can be searched in the form of expansion [2]

$$h_1 = \varepsilon\left(h_{10} + \varepsilon h_{11} + \varepsilon^2 h_{12} + \ldots\right), \qquad (77)$$

where

$$h_{10} = A\exp(i\theta) + A^*\exp(-i\theta), \qquad (78)$$

A^* - complex conjugate amplitude; and the phase variable in this case can be represented in the form $\theta = \int k d\xi - \nu t$, where, in turn, k is the wave number, ν is the perturbation frequency.

In order to transform (76), the multi-scale decomposition algorithm of differential operators [19] can be used. In accordance with this technique the following representations for derivatives are valid:

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots ,$$

$$\frac{\partial}{\partial \xi} \equiv \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial X_1} + \varepsilon^2 \frac{\partial}{\partial X_2} + \ldots ,$$

$$\frac{\partial^3}{\partial \xi^3} \equiv \frac{\partial^3}{\partial \xi^3} + \varepsilon\left\{2\frac{\partial}{\partial \xi}\frac{\partial^2}{\partial \xi \partial X_1} + 2\frac{\partial^2}{\partial \xi^2}\frac{\partial}{\partial X_1}\right\} + \mathrm{O}\left(\varepsilon^2\right),$$

where $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$, $X_1 = \varepsilon \xi$, $X_2 = \varepsilon^2 \xi$.

Determinant (71) plays a role of a dispersion relation. The parameter I can be expanded in the vicinity of the critical value in a Taylor series according to the procedure [19, 23]

$$I = I(k_{cr}) + \frac{1}{2}\varepsilon^2 I^4(k_{cr}) + \ldots . \qquad (79)$$

According to the known classification [22, 23], the instability of the solution of problem (76) belongs to the category of dissipative instability. Therefore, the removal of secular terms leads to an amplitude equation of the type of the Landau-Ginzburg equation [19]:

$$\frac{\partial A}{\partial T_2} + i\gamma_1(k,\nu)\frac{\partial^2 A}{\partial \xi^2} = i\gamma_2(k,\nu)A^2 A^* + \gamma_3(k,\nu)BA \qquad (80)$$

The appearance of an additional nonlinearity of the so-called BA-type [19, 23] is due to the presence of an average background flux, which in our case has a clear physical meaning, namely: it appears due to an increase in the condensate flow rate during a phase transition. The function B affects the following approximations and for h_{11} it can be written [23]

$$h_{11} = B(X_1, T_1) + C(k,\nu)\left[A^2\exp(2i\theta) + AA^*\exp(-2i\theta)\right]. \qquad (81)$$

The equation for the function B has the following type [23]

$$\frac{\partial B}{\partial T_1} - \beta_2 \frac{\partial B}{\partial X_1} = \beta_1 \frac{\partial\left(AA^*\right)}{\partial X_1}, \qquad (82)$$

And for the amplitude of the zero approximation, in addition to equation (82), a relation of the special type [19] can be supposed

$$\frac{\partial A}{\partial T_1} = i\frac{\nu_0''}{2}\frac{\partial^2 A}{\partial X_1^2} + i\beta\left(A^2 A^*\right), \qquad (83)$$

where the value $\nu_0'' = d^2\nu_0 / dk^2$ is determined on the neutral line [19].

4 Conclusion

The nature of the propagation waves and evolution of nonlinear waves characteristics in systems with mass sources substantially depends on both the intensity of the sources and form of the boundary conditions.

Asymptotic analysis showed that an equation of nonlinear waves propagation in thin layer of inviscid liquid such as the perturbed KdV equation for shallow water can be derived only in the case when the order of smallness of the parameter characterizing the intensity of the bottom mass source is higher than the order of the small parameter characterizing the ratio between the liquid layer thickness and the length of the surface wave in the long-wave approximation.

Besides, in the case of a linear relationship approximation between the weak source intensity and the flow rate the solution of a solitary wave type has been obtained.

If the same or a lower order for the mass source is accepted than the structure of the evolution equation is destroyed. This phenomenon can be interpreted as damping influence of the mass source.

The novel modifications of perturbed KdV equations (23) and (32) for a thin layer of the perfect liquid with accounting the joint influence of a weak source at the bottom and surface activity have been derived. It can be assumed that the equations obtained allow correctly describing how the phenomenon of surface activity affect the flow pattern and how the nonlinear waves arise in the presence of contaminating surfactants in the liquid. However, this issue should be investigated further more thoroughly.

The study of the influence of possible sources of natural (springs) or technogenic origin in the near-bottom zone and surface activity could be of practical interest in environment studies.

The novel modification of perturbed KdV equation (76) for describing non-linear waves in the thin viscid condensate films under the condition of slow consumption increase is also derived. The obtained solvability condition (71) plays a role of the general dispersion relation for nonlinear waves in thin condensate films.

The solutions of the obtained equation may be useful for detail describing the formation of solitary waves at a sufficient distance from the starting point of the condensation process.

The development of the asymptotic methods proposed in this paper for describing nonlinear waves in thin layers both of perfect and viscid fluids in the presence of mass sources both on the bottom and at the free moving boundary allows explaining the phenomenon of solitary waves propagation in modelled processes. These phenomena have been observed before in full-scale experiments.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Brener A.M.- Ideas; formulation of research goals and aims; development of methodology.

Yegenova A. - Creation of models.

Botayeva S. - Participation in the literature review and creation of models