Unsteady MHD Second Grade Fluids Flow in a Porous Medium with Ramped Wall Temperature

ZULKHIBRI ISMAIL
Universiti Malaysia Pahang
Faculty of Industrial Sciences & Technology
Lebuh Raya Tun Razak, 26300 Kuantan, Pahang
MALAYSIA
zulkhibri@ump.edu.my

AHMAD QUSHAIRI MOHAMAD
Universiti Teknologi Malaysia
Department of Mathematical Sciences
Faculty of Science, 81310 Johor Bahru
MALAYSIA
ahmadqushairi91@yahoo.com

ILYAS KHAN
College of Engineering Majmaah University
Basic Sciences Department
P.O. Box 66, Majmaah 11952
SAUDI ARABIA
ilyaskhanqau@yahoo.com

SHARIDAN SHAFIE
Universiti Teknologi Malaysia
Department of Mathematical Sciences
Faculty of Science, 81310 Johor Bahru
MALAYSIA
sharidan@utm.my

Abstract: In this paper, the unsteady MHD flow of second grade fluids in a porous medium are analyzed. It is assumed that the bounding infinite inclined plate has a ramped wall temperature with the presence of heat and mass diffusion. Closed-form solutions in a general form are obtained by using the Laplace transform technique. The obtained results for velocity is found to satisfy all the imposed initial and boundary conditions. It can be reduced to known solutions from the literature as limiting cases.

Key–Words: Double diffusion, MHD, Porous medium, Inclined plate, Second grade fluids, Laplace transform

1 Introduction

Free convection flow past an inclined plate is essential due to its wide applications in modern technology. [1] investigated the effect of thermal radiation on unsteady MHD free convection flow of an optically thin gray gas past an infinite inclined isothermal plate. In recent years, ismail et al. and ismail et al. [2, 3] performed the effects radiation on MHD free convection flow in a porous medium past an infinite inclined plate with ramped wall temperature.

Non-Newtonian fluids are important because of its significant application such as peristaltic transport [4], polymers processing, biomechanics, enhanced oil recovery and food products [5]. Samiulhaq et al. [6] examined a free convection flow of a second grade fluid with ramped wall temperature using Laplace transform method. Most recently, samiulhaq et al. [7] considered a porous medium with the flow of free convection second grade fluids.

However, a little work has been studied regarding second grade fluids with heat and mass transfer effects over an inclined plate, which occurs frequently in nature, hence the motivation of this paper. The objective of this present paper is to solve the double diffusions of second grade fluids on unsteady MHD free convection flow in a porous medium past an infinite inclined plate with ramped temperature using Laplace transform technique.

2 Mathematical Formulation

Consider the unsteady MHD of second grade incompressible fluids with combined heat and mass transfer by natural convection flow, near an infinite inclined plate embedded in a saturated porous medium. The x^*-axis is along to the plate with the inclination angle ϕ to the vertical and the y^*- axis is taken normal to the plate. The plate is assumed to be electrically conducting with a uniform magnetic field $\mathbf{B}$ of strength B_0, applied in a direction perpendicular to the plate. The magnetic Reynolds number is assumed to be small in order to neglect the effect of applied magnetic field. The radiation effect is also taken into account. Initially, for time $t \leq 0$, both the fluid and the plate are at rest with the constant temperature T^*_∞ and constant concentration C^*_∞. At a time $t^* > 0$, the temperature is raised or lowered to $T^*_\infty + (T^*_w - T^*_\infty)\frac{t^*}{t_0}$ when $t^* \leq t_0$. Thereafter, it is maintained at a constant temperature T^*_w when $t^* > t_0$. The concentration is raised to a constant concentration C^*_w. The physical variables become functions of the time t^* and the space y^* only, since plate is infinite in x^* and z^*-directions. The geometry of the problem is shown in Figure 1.

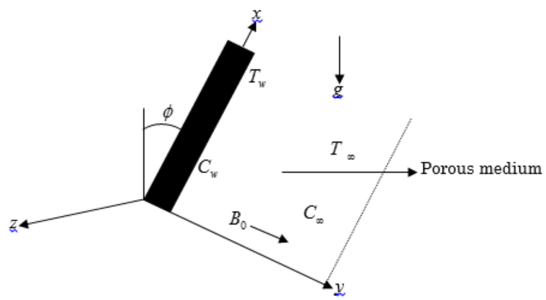

Figure 1: Geometry diagram and coordinate system

From the above assumptions, the unsteady MHD free convection viscous fluids flow past an infinite inclined plate in a porous medium is governed by

$$\begin{aligned}\frac{\partial u^*}{\partial t^*} &= v\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\alpha_1}{\rho}\frac{\partial^3 u^*}{\partial y^{*2}\partial t^*} - \frac{\sigma B_0^2}{\rho}u^* \\ &- \frac{\theta}{K^*}\left(v + \alpha_1\frac{\partial}{\partial t^*}\right)u^* \\ &+ g\beta_T\cos\phi\left(T^* - T^*_\infty\right) \\ &+ g\beta_C\cos\phi\left(C^* - C^*_\infty\right),\end{aligned} \tag{1}$$

$$\frac{\partial T^*}{\partial t^*} = \frac{k}{\rho c_p}\frac{\partial^2 T^*}{\partial y^{*2}} - \frac{1}{\rho c_p}\frac{\partial {q^*}_r}{\partial y^*}, \tag{2}$$

$$\frac{\partial C^*}{\partial t^*} = D\frac{\partial^2 C^*}{\partial y^{*2}}, \tag{3}$$

with the following initial and boundary conditions:

$$\begin{gathered}u^* = 0,\ T^* = T^*_\infty,\ C^* = C^*_\infty, \\ \text{at} \quad y^* \geq 0 \quad \text{and} \quad t^* \leq 0, \\ u^* = 0,\ C^* = C^*_w, \\ \text{at} \quad y^* = 0 \quad \text{and} \quad t^* > 0, \\ T^* = T^*_\infty + \left(T^*_w - T^*_\infty\right)\frac{t^*}{t_0}, \\ \text{at} \quad y^* = 0 \quad \text{and} \quad 0 < t^* \leq t_0, \\ T^* = T^*_w, \quad \text{at} \quad y^* = 0 \quad \text{and} \quad t^* > t_0, \\ u^*, C^*, T^* \to 0, \text{as} \quad y^* \to \infty \quad \text{and} \quad t^* > 0,\end{gathered} \tag{4}$$

where u^*, T^* and C^* denote the velocity, temperature and concentration respectively, v is the kinematic viscosity, α_1 represents the material parameter of second grade fluid, σ is the electrical conductivity of the fluid, ρ is the fluid density, φ is a porosity of porous medium, K^* is the permeability of the porous medium, g is the acceleration due to the gravity, β_T and β_C are the thermal expansion and concentration expansion, k is the fluid thermal diffusivity, c_p is the specific heat, q_r is the radiative heat flux and D is the mass diffusion. It is assumed that the radiative heat flux term is produced by plates temperature T^*_∞ and T^*_w and simplified by using Rosseland approximation,

$$\frac{\partial q_r}{\partial y^*} = 4\alpha_0\sigma^*\left(T^{*4} - T^{*4}_\infty\right), \tag{5}$$

where α_0 is the mean radiation absorption coefficient and σ^* is the Stefan-Boltzmann constant. We assume that the temperature differences within the flow are sufficiently small such that T^{*4} may be expressed as a linear function of the temperature. Using Taylor series by expanding T^{*4} about T^*_∞ and neglecting higher-order terms, thus

$$T^{*4} \cong 4T^3_\infty T - 3T^4_\infty. \tag{6}$$

Introducing the following dimensionless variables

$$\begin{gathered}y = \frac{y^*}{L},\ t = \frac{t^*(vg)^{1/3}}{L},\ u = \frac{u^*}{(vg)^{1/3}}, \\ T = \frac{T^* - T^*_\infty}{T^*_w - T^*_\infty},\ C = \frac{C^* - C^*_\infty}{C^*_w - C^*_\infty}, \\ L = \frac{v^{2/3}}{g^{1/3}},\ \frac{1}{K} = \frac{\varphi L^2}{K^*},\ M = \frac{\sigma^* B_o^2 L^2}{\mu}.\end{gathered} \tag{7}$$

Substituting equation (6) and equation (7) into equations (1) to (3) by eliminate the $*$ notation, the dimensionless momentum equation can be written as

$$\begin{aligned}c\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2\partial t} - \\ &bu + GrT\cos\phi + GmC\cos\phi\end{aligned}, \tag{8}$$

$$\frac{\partial T}{\partial t} = \frac{1}{\Pr}\left(\frac{\partial^2 T}{\partial y^2} - RT\right), \tag{9}$$

$$\frac{\partial C}{\partial t} = \frac{1}{Sc}\frac{\partial^2 C^*}{\partial y^{*2}}, \tag{10}$$

with the dimensional initial and boundary conditions are

$$\begin{gathered}u, C, T = 0, \quad \text{at} \quad y \geq 0 \quad \text{and} \quad t \leq 0, \\ u = 0,\ C = 1, \quad \text{at} \quad y = 0 \quad \text{and} \quad t > 0, \\ T = t, \quad \text{at} \quad y = 0 \quad \text{and} \quad 0 < t \leq 1, \\ T = 1, \quad \text{at} \quad y = 0 \quad \text{and} \quad t > 1, \\ u, C, T \to 0, \text{as} \quad y \to \infty \quad \text{and} \quad t > 0.\end{gathered} \tag{11}$$

Here, u, T, C, K_1, M, R, Gr, Gm, Pr, α and Sc are nondimensional fluid velocity, temperature, concentration, permeability of the porous medium, magnetic parameter known as Hartmann number, radiation parameter, thermal Grashof number and the mass

Grashof number, Prandtl number, material module of second grade fluids and Schmidt number, respectively. The constants appearing in the above equations are defined as

$$\alpha = \frac{\alpha_1}{L^2},\ b = M + \frac{1}{K},\ c = 1 + \frac{\alpha_1 \varphi L^2}{K},$$
$$Sc = \frac{\upsilon}{D},\ \Pr = \frac{\mu c_p}{k},\ R = \frac{16\alpha\sigma^* T_\infty^* L^2}{k},$$
$$M = \frac{\sigma B_0^2 L^2}{\rho\upsilon},\ \frac{K^*}{L^2} = K, \tag{12}$$
$$Gr = \frac{g^{1/3}\beta_T \left(T_w^* - T_\infty^*\right) L}{\upsilon^{2/3}},$$
$$Gm = \frac{g^{1/3}\beta_m \left(C_w^* - C_\infty^*\right) L}{\upsilon^{2/3}}.$$

3 Problem Solution

The analytical solutions for the system of equations (8)–(10) with the initial and boundary conditions (2) will be determined by the Laplace transform method. The following transformed equations is s-domain are obtained

$$(b + cs)\,\bar{u} - Gr\bar{T}\cos\phi - Gm\bar{C}\cos\phi = (1 + \alpha s)\,\frac{\partial^2 \bar{u}}{\partial y^2}, \tag{13}$$

$$\frac{\partial^2 \bar{T}}{\partial y^2} - (\Pr s + R)\,\bar{T} = 0, \tag{14}$$

$$\frac{\partial^2 \bar{C}}{\partial y^2} - Scs\bar{C} = 0. \tag{15}$$

The equations (14) and (14) are uncoupled from equation (13) and their solutions are

$$\overline{T}(y,s) = \left(\frac{1 - e^{-s}}{s^2}\right) e^{-y\sqrt{\Pr(\Phi+s)}}, \tag{16}$$

$$\overline{C}(y,s) = \frac{1}{s} e^{-y\sqrt{Scs}}, \tag{17}$$

where

$$\Phi = \frac{R}{\Pr}.$$

In order to get T and C, by using the inverse Laplace transform,

$$C(y,t) = \operatorname{erfc}\left(\frac{y\sqrt{Sc}}{2\sqrt{t}}\right), \tag{18}$$

and

$$T_r(y,t) = T_1(y,t) - T_1(y,t-1)\,H(t-1), \tag{19}$$

where

$$T_1(y,t) = \left(\begin{array}{c} \frac{t}{2} + \\ \frac{y}{4}\sqrt{\frac{\Pr}{\Phi}} \end{array}\right) e^{y\sqrt{\Phi \Pr}} \operatorname{erfc}\left(\begin{array}{c} \frac{y}{2}\sqrt{\frac{\Pr}{t}} \\ +\sqrt{\Phi t} \end{array}\right) + \left(\begin{array}{c} \frac{t}{2} - \\ \frac{y}{4}\sqrt{\frac{\Pr}{\Phi}} \end{array}\right) e^{-y\sqrt{\Phi \Pr}} \operatorname{erfc}\left(\begin{array}{c} \frac{y}{2}\sqrt{\frac{\Pr}{t}} \\ -\sqrt{\Phi t} \end{array}\right).$$

In order to find the solution of equation (13), let

$$\bar{u}(y,s) = \bar{u}_1 + \bar{u}_2, \tag{20}$$

where

$$\bar{u}_1(y,s) = \frac{Gr\cos\phi}{(\Pr s + \Pr\Phi)(1+\alpha s) - (b+cs)} \frac{1-e^{-s}}{s^2}\left(e^{-y\sqrt{\frac{b+cs}{1+\alpha s}}} - e^{-y\sqrt{\Pr}\sqrt{\Phi+s}}\right), \tag{21}$$

and

$$\bar{u}_2(y,s) = \frac{Gm\cos\phi}{s[Scs(1+\alpha s) - (b+cs)]} \left(e^{-y\sqrt{\frac{b+cs}{1+\alpha s}}} - e^{-y\sqrt{Scs}}\right). \tag{22}$$

To find the solution for $\overline{u}(y,s)$, we solve $\bar{u}_1$ and $\bar{u}_2$ separately. Let

$$\begin{aligned} \bar{u}_{11} &= \frac{Gr\cos\phi}{s}\frac{1}{\alpha\Pr\left[(s+m_1)^2 - m_2^2\right]}, \\ \bar{u}_{12} &= \frac{1}{s}\left(e^{-y\sqrt{\frac{b+cs}{1+\alpha s}}} - e^{-y\sqrt{\Pr}\sqrt{\Phi+s}}\right), \\ \bar{u}_{21} &= \frac{Gm\cos\phi}{\alpha Sc\left[(s+m_3)^2 - m_4^2\right]}, \\ \bar{u}_{22} &= \frac{1}{s}\left(e^{-y\sqrt{\frac{b+cs}{1+\alpha s}}} - e^{-y\sqrt{Scs}}\right), \end{aligned} \tag{23}$$

where

$$\begin{aligned} m_1 &= \frac{\alpha\Pr\Phi + \Pr - c}{2\alpha\Pr}, \\ m_2 &= \frac{\sqrt{(\alpha\Pr\Phi + \Pr - c)^2 - 4\alpha\Pr(\Pr\Phi - b)}}{2\alpha\Pr}, \\ m_3 &= \frac{Sc - c}{2\alpha Sc}, \\ m_4 &= \frac{\sqrt{(Sc-c)^2 - 4\alpha b Sc}}{2\alpha Sc}. \end{aligned}$$

The inverse Laplace transforms for $\bar{u}_{11}$ and $\bar{u}_{21}$ are given by

$$u_{11}(t) = \frac{Gr\cos\phi}{\alpha\Pr}\frac{1}{m_2^2 - m_1^2}\left[\frac{e^{-m_1 t}}{m_2}\left(\begin{array}{c} m_1\sinh m_2 t + \\ m_2\cosh m_2 t \end{array}\right) - 1\right], \tag{24}$$

and

$$u_{21}(t) = \frac{Gm\cos\phi}{\alpha m_4 Sc} e^{-m_3 t}\sinh m_4 t. \tag{25}$$

In order to determine the inverse Laplace transforms of $\bar{u}_{12}$ and $\bar{u}_{22}$, we consider the following function:

$$F(y,s) = \frac{1}{s} e^{-y\sqrt{\frac{b+cs}{1+\alpha s}}}. \tag{26}$$

The inverse Laplace transform for the equation (26) is given by

$$f(y,t)=\frac{b}{\alpha}\int_0^{\infty}\int_0^{t}\left[\begin{array}{c} e^{-\frac{uc+t}{\alpha}}\operatorname{erfc}\left(\frac{y}{2\sqrt{u}}\right) \\ I_0\left(\frac{2}{\alpha}\sqrt{us\left(c-\alpha b\right)}\right)\end{array}\right]dsdu$$
$$+\frac{c}{\alpha}\int_0^{\infty}\left[\begin{array}{c} e^{-\frac{uc+t}{\alpha}}\operatorname{erfc}\left(\frac{y}{2\sqrt{u}}\right) \\ I_0\left(\frac{2}{\alpha}\sqrt{ut\left(c-\alpha b\right)}\right)\end{array}\right]du. \quad (27)$$

Thus

$$u_{12}(y,t)=f(y,t)-\frac{1}{2}\left[\begin{array}{c} e^{y\sqrt{\Phi \Pr}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}+\Phi t\right)+ \\ e^{-y\sqrt{\Phi \Pr}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}-\Phi t\right)\end{array}\right], \quad (28)$$

and

$$u_{22}(y,t)=f(y,t)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{Sc}{t}}\right), \quad (29)$$

where $I_0(\cdot)$ is modified Bessel function of the first kind of order zero and $\operatorname{erfc}(\cdot)$ is complimentary error function. Consequently, the expression for velocity in the domain t can be written in simple form as

$$\begin{aligned} u(y,t) &= U_1(y,t)H(t)- \\ &\quad U_1(y,t-1)H(t-1) \\ &\quad +U_2(y,t)H(t), \end{aligned} \quad (30)$$

where

$$\begin{aligned} U_1(y,t) &= (u_{11}*u_{12})(t) \\ &= \int_0^{u} u_{11}(t-s)\,u_{12}(y,s)\,ds, \end{aligned}$$

and

$$\begin{aligned} U_2(y,t) &= (u_{21}*u_{22})(t) \\ &= \int_0^{u} u_{21}(t-s)\,u_{22}(y,s)\,ds. \end{aligned}$$

The symbol of $(u_{11}*u_{12})(t)$ and $(u_{21}*u_{22})(t)$ denotes convolution product of two functions.

4 Graphical Results and Discussion

Analytical solutions of unsteady MHD free convection flow in a porous medium are investigated. In this case, we consider second grade fluids past an infinite inclined plate with ramped wall temperature. The problem solutions are solved by using Laplace transform technique. Limiting case show that the present momentum solution was reduced to published result. As expected, the result are found identical, which show the validity of the obtainable solution.

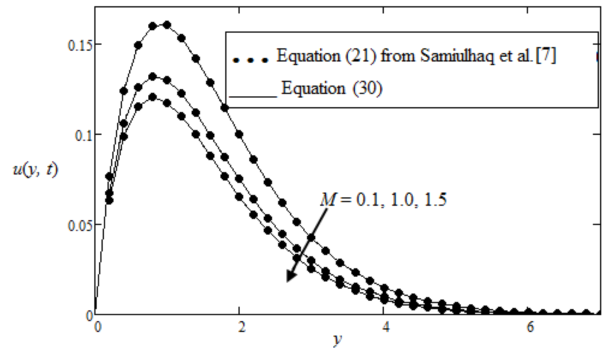

Figure 2: comparison of velocity $u(y,t)$ in equation (30) with equation (21) from Samiulhaq et al. [7]

From Figure 2, by neglecting the effect of mass transfer and inclination angle, the momentum equation from (30) is identical to the equation (21) by Samiulhaq et al [7]. It is worth to mention that equations (18), (19) and (30) satisfy all the imposed initial and boundary conditions. Hence, this also provides a useful mathematical check to our calculi.

The obtained solutions are also studied numerically in order to determine the effects of several involved parameters such as second grade parameter α, and Hartmann number M. Figure 3 shows the effect of second grade parameters α on velocity profiles. It is noticed that the fluid velocity decreases and then increases on decreasing second grade parameters.

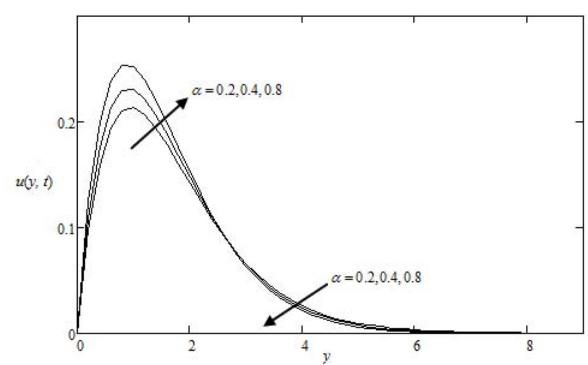

Figure 3: velocity profiles at various second grade parameters α

The variation of velocity for different values of Hartmann numbers M is plotted in Figure 4. We can see that the application of transverse magnetic field will result a resistive type force namely the Lorentz force. This force tends to resist the fluid flow and thus reducing the velocity.

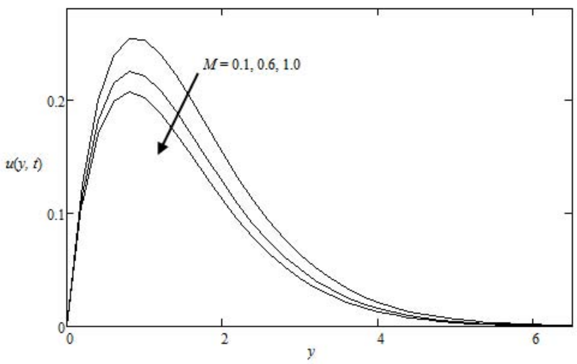

Figure 4: velocity profiles at various Hartmann numbers M

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