The Velocity Potential PDE in an Orthogonal Curvilinear Coordinate System

RICHARD SELESCU
Flow Physics Department, Experimental Aerodynamics Compartment, Trisonic Wind Tunnel Laboratory
"Elie Carafoli" National Institute for Aerospace Research – INCAS (under the Aegis of the Romanian Academy)
Bucharest, Sector 6, Bd. Iuliu Maniu, No. 220, Code 061126
ROMANIA
e-mail: rselescu@gmail.com web page: http://www.incas.ro

Abstract: This work studies and clarifies some local physical phenomena in fluid mechanics, in the form of an intrinsic analytic study, regarding the PDEs of the velocity potential and (especially) 2-D "quasi-potential" (their simpler and special forms), over the "isentropic" or the 3-D (**V**, **Ω**) surfaces and along the "isentropic & isotachic" space curves, written for any potential and even rotational flow of an inviscid compressible fluid for both steady and unsteady motions. It continues a series of works presented at some conferences and a congress during 2006 – 2012, representing a real deep insight into the still hidden theory of the isoenergetic flow. Applying the advantages offered by the special virtual surfaces ("isentropic" and "polytropic") and space curves (intersection lines of these surfaces) introduced in the previous works, a simpler PDE of the 2nd order in only two variables, and more, a Laplace's PDE (for any rotational "pseudo-flow", using a new smart intrinsic coordinate system), instead of the general PDE (Steichen, 1909, for *plane potential supersonic* flows only) of the 2nd order in three variables. So far, this equation was known as being written for *potential* flows only. A model extension for rotational flows of a viscous compressible fluid was given.

Key-Words: rotational flows; steady and unsteady flows; inviscid and viscous fluids; compressible fluids; isentropic and polytropic surfaces; Selescu's isentropic & isotachic vector ($d\mathbf{R}_{ij}$), quasi-Laplace lines (quasi-isothermal quasi-potential)

1 Steichen's vector equation; nomenclature

Joining the continuity and physical equations to the motion equation (Euler) for an inviscid compressible fluid steady irrotational (isentropic) flow (of a small fluid particle), and using the local speed of sound a definition, one gets:

$$\nabla\mathbf{V} = \frac{\mathbf{V}}{2a^2}\nabla\left(V^2\right), \text{ with } \mathbf{V} = \nabla\Phi, \text{ so obtaining (see [1])}$$

$$\Delta\Phi = \frac{\nabla\Phi}{2a^2}\nabla\left[(\nabla\Phi)^2\right] \quad \text{(Steichen), where, symbolically:}$$

$$\Delta = \frac{1}{\prod_{j=1}^{3} h_j}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\left(\prod_{j=1}^{3} h_j\right)\cdot\frac{1}{h_i^2}\frac{\partial}{\partial x_i}\right] = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \text{ and}$$

$$\nabla = \sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial}{\partial x_i} = \mathbf{k}_x\frac{\partial}{\partial x}+\mathbf{k}_y\frac{\partial}{\partial y}+\mathbf{k}_z\frac{\partial}{\partial z} \quad \text{– del (nabla)}$$

(Laplace's and Hamilton's operators, respectively), in a triorthogonal system of curvilinear coordinates x_i; $\mathbf{k}_i$ – a 3-D basis (versors of Ox_i axes); h_i – Lamé's coefficients; **V** – the local velocity of translation (of the fluid particle) – intensity of the local fluid field; $V = |\mathbf{V}|$; Φ – the velocity potential; $a^2 = (dp/d\rho)_{S=S_0} = \gamma\mathcal{R}T$, with γ, $\mathcal{R}$ – constants (γ, $\mathcal{R}$, p, ρ are defined later); S – the specific entropy of the fluid particle; T – the static temperature (absolute) of the fluid particle. For a general (rotational) inviscid flow we also introduce: $\mathbf{\Omega} = \nabla\times\mathbf{V} = 2\,\boldsymbol{\omega}$ – the vorticity, with: $\boldsymbol{\omega}$ – the local velocity of rotation (of the small fluid particle); i – the specific enthalpy; $i_0 = i + V^2/2$ – the total specific enthalpy.

2 Steichen's PDE of the velocity potential in an orthogonal curvilinear coordinate system

So far, to the best of the author's knowledge, nowhere in the world literature, except for some particular cases (like the cylindrical and spherical coordinate systems), this equation was expressed. This author established Steichen's PDE for a steady (and then unsteady) flow's velocity potential in its general expanded form ([2] – [5]), considering a curvilinear coordinate system Oξηζ (with h_ξ, h_η, h_ζ – Lamé's coefficients). The vector form of this equation (usually written *for irrotational flows only*) can be expanded as follows (now *for rotational flows also*):

$$\nabla\mathbf{V}_i = \frac{\mathbf{V}_i}{2a_i^2}\nabla\left(V_i^2\right), \text{ with } \mathbf{V}_i = \nabla\Phi, \text{ but now } \mathbf{\Omega}_i \neq 0;$$

$$\Rightarrow \Delta\Phi_i = \frac{\nabla\Phi_i}{2a_i^2}\nabla\left[(\nabla\Phi_i)^2\right], \text{ or } \Delta\Phi_i = \frac{1}{a_i^2}\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i,$$

$$\text{where: } \Delta\Phi_i = \frac{\partial^2\Phi_i}{\partial x^2}+\frac{\partial^2\Phi_i}{\partial y^2}+\frac{\partial^2\Phi_i}{\partial z^2} \text{ and, symbolically}$$

$$\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i = \left(\frac{\partial\Phi_i}{\partial x}\cdot\frac{\partial}{\partial x}+\frac{\partial\Phi_i}{\partial y}\cdot\frac{\partial}{\partial y}+\frac{\partial\Phi_i}{\partial z}\cdot\frac{\partial}{\partial z}\right)^{(2)}\Phi_i$$

$$= \left[\left(\frac{\partial\Phi_i}{\partial x}\right)^2\frac{\partial^2}{\partial x^2}+\left(\frac{\partial\Phi_i}{\partial y}\right)^2\frac{\partial^2}{\partial y^2}+\left(\frac{\partial\Phi_i}{\partial z}\right)^2\frac{\partial^2}{\partial z^2}\right.$$

$$\left.+2\frac{\partial\Phi_i}{\partial x}\frac{\partial\Phi_i}{\partial y}\frac{\partial^2}{\partial x\partial y}+2\frac{\partial\Phi_i}{\partial y}\frac{\partial\Phi_i}{\partial z}\frac{\partial^2}{\partial y\partial z}+2\frac{\partial\Phi_i}{\partial z}\frac{\partial\Phi_i}{\partial x}\frac{\partial^2}{\partial z\partial x}\right]\Phi_i \quad .$$

Φ_i is the velocity quasi-potential – over the $(\mathbf{V}, \mathbf{\Omega})_i$ sheets only, with $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = TdS = 0$; $d\mathbf{R} \in (\mathbf{V}, \mathbf{\Omega})$ plane, due to Crocco's equation for steady isoenergetic (i_0 = const.; $\nabla i_0 = 0$) flows (see [6]): $\mathbf{\Omega} \times \mathbf{V} = T\nabla S$; $d\mathbf{R}$ is a virtual elementary displacement. The $(\mathbf{V}, \mathbf{\Omega})_i$ sheet (envelope of the planes above, containing streamlines and vortex lines) is *isentropic* ($S = S_{0i}$ = const.), allowing to introduce a Φ_i (see [2] – [5] and section 10). The speed of sound a_i is given by $a_i^2 = (dp/d\rho)_{S=S_{0i}} = \gamma\mathcal{R}T_i = (\gamma-1)(W^2 - V_i^2)/2 = (\gamma-1)\left[W^2 - (\partial\Phi_i/\partial x)^2 - (\partial\Phi_i/\partial y)^2 - (\partial\Phi_i/\partial z)^2\right]/2$, where: p – the fluid static pressure; ρ – the fluid density; γ – the adiabatic exponent (ratio of specific heats, C_p/C_v); W – the gas maximum speed (corresponding to the expansion into a vacuum) – an invariant quantity. For a perfect (an ideal) gas: $p = \mathcal{R}\rho T$, with $\mathcal{R} = C_p - C_v$ = const. The isentropic surfaces are analogous to D. Bernoulli's (Lamb's) ones ([2] – [5], [7]) for a barotropic fluid ($B = V^2/2 + \int dp/\rho + gz = B_{0i}$, with: g – the acceleration of gravity). *All the points* M(x, y, z) *at which this new PDE* ($\mathbf{\Omega} \neq 0$) *is satisfied belong to a certain* $(\mathbf{V}, \mathbf{\Omega})_i$ *isentropic sheet*. In the Oξηζ curvilinear coordinate system, and in a certain orthogonal system of curvilinear coordinates q_i ($Oq_1q_2q_3$) with the 3-D basis $\mathbf{k}_i$ we have, resp.:

$$\Delta\Phi = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi}{\partial\zeta}\right)\right]$$
$$= \left(1\Big/\prod_{j=1}^{3} h_j\right)\cdot\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\left(\prod_{j=1}^{3} h_j\right)\cdot\frac{1}{h_i}\frac{\partial\Phi}{h_i\partial q_i}\right] \quad ;$$

$$\nabla\Phi = \mathbf{k}_\xi\frac{1}{h_\xi}\frac{\partial\Phi}{\partial\xi} + \mathbf{k}_\eta\frac{1}{h_\eta}\frac{\partial\Phi}{\partial\eta} + \mathbf{k}_\zeta\frac{1}{h_\zeta}\frac{\partial\Phi}{\partial\zeta} = \sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial\Phi}{\partial q_i};$$

$$\nabla(V^2) = \nabla\left[(\nabla\Phi)^2\right] = \sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial}{\partial q_i}\left[\sum_{j=1}^{3}\left(\frac{1}{h_j}\frac{\partial\Phi}{\partial q_j}\right)^2\right] \quad ;$$

$$a^2 = \frac{\gamma-1}{2}(W^2 - V^2) = \frac{\gamma-1}{2}\left[W^2 - (\nabla\Phi)^2\right]$$
$$= \frac{\gamma-1}{2}\left[W^2 - \frac{1}{h_\xi^2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2 - \frac{1}{h_\eta^2}\left(\frac{\partial\Phi}{\partial\eta}\right)^2 - \frac{1}{h_\zeta^2}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\right]$$
$$= \frac{\gamma-1}{2}\left[W^2 - \sum_{i=1}^{3}\left(\frac{1}{h_i}\frac{\partial\Phi}{\partial q_i}\right)^2\right] \quad .$$

So, Steichen's vector equation in section 1 becomes:

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi}{\partial\zeta}\right)\right]\times$$
$$\times\left[W^2 - \frac{1}{h_\xi^2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2 - \frac{1}{h_\eta^2}\left(\frac{\partial\Phi}{\partial\eta}\right)^2 - \frac{1}{h_\zeta^2}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\right]$$
$$= \frac{2}{\gamma-1}\left\{\frac{1}{h_\xi^4}\left(\frac{\partial\Phi}{\partial\xi}\right)^2\frac{\partial^2\Phi}{\partial\xi^2} + \frac{1}{h_\eta^4}\left(\frac{\partial\Phi}{\partial\eta}\right)^2\frac{\partial^2\Phi}{\partial\eta^2} + \frac{1}{h_\zeta^4}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\frac{\partial^2\Phi}{\partial\zeta^2}\right.$$
$$+ \frac{2}{h_\xi^2 h_\eta^2}\frac{\partial\Phi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\xi\partial\eta} + \frac{2}{h_\eta^2 h_\zeta^2}\frac{\partial\Phi}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}\frac{\partial^2\Phi}{\partial\eta\partial\zeta} + \frac{2}{h_\zeta^2 h_\xi^2}\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\zeta\partial\xi}$$
$$-\left[\frac{1}{h_\xi^4}\frac{\partial \ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi}{\partial\xi}\right)^3 + \frac{1}{h_\eta^4}\frac{\partial \ln h_\eta}{\partial\eta}\left(\frac{\partial\Phi}{\partial\eta}\right)^3 + \frac{1}{h_\zeta^4}\frac{\partial \ln h_\zeta}{\partial\zeta}\left(\frac{\partial\Phi}{\partial\zeta}\right)^3\right.$$
$$+ \frac{1}{h_\xi^2 h_\eta^2}\left(\frac{\partial \ln h_\xi}{\partial\eta}\frac{\partial\Phi}{\partial\xi} + \frac{\partial \ln h_\eta}{\partial\xi}\frac{\partial\Phi}{\partial\eta}\right)\frac{\partial\Phi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}$$
$$+ \frac{1}{h_\eta^2 h_\zeta^2}\left(\frac{\partial \ln h_\eta}{\partial\zeta}\frac{\partial\Phi}{\partial\eta} + \frac{\partial \ln h_\zeta}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}\right)\frac{\partial\Phi}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}$$
$$\left.\left. + \frac{1}{h_\zeta^2 h_\xi^2}\left(\frac{\partial \ln h_\zeta}{\partial\xi}\frac{\partial\Phi}{\partial\zeta} + \frac{\partial \ln h_\xi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right)\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]\right\} \quad , \tag{1}$$

– a nonlinear PDE of the 2nd order in 3 variables, or (using the classical symbols Σ and Π applied to some variables indexed with the subscripts "i" and "j") in the compact scalar form below:

$$\frac{\gamma-1}{2}\frac{1}{\prod_{j=1}^{3} h_j}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\left(\prod_{j=1}^{3} h_j\right)\cdot\frac{1}{h_i}\frac{\partial\Phi}{h_i\partial q_i}\right]\cdot\left[W^2 - \sum_{i=1}^{3}\left(\frac{\partial\Phi}{h_i\partial q_i}\right)^2\right]$$
$$= \sum_{i=1}^{3}\left\{\left[\left(\frac{\partial\Phi}{h_i\partial q_i}\right)\frac{\partial}{h_i\partial q_i}\right]^{(2)}\Phi - \left[\frac{\partial \ln h_i}{h_i\partial q_i}\left(\frac{\partial\Phi}{h_i\partial q_i}\right)^3 + \ldots\right]\right\} \quad . \tag{2}$$

In order to express the right-hand side of Eqs. (1), (2) in a more compact form, we must prepare the scalar product in the right-hand side of Steichen's equation, so having:

$$\mathbf{V}\cdot\nabla(V^2) = \nabla\Phi\cdot\nabla\left[(\nabla\Phi)^2\right] = \sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial\Phi}{\partial q_i}\cdot\sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial}{\partial q_i}\left[\sum_{j=1}^{3}\left(\frac{1}{h_j}\frac{\partial\Phi}{\partial q_j}\right)^2\right]$$
$$= 2\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\Phi}{h_i\partial q_i}\frac{\partial\Phi}{h_j\partial q_j}\left[\frac{\partial^2\Phi}{h_i\partial q_i\cdot h_j\partial q_j} - \frac{1}{2}\left(\frac{\partial \ln h_j}{\partial q_i}\frac{\partial\Phi}{\partial q_j} + \frac{\partial \ln h_i}{\partial q_j}\frac{\partial\Phi}{\partial q_i}\right)\right],$$

so, finally, the most compact symbolic expansion for the scalar equation of the velocity potential Φ is as follows:

$$\frac{\gamma-1}{2}\frac{1}{\prod_{j=1}^{3} h_j}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\left(\prod_{j=1}^{3} h_j\right)\cdot\frac{1}{h_i}\frac{\partial\Phi}{h_i\partial q_i}\right]\cdot\left[W^2 - \sum_{i=1}^{3}\left(\frac{\partial\Phi}{h_i\partial q_i}\right)^2\right]$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\Phi}{h_i\partial q_i}\frac{\partial\Phi}{h_j\partial q_j}\left[\frac{\partial^2\Phi}{h_i\partial q_i\cdot h_j\partial q_j} - \frac{1}{2}\left(\frac{\partial \ln h_j}{\partial q_i}\frac{\partial\Phi}{\partial q_j} + \frac{\partial \ln h_i}{\partial q_j}\frac{\partial\Phi}{\partial q_i}\right)\right].$$

3 The 2-D velocity "quasi-potential" PDE (for a rotational flow) in a smart intrinsic orthogonal curvilinear coordinate system

In a system of 3-orthogonal intrinsic coordinates Oξηζ tied to the isentropic $(\mathbf{V}, \mathbf{\Omega})_i$ surfaces (or Oλμν, with λ, μ, ν – lengths of the orthogonal arcs, with λ and μ contained in the local plane tangent to $(\mathbf{V}, \mathbf{\Omega})_i$ and ν directed along the normal), Laplace's and Hamilton's operators (Δ and ∇), as well as the speed of sound a_i are given by the general expressions below (Φ_i depends on ξ, η and ζ_{0i}, or on λ, μ and ν_{0i}, where: $d\lambda = h_\xi d\xi$; $d\mu = h_\eta d\eta$ and $d\nu = h_\zeta d\zeta$ – the elementary arc lengths):

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta}\right)\right]$$

$$= \left(1\Big/\prod_{k=1}^{3} h_k\right)\cdot\sum_{j=1}^{3}\frac{\partial}{\partial q_j}\left[\left(\prod_{k=1}^{3} h_k\right)\cdot\frac{1}{h_j^2}\frac{\partial\Phi_i}{\partial q_j}\right] \quad ;$$

$$\nabla\Phi_i = \mathbf{k}_\xi\frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}+\mathbf{k}_\eta\frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}+\mathbf{k}_\zeta\frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta}=\sum_{j=1}^{3}\frac{\mathbf{k}_j}{h_j}\frac{\partial\Phi_i}{\partial q_j} ;$$

$$\nabla\left(V_i^2\right)=\nabla\left[(\nabla\Phi_i)^2\right]=\sum_{j=1}^{3}\frac{\mathbf{k}_j}{h_j}\frac{\partial}{\partial q_j}\left[\sum_{k=1}^{3}\left(\frac{1}{h_k}\frac{\partial\Phi_i}{\partial q_k}\right)^2\right] \quad ;$$

$$a_i^2=\frac{\gamma-1}{2}\left[W^2-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2-\frac{1}{h_\eta^2}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2-\frac{1}{h_\zeta^2}\left(\frac{\partial\Phi_i}{\partial\zeta}\right)^2\right];$$

but $V_{\zeta i}=0 \;\Rightarrow\; \dfrac{\partial\Phi_i}{\partial\zeta}=0$, therefore

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right]\Bigg|_{\zeta=\zeta_{0i}} \text{, or:}$$

$$\Delta\Phi_i = \frac{1}{h_\zeta}\left[\frac{1}{h_\eta}\frac{\partial}{\partial\lambda}\left(h_\eta h_\zeta\frac{\partial\Phi_i}{\partial\lambda}\right)+\frac{1}{h_\xi}\frac{\partial}{\partial\mu}\left(h_\zeta h_\xi\frac{\partial\Phi_i}{\partial\mu}\right)\right]\Bigg|_{\nu=\nu_{0i}} ;$$

$$\nabla\Phi_i = \mathbf{k}_\xi\frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}+\mathbf{k}_\eta\frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\Bigg|_{\zeta=\zeta_{0i}}=\mathbf{k}_\xi\frac{\partial\Phi_i}{\partial\lambda}+\mathbf{k}_\eta\frac{\partial\Phi_i}{\partial\mu}\Bigg|_{\nu=\nu_{0i}} ,$$

$$\nabla\left(V_i^2\right)=\nabla\left[(\nabla\Phi_i)^2\right]=\sum_{j=1}^{2}\frac{\mathbf{k}_j}{h_j}\frac{\partial}{\partial q_j}\left[\sum_{k=1}^{2}\left(\frac{1}{h_k}\frac{\partial\Phi_i}{\partial q_k}\right)^2\right]\Bigg|_{q_3=q_{30i}(\zeta=\zeta_{0i})} ;$$

$$a_i^2=\frac{\gamma-1}{2}\left[W^2-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2-\frac{1}{h_\eta^2}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\right]\Bigg|_{\zeta=\zeta_{0i}} \text{, or:}$$

$$a_i^2=\frac{\gamma-1}{2}\left[W^2-\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2-\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2\right]\Bigg|_{\nu=\nu_{0i}} .$$

Analogously, in Steichen's equation – a nonlinear PDE of the 2nd order in three variables – ξ, η and ζ (written for a rotational flow – $\mathbf{\Omega} \neq 0$, but *on the* "i" *isentropic surface* $\zeta = \zeta_{0i}$) all the terms containing the partial derivative with respect to ζ of the quasi-potential function Φ_i disappear ($\partial\Phi_i/\partial\zeta = 0$) and its derivatives with respect to ξ, η and ζ disappear also, thus a nonlinear PDE of the second order in only two variables – ξ and η – being obtained (a simpler form for the 2-D velocity quasi-potential PDE):

$$\left[\frac{1}{h_\xi^2}\frac{\partial^2\Phi_i}{\partial\xi^2}+\frac{1}{h_\eta^2}\frac{\partial^2\Phi_i}{\partial\eta^2}+\frac{1}{h_\xi^2}\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_i}{\partial\xi}+\frac{1}{h_\eta^2}\frac{\partial\ln(h_\zeta h_\xi)}{\partial\eta}\frac{\partial\Phi_i}{\partial\eta}\right]_{\zeta=\zeta_{0i}}\times\left[W^2-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2-\frac{1}{h_\eta^2}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\right]_{\zeta=\zeta_{0i}}$$

$$=\frac{2}{\gamma-1}\left\{\frac{1}{h_\xi^4}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2\frac{\partial^2\Phi_i}{\partial\xi^2}+\frac{1}{h_\eta^4}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\frac{\partial^2\Phi_i}{\partial\eta^2}\right.$$

$$+\frac{2}{h_\xi^2h_\eta^2}\frac{\partial\Phi_i}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta}\frac{\partial^2\Phi_i}{\partial\xi\partial\eta}-\left[\frac{1}{h_\xi^4}\frac{\partial\ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^3+\frac{1}{h_\eta^4}\frac{\partial\ln h_\eta}{\partial\eta}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^3\right.$$

$$\left.\left.+\frac{1}{h_\xi^2h_\eta^2}\frac{\partial\ln h_\xi}{\partial\eta}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2\frac{\partial\Phi_i}{\partial\eta}+\frac{1}{h_\xi^2h_\eta^2}\frac{\partial\ln h_\eta}{\partial\xi}\frac{\partial\Phi_i}{\partial\xi}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\right]\right\}\Bigg|_{\zeta=\zeta_{0i}} .$$

4 The case of the unsteady flow (see [5])

For the unsteady flow of a compressible fluid, the velocity potential equation has the vector form below (t – the time):

$$\nabla\mathbf{V}-\frac{\mathbf{V}}{2a^2}\nabla\left(V^2\right)=\frac{1}{a^2}\left[\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial}{\partial t}\left(V^2\right)\right] \text{ with } \mathbf{V}=\nabla\Phi;$$

$$\Delta\Phi-\frac{\nabla\Phi}{2a^2}\nabla\left[(\nabla\Phi)^2\right]=\frac{1}{a^2}\left\{\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial}{\partial t}\left[(\nabla\Phi)^2\right]\right\} \quad \text{or}$$

$$\Delta\Phi\cdot a^2=\frac{1}{2}\nabla\Phi\cdot\nabla\left[(\nabla\Phi)^2\right]+\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial}{\partial t}\left[(\nabla\Phi)^2\right] \quad ,$$

($\mathbf{V}$, V, a and Φ are now local instantaneous values) also differing from the corresponding one for a steady flow (see section 1) by the last two terms, or in expanded form:

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi}{\partial\zeta}\right)\right]\times$$

$$\times\left[2C_i(t)-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2-\frac{1}{h_\eta^2}\left(\frac{\partial\Phi}{\partial\eta}\right)^2-\frac{1}{h_\zeta^2}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2-2\frac{\partial\Phi}{\partial t}\right]$$

$$=\frac{2}{\gamma-1}\left\{\frac{1}{h_\xi^4}\left(\frac{\partial\Phi}{\partial\xi}\right)^2\frac{\partial^2\Phi}{\partial\xi^2}+\frac{1}{h_\eta^4}\left(\frac{\partial\Phi}{\partial\eta}\right)^2\frac{\partial^2\Phi}{\partial\eta^2}+\frac{1}{h_\zeta^4}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\frac{\partial^2\Phi}{\partial\zeta^2}\right.$$

$$+\frac{2}{h_\xi^2h_\eta^2}\frac{\partial\Phi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\xi\partial\eta}+\frac{2}{h_\eta^2h_\zeta^2}\frac{\partial\Phi}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}\frac{\partial^2\Phi}{\partial\eta\partial\zeta}+\frac{2}{h_\zeta^2h_\xi^2}\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\zeta\partial\xi}$$

$$-\left[\frac{1}{h_\xi^4}\frac{\partial\ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi}{\partial\xi}\right)^3+\frac{1}{h_\eta^4}\frac{\partial\ln h_\eta}{\partial\eta}\left(\frac{\partial\Phi}{\partial\eta}\right)^3+\frac{1}{h_\zeta^4}\frac{\partial\ln h_\zeta}{\partial\zeta}\left(\frac{\partial\Phi}{\partial\zeta}\right)^3\right.$$

$$+\frac{1}{h_\xi^2h_\eta^2}\left(\frac{\partial\ln h_\xi}{\partial\eta}\frac{\partial\Phi}{\partial\xi}+\frac{\partial\ln h_\eta}{\partial\xi}\frac{\partial\Phi}{\partial\eta}\right)\frac{\partial\Phi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}$$

$$+\frac{1}{h_\eta^2h_\zeta^2}\left(\frac{\partial\ln h_\eta}{\partial\zeta}\frac{\partial\Phi}{\partial\eta}+\frac{\partial\ln h_\zeta}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}\right)\frac{\partial\Phi}{\partial\eta}\frac{\partial\Phi}{\partial\zeta}$$

$$\left.+\frac{1}{h_\zeta^2h_\xi^2}\left(\frac{\partial\ln h_\zeta}{\partial\xi}\frac{\partial\Phi}{\partial\zeta}+\frac{\partial\ln h_\xi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right)\frac{\partial\Phi}{\partial\zeta}\frac{\partial\Phi}{\partial\xi}\right]$$

$$\left.+\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial}{\partial t}\left[\frac{1}{h_\xi^2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2+\frac{1}{h_\eta^2}\left(\frac{\partial\Phi}{\partial\eta}\right)^2+\frac{1}{h_\zeta^2}\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\right]\right\},$$

which reduces on a certain $(\mathbf{V}, \mathbf{\Omega})_i$ surface $\zeta = \zeta_{0i}$ to a simpler form for the 2-D velocity quasi-potential PDE:

$$\left[\frac{1}{h_\xi^2}\frac{\partial^2\Phi_i}{\partial\xi^2}+\frac{1}{h_\eta^2}\frac{\partial^2\Phi_i}{\partial\eta^2}+\frac{1}{h_\xi^2}\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_i}{\partial\xi}+\frac{1}{h_\eta^2}\frac{\partial\ln(h_\zeta h_\xi)}{\partial\eta}\frac{\partial\Phi_i}{\partial\eta}\right]\times$$

$$\times\left[2C_i(t)-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2-\frac{1}{h_\eta^2}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2-2\frac{\partial\Phi_i}{\partial t}\right]$$

$$=\frac{2}{\gamma-1}\left\{\frac{1}{h_\xi^4}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2\frac{\partial^2\Phi_i}{\partial\xi^2}+\frac{1}{h_\eta^4}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\frac{\partial^2\Phi_i}{\partial\eta^2}+\frac{2}{h_\xi^2h_\eta^2}\frac{\partial\Phi_i}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta}\frac{\partial^2\Phi_i}{\partial\xi\partial\eta}\right.$$
$$-\left[\frac{1}{h_\xi^4}\frac{\partial\ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^3+\frac{1}{h_\eta^4}\frac{\partial\ln h_\eta}{\partial\eta}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^3\right.$$
$$\left.+\frac{1}{h_\xi^2h_\eta^2}\left(\frac{\partial\ln h_\xi}{\partial\eta}\frac{\partial\Phi_i}{\partial\xi}+\frac{\partial\ln h_\eta}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta}\right)\frac{\partial\Phi_i}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta}\right]$$
$$\left.+\frac{\partial^2\Phi_i}{\partial t^2}+\frac{\partial}{\partial t}\left[\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2+\frac{1}{h_\eta^2}\left(\frac{\partial\Phi_i}{\partial\eta}\right)^2\right]\right\} .$$

5 Simpler forms of the 2-D velocity "quasi-potential" PDE over the $(\mathbf{V}, \mathbf{\Omega})_i$ surfaces

On a certain $\nu = \nu_{0i}$ isentropic surface $(\mathbf{V}, \mathbf{\Omega})_i$ for a steady flow, using the elementary arc lengths $d\lambda$ and $d\mu$, one gets:

$$\left[\frac{\partial^2\Phi_i}{\partial\lambda^2}+\frac{\partial^2\Phi_i}{\partial\mu^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{\partial\Phi_i}{\partial\lambda}+\frac{\partial\ln(h_\zeta h_\xi)}{\partial\mu}\frac{\partial\Phi_i}{\partial\mu}\right]\times$$
$$\times\left[W^2-\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2-\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2\right]$$
$$=\frac{2}{\gamma-1}\left\{\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2\frac{\partial^2\Phi_i}{\partial\lambda^2}+\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2\frac{\partial^2\Phi_i}{\partial\mu^2}+2\frac{\partial\Phi_i}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\frac{\partial^2\Phi_i}{\partial\lambda\partial\mu}\right.$$
$$-\left[\frac{\partial\ln h_\xi}{\partial\lambda}\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^3+\frac{\partial\ln h_\eta}{\partial\mu}\left(\frac{\partial\Phi_i}{\partial\mu}\right)^3\right.$$
$$\left.\left.+\left(\frac{\partial\ln h_\xi}{\partial\mu}\frac{\partial\Phi_i}{\partial\lambda}+\frac{\partial\ln h_\eta}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\right)\frac{\partial\Phi_i}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\right]\right\} ,$$

and the corresponding PDE of the 2-D velocity quasi-potential for the unsteady flow case becomes simpler too:

$$\left[\frac{\partial^2\Phi_i}{\partial\lambda^2}+\frac{\partial^2\Phi_i}{\partial\mu^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{\partial\Phi_i}{\partial\lambda}+\frac{\partial\ln(h_\zeta h_\xi)}{\partial\mu}\frac{\partial\Phi_i}{\partial\mu}\right]\times$$
$$\times\left[2C_i(t)-\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2-\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2-2\frac{\partial\Phi_i}{\partial t}\right]$$
$$=\frac{2}{\gamma-1}\left\{\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2\frac{\partial^2\Phi_i}{\partial\lambda^2}+\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2\frac{\partial^2\Phi_i}{\partial\mu^2}+2\frac{\partial\Phi_i}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\frac{\partial^2\Phi_i}{\partial\lambda\partial\mu}\right.$$
$$-\left[\frac{\partial\ln h_\xi}{\partial\lambda}\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^3+\frac{\partial\ln h_\eta}{\partial\mu}\left(\frac{\partial\Phi_i}{\partial\mu}\right)^3\right.$$
$$\left.+\left(\frac{\partial\ln h_\xi}{\partial\mu}\frac{\partial\Phi_i}{\partial\lambda}+\frac{\partial\ln h_\eta}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\right)\frac{\partial\Phi_i}{\partial\lambda}\frac{\partial\Phi_i}{\partial\mu}\right]$$
$$\left.+\frac{\partial^2\Phi_i}{\partial t^2}+\frac{\partial}{\partial t}\left[\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2+\left(\frac{\partial\Phi_i}{\partial\mu}\right)^2\right]\right\} . \qquad (3)$$

Choosing the ξ intrinsic coordinate along the streamlines direction (assumed as being known), the new PDE becomes an ODE ($\partial\Phi_i/\partial\xi = d\Phi_i/d\xi$) of the 2nd order:

$$\frac{1}{h_\xi^2}\left[\frac{d^2\Phi_i}{d\xi^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{d\Phi_i}{d\xi}\right]\times\left[W^2-\frac{1}{h_\xi^2}\left(\frac{d\Phi_i}{d\xi}\right)^2\right]$$
$$=\frac{2}{\gamma-1}\frac{1}{h_\xi^4}\left[\left(\frac{d\Phi_i}{d\xi}\right)^2\frac{d^2\Phi_i}{d\xi^2}-\frac{\partial\ln h_\xi}{\partial\xi}\left(\frac{d\Phi_i}{d\xi}\right)^3\right] ,$$

representing the simplest form for the equation of the now 1-D velocity quasi-potential Φ_i for a steady flow, and more, using the elementary arc length $d\lambda = h_\xi d\xi$:

$$\left[\frac{d^2\Phi_i}{d\lambda^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{d\Phi_i}{d\lambda}\right]\times\left[W^2-\left(\frac{d\Phi_i}{d\lambda}\right)^2\right]$$
$$=\frac{2}{\gamma-1}\left[\left(\frac{d\Phi_i}{d\lambda}\right)^2\frac{d^2\Phi_i}{d\lambda^2}-\frac{\partial\ln h_\xi}{\partial\lambda}\left(\frac{d\Phi_i}{d\lambda}\right)^3\right] ,$$

thus the streamlines being *just the characteristic lines*. Analogously, for an unsteady flow, the new PDE below becomes the simplest one also (Φ_i depends on ξ and t):

$$\frac{\gamma-1}{2h_\xi^2}\left[\frac{\partial^2\Phi_i}{\partial\xi^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_i}{\partial\xi}\right]\times\left[2C_i(t)-\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2-2\frac{\partial\Phi_i}{\partial t}\right]$$
$$=\frac{1}{h_\xi^4}\left[\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2\frac{\partial^2\Phi_i}{\partial\xi^2}-\frac{\partial\ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^3\right]+\frac{\partial^2\Phi_i}{\partial t^2}+\frac{\partial}{\partial t}\left[\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_i}{\partial\xi}\right)^2\right]$$

and more, with the elementary arc length $d\lambda = h_\xi d\xi$:

$$\frac{\gamma-1}{2}\left[\frac{\partial^2\Phi_i}{\partial\lambda^2}+\frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{\partial\Phi_i}{\partial\lambda}\right]\times\left[2C_i(t)-\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2-2\frac{\partial\Phi_i}{\partial t}\right]$$
$$=\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2\frac{\partial^2\Phi_i}{\partial\lambda^2}-\frac{\partial\ln h_\xi}{\partial\lambda}\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^3+\frac{\partial^2\Phi_i}{\partial t^2}+\frac{\partial}{\partial t}\left(\frac{\partial\Phi_i}{\partial\lambda}\right)^2 .$$

6 Special forms of the 2-D velocity "quasi-potential" PDE along the quasi-Laplace lines of a compressible fluid rotational flow

On a certain virtual "i" surface, for an inviscid fluid flow we have: $\nabla(V^2/2)|_i + (\mathbf{\Omega}\times\mathbf{V})|_i = -(\nabla p/\rho)|_i$. Performing a scalar multiplication of this relation by a virtual elementary displacement $d\mathbf{R}_i \in (\mathbf{V}, \mathbf{\Omega})|_i$ (therefore an isentropic virtual surface), we obtain: $d(V^2/2)|_i = -(dp/\rho)|_i$, with: $p = K_i\rho^\gamma$, so a first integrable form. On the other hand, on any polytropic virtual surface we have: $(p/\rho^n)_j = (p/\rho^n)_1 = \text{const.}_j$, or $p = \text{const.}_j\cdot\rho^n$, and so $\nabla p = \text{const.}_j\cdot\nabla(\rho^n) = \text{const.}_j\cdot\rho^{n-1}\nabla\rho$. Performing its scalar multiplication by a virtual elementary displacement $d\mathbf{R}_j$ contained in the plane tangent to the polytropic "j" surface, we obtain: $dp|_j = \text{const.}_j\cdot(\rho^{n-1}d\rho)|_j$. So, along the intersection "ij" lines (with: $d\mathbf{R}_{ij} = k\cdot\nabla S|_i\times\nabla(p/\rho^n)|_j$) of the two surface families we have: $d(V^2/2)|_{ij} = -(dp/\rho)|_{ij} = -\text{const.}_j(\rho^{n-2}d\rho)|_{ij}$. Analogously we have: $p = \text{const.}_j\cdot T^{n/(n-1)}$ and so: $\nabla p = \text{const.}_j\cdot\nabla(T^{n/(n-1)}) = \text{const.}_j\, T^{1/(n-1)}\nabla T$, and: $dp|_j = \text{const.}_j\cdot(T^{1/(n-1)}dT)|_j$. Taking into account that through any intersection "ij" line of the two surface families above there is a "star (pencil) of sheets" passing (for various particular values of the polytropic exponent "n", e.g.: isentropic, isothermal & isotachic, isobaric, isochoric and general polytropic one), we can write along these lines: $dS|_{ij} = d(V^2/2)|_{ij} = -(dp/\rho)|_{ij} = -\text{const.}_j\,(\rho^{n-2}d\rho)|_{ij} = -\text{const.}_j\,[(T^{1/(n-1)}dT)/\rho]|_{ij} = 0$, or simpler: $dS|_{ij} = dT|_{ij} = dV|_{ij} = dp|_{ij} = d\rho|_{ij} = d(p/\rho^n)|_{ij} = 0$, so having: $d\mathbf{R}_{ij} = k\cdot\nabla S\times\nabla T = k_1\cdot\nabla S\times\nabla V = k_2\cdot\nabla S\times\nabla p = k_3\cdot\nabla S\times\nabla\rho = k_4\cdot\nabla S\times\nabla(p/\rho^n)$. Through any point of this rotational flow, such of "star of integral sheets" is passing. Applying·a·scalar·multiplication·of·Steichen's·PDE of the compressible 2-D velocity "quasi-potential" Φ_{ij}, $\nabla\mathbf{V}_{ij} = \mathbf{V}_{ij}\cdot\nabla(V_{ij}^2)/2a_{ij}^2$ (with $\mathbf{V}_{ij} = \nabla\Phi_{ij}$) by the $d\mathbf{R}_{ij}$ above, one gets $\nabla\mathbf{V}_{ij}\cdot d\mathbf{R}_{ij} = \mathbf{V}_{ij}\cdot d(V_{ij}^2)/2a_{ij}^2$ with $d(V_{ij}^2) = 0$, and so $\nabla\mathbf{V}_{ij} = 0$ (because $d\mathbf{R}_{ij} \neq 0$). Introducing the scalar function $\Phi_{ij}(\xi,\eta)$ one obtains a PDE *identical to Laplace's one*: $\Delta\Phi_{ij} = 0$, so $\Phi_{ij}(\xi,\eta)$ being now a harmonic function. This simpler PDE is valid for a certain rotational flow ($\mathbf{V}_{ij} = \nabla\Phi_{ij}$; $\boldsymbol{\Omega}_{ij} = \nabla\times\mathbf{V}_{ij} \neq 0$; $(p/\rho^n)_{ij} = \text{const.}$) – a "quasi-incompressible" fluid behavior along the intersection lines of any isentropic virtual sheet with any polytropic one. Over flow's isothermal & isotachic virtual surfaces we have: $|\mathbf{V}| = V = V_j = \text{const.}_j$ or $\nabla V\cdot d\mathbf{R} = dV = 0$ (the fluid has a "quasi-uniform" behavior). The vector equation $\mathbf{V}\cdot\nabla(V^2) = 0$ (a zero-scalar product in the right-hand side of Steichen's equation), expressed, say, in the Cartesian system in the expanded form below:

$$V_x^2\frac{\partial V_x}{\partial x} + V_y^2\frac{\partial V_y}{\partial y} + V_z^2\frac{\partial V_z}{\partial z} + V_xV_y\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + V_yV_z\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right) + V_zV_x\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right) = 0 \quad ,$$

(not implying the existence of a potential) has the solutions:
1) $\mathbf{V} = 0$ ($V_x = V_y = V_z = 0$), representing a trivial solution:
a) the equilibrium (fluid statics) case, and
b) the flow stagnation lines ($p = p_0$; $T = T_0$), for the plane flows and for some special axisymmetric flows;
2) $\nabla(V^2) = 0$, where $\nabla(V^2) = 2V\cdot\nabla V = 0$ (the acceleration being $\mathbf{a} = \boldsymbol{\Omega}\times\mathbf{V} = 2\boldsymbol{\omega}\times\mathbf{V} = T\nabla S$, like a Coriolis one, the vectors $\mathbf{V}$, $\boldsymbol{\Omega}$ and $\mathbf{a}$ forming an orthogonal system):
a) the flow of an incompressible fluid ($\rho = \text{const.} \Rightarrow \nabla\rho = 0$, and so: $\nabla(V^2) = -2a^2\cdot\nabla\rho/\rho = 0$) – a liquid flow model;
b) a parallel and uniform flow of a compressible fluid ($\mathbf{V} = \text{const.}$, with $\boldsymbol{\Omega} = 0$) – a trivial solution also, and
c) the flow's isotachic surfaces $\nabla V = 0$, and especially $d(V_{ij}^2) = 0$ (along the intersection "ij" lines above), which seems to be the most important solution (Selescu), due to the fact that over any "i" isentropic surface we have $\mathbf{V}_i = \nabla\Phi_i$, and so $\mathbf{V}_{ij} = \nabla\Phi_{ij}$ also (a quasi-potential flow);
3) $\mathbf{V}\perp\nabla V$, or $\mathbf{V}\perp\nabla(V^2)$ – a very particular solution; $\nabla(V^2)\perp$ (the isotachic & isothermal surfaces), hence the velocity $\mathbf{V}$ lies just on (is tangent to) the isotachic & isothermal surfaces (having no component along the normal), representing flows due to vortex distributions:
a) the plane flow of a compressible fluid due to a straight infinite vortex filament, having two regions: subsonic and supersonic, separated by a circular critical line, along which the Mach number reaches the value 1, all streamlines being concentric circles, including a solid cylindrical nucleus (a no-motion region); on its surface the maximum speed W is reached (see fig. 1); b) the superposition of the flow above with a uniform flow, parallel to vortex filament, all streamlines being co-axial circular cylindrical helices with the same pitch; c) the same as above, but with a certain direction of the uniform parallel flow; d) some flows of an inviscid compressible fluid due to an infinite sequence of very close vortex filaments, leading to streamlines patterns identical (and velocity distributions somehow related) to those for the well-known models of plane and axisymmetric flows *of a viscous fluid*, like Couette and Hagen–Poiseuille ones; for an extension to the viscous fluid flow see sections 10, 11).

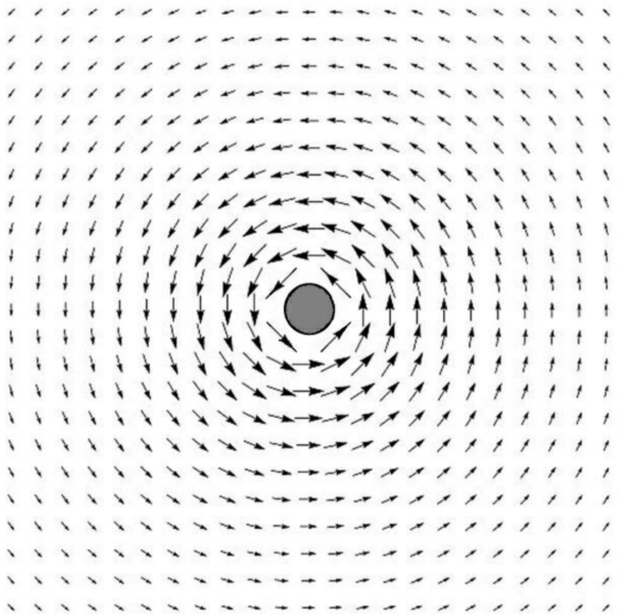


Fig. 1. The compressible fluid flow pattern (the velocity distribution) due to a straight infinite vortex filament – the free (irrotational) vortex plane flow

The vector field given in figure 1 corresponds to the velocity field due to a straight infinite vortex filament directed along the z - axis (normal to the x, y - plane):

$$\mathbf{V}(x,y,z) = \frac{\Gamma}{2\pi}\left(-\mathbf{k}_x\frac{y}{x^2+y^2} + \mathbf{k}_y\frac{x}{x^2+y^2} + \mathbf{k}_z\cdot 0\right).$$

Γ is the vortex constant intensity (the velocity circulation). The lengths of the velocity vectors induced by the vortex filament are proportional to their intensities (moduli) $|\mathbf{V}(x,y,z)| = V = \Gamma/[2\pi\cdot(x^2+y^2)^{1/2}] = \Gamma/(2\pi R)$ (a hyperbolic variation law). Their directions are tangent to the concentric circles having the center in the origin, and their senses are represented by arrowheads, corresponding to a left-hand screw (counter-clockwise). Along a certain circle (streamline) $R = (x^2+y^2)^{1/2} = \text{const.}$, the velocity vector has a constant modulus. So, these circles are the flow isotachs (isotachic lines). The critical speed c (for which $V = a$) is reached along a circle of radius $R_c = \Gamma/2\pi c = \Gamma/2\pi\cdot[(\gamma+1)\rho_0/2\gamma p_0]^{1/2}$, and the maximum speed W (for which V = W) is reached along another circle (the wall of the solid cylindrical nucleus – the no-motion region), of minimum radius $R_W = \Gamma/2\pi W = \Gamma/2\pi\cdot[(\gamma-1)\rho_0/2\gamma p_0]^{1/2} = [(\gamma-1)/(\gamma+1)]^{1/2}\cdot R_c$, so that for the region $R \in (R_W, R_c)$ the plane flow is supersonic (V decreases from W to c) and for the semi-infinite region $R > R_c$ the flow is subsonic (V decreases from c to 0). This field includes a vortex at its center (singularity), so it is rotational. However, any simply-connected subset that excludes the vortex line will have zero curl, $\mathbf{\Omega} = 0$ (no vorticity), the fluid particles performing circular translations only. We give below a detailed explanation of solutions 2 and 3. Intersecting the isentropic ($\nabla S\cdot d\mathbf{R}_i = dS_i = 0$) virtual surfaces $S = S_{0i} = \text{const.}_i$ ($\zeta = \zeta_{0i}$) with the special isotachic ones ($|\mathbf{V}| = V = V_j = \text{const.}_j$) one obtains a family (net) of space curves along which Steichen's PDE of the compressible 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta)$ becomes again simpler, identical to Laplace's one: $\Delta\Phi_{ij} = 0$, $\Phi_{ij}(\xi, \eta)$ being now a harmonic function.

So, the solution $\mathbf{V}_{ij}$ of the general Steichen's PDE $\nabla\mathbf{V}_{ij} = \mathbf{V}_{ij}\cdot\nabla(V_{ij}^2)/2a_{ij}^2$, written for a special rotational flow ($\mathbf{V}_{ij} = \nabla\Phi_{ij}$; $\mathbf{\Omega}_{ij} = \nabla\times\mathbf{V}_{ij} \neq 0$; $|\mathbf{V}_{ij}| = V_{ij} = \text{const.}_{ij}$, with $\mathbf{V}_{ij}\cdot\nabla(V_{ij}^2) = 0$), is now the solution of the system:

$$\begin{cases}\nabla\mathbf{V}_i = \mathbf{V}_i\cdot\nabla(V_i^2)/2a_i^2 \quad , \quad \text{with} \quad \mathbf{V}_i = \nabla\Phi_i(\xi,\eta) \ ; \\ \mathbf{V}_j\cdot\nabla(V_j^2) = 0 \quad ,\end{cases}$$

or, along the "ij" lines ($\mathbf{V}_i \equiv \mathbf{V}_j \equiv \mathbf{V}_{ij}$, but not directed along the "ij" lines; $a_i = a_j = a_{ij}$; $\Phi_i = \Phi_{ij}$):

$$\begin{cases}\nabla\mathbf{V}_{ij} = \mathbf{V}_{ij}\cdot\nabla(V_{ij}^2)/2a_{ij}^2 \quad , \text{ with } \quad \mathbf{V}_{ij} = \nabla\Phi_{ij}(\xi,\eta) \ ; \\ \mathbf{V}_{ij}\cdot\nabla(V_{ij}^2)\cdot d\mathbf{R}_{ij} = 0 \quad , \quad \text{or, mainly :}\end{cases}$$

$$\begin{cases}\Delta\Phi_{ij} = 0 \quad ; \\ d(V_{ij}^2) = 0 \quad ,\end{cases} \tag{4}$$

this leading to $\nabla\mathbf{V}_{ij} = 0$ and so obtaining the simpler form: $\nabla\mathbf{V}_{ij} = \Delta\Phi_{ij} = 0$; with $\mathbf{V}_{ij} = \nabla\Phi_{ij} = (\nabla\times\mathbf{\Psi}_{ij})/\rho$; $\mathbf{\Psi}_{ij} \neq \nabla G_{ij}$ – the 3-D stream function vector (Selescu, see section 7 in [7], for the mass flux density vector $\rho\mathbf{V}_{ij}$):

$$\Delta\Phi_{ij} = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_{ij}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_{ij}}{\partial\eta}\right)\right]\Bigg|_{\begin{cases}\zeta=\zeta_{0i}\\ V=V_j\end{cases}} = 0,$$

$$\text{or} \quad \left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_{ij}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_{ij}}{\partial\eta}\right)\right]\Bigg|_{\begin{cases}\zeta=\zeta_{0i}\\ V=V_j\end{cases}} = 0, \quad \text{or more}$$

$$\left[\frac{1}{h_\xi^2}\frac{\partial^2\Phi_{ij}}{\partial\xi^2} + \frac{1}{h_\eta^2}\frac{\partial^2\Phi_{ij}}{\partial\eta^2} + \frac{1}{h_\xi^2}\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_{ij}}{\partial\xi} + \frac{1}{h_\eta^2}\frac{\partial\ln(h_\zeta h_\xi)}{\partial\eta}\frac{\partial\Phi_{ij}}{\partial\eta}\right]\Bigg|_{\begin{cases}\zeta=\zeta_{0i}\\ V=V_j\end{cases}} = 0$$

$$\left[\frac{\partial^2\Phi_{ij}}{\partial\lambda^2} + \frac{\partial^2\Phi_{ij}}{\partial\mu^2} + \frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{\partial\Phi_{ij}}{\partial\lambda} + \frac{\partial\ln(h_\zeta h_\xi)}{\partial\mu}\frac{\partial\Phi_{ij}}{\partial\mu}\right]\Bigg|_{\begin{cases}v=v_{0i}\\ V=V_j\end{cases}} = 0 \quad ,$$

for a steady flow, *therefore an elliptic PDE, irrespective of the "pseudo-flow" character* (subsonic or supersonic), this representing the annulment of the first factor (inside the first square brackets) in the left-hand side of PDE (3). For an unsteady flow Steichen's equation takes the form

$$\nabla\mathbf{V}_{ij} - \frac{\mathbf{V}_{ij}}{2a_{ij}^2}\nabla\left(V_{ij}^2\right) = \frac{1}{a_{ij}^2}\left[\frac{\partial^2\Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t}\left(V_{ij}^2\right)\right] \quad , \quad \text{with :}$$

$$\mathbf{V}_{ij}\cdot\nabla\left(V_{ij}^2\right) = 0 \quad , \quad \text{this resulting in :}$$

$$\Delta\Phi_{ij} = \frac{1}{a_{ij}^2}\left\{\frac{\partial^2\Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t}\left[\left(\nabla\Phi_{ij}\right)^2\right]\right\} \quad ,$$

$$\text{or :} \quad \Delta\Phi_{ij}\cdot a_{ij}^2 = \frac{\partial^2\Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t}\left[\left(\nabla\Phi_{ij}\right)^2\right] \quad , \quad \text{with :}$$

$$a_{ij}^2 = \frac{\gamma-1}{2}\left[2C_i(t) - \frac{1}{h_\xi^2}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2 - \frac{1}{h_\eta^2}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2 - 2\frac{\partial\Phi_{ij}}{\partial t}\right],$$

so obtaining along the space curves ($S = S_{0i}$; $V = V_{0j}$, i.e. coupling the cases 4 and 5 in subsection 1.2 in [7]), for the function $\Phi_{ij}(\xi, \eta, t)$ the following PDE:

$$\left[\frac{1}{h_\xi^2}\frac{\partial^2\Phi_{ij}}{\partial\xi^2} + \frac{1}{h_\eta^2}\frac{\partial^2\Phi_{ij}}{\partial\eta^2} + \frac{1}{h_\xi^2}\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_{ij}}{\partial\xi} + \frac{1}{h_\eta^2}\frac{\partial\ln(h_\zeta h_\xi)}{\partial\eta}\frac{\partial\Phi_{ij}}{\partial\eta}\right]\times$$
$$\times\left[2C_i(t) - \frac{1}{h_\xi^2}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2 - \frac{1}{h_\eta^2}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2 - 2\frac{\partial\Phi_{ij}}{\partial t}\right]$$
$$= \frac{2}{\gamma-1}\left\{\frac{\partial^2\Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{1}{h_\xi^2}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2 + \frac{1}{h_\eta^2}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2\right]\right\} \quad ,$$

$$\text{or more}\left[\frac{\partial^2\Phi_{ij}}{\partial\lambda^2} + \frac{\partial^2\Phi_{ij}}{\partial\mu^2} + \frac{\partial\ln(h_\eta h_\zeta)}{\partial\lambda}\frac{\partial\Phi_{ij}}{\partial\lambda} + \frac{\partial\ln(h_\zeta h_\xi)}{\partial\mu}\frac{\partial\Phi_{ij}}{\partial\mu}\right]\times$$
$$\times\left[2C_i(t) - \left(\frac{\partial\Phi_{ij}}{\partial\lambda}\right)^2 - \left(\frac{\partial\Phi_{ij}}{\partial\mu}\right)^2 - 2\frac{\partial\Phi_{ij}}{\partial t}\right]$$
$$= \frac{2}{\gamma-1}\left\{\frac{\partial^2\Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t}\left[\left(\frac{\partial\Phi_{ij}}{\partial\lambda}\right)^2 + \left(\frac{\partial\Phi_{ij}}{\partial\mu}\right)^2\right]\right\} \quad .$$

More, over flow's isothermal & isotachic virtual surfaces ($\nabla V = 0$) one can write $\mathbf{V} = \nabla\times\mathbf{\Psi}_{inc}$, this time $\mathbf{\Psi}_{inc}$ ($\neq \nabla G_{inc}$) being a true 3-D stream function vector due to the fact that over these surfaces the fluid has *an incompressible behavior*.

Until now, we used only the first equation of system (4): $\Delta\Phi_{ij} = 0$. The second equation ($dV_{ij}^2 = 0$, or $|\mathbf{V}_{ij}| = \text{const.}_{ij}$) leads to the following vector condition ($\mathbf{V}$ lies on the isentropic "i" surface $\zeta = \zeta_{0i}$, so being: $\mathbf{V} = \mathbf{k}_\xi V_\xi + \mathbf{k}_\eta V_\eta$):

$$\frac{\mathbf{k}_\xi}{h_\xi}\frac{\partial(V_\xi^2 + V_\eta^2)_{ij}}{\partial\xi} + \frac{\mathbf{k}_\eta}{h_\eta}\frac{\partial(V_\xi^2 + V_\eta^2)_{ij}}{\partial\eta} = 0 \quad , \text{ and so to:}$$

$$\frac{\partial(V_\xi^2 + V_\eta^2)_{ij}}{\partial\xi} = 0 \text{ and } \frac{\partial(V_\xi^2 + V_\eta^2)_{ij}}{\partial\eta} = 0 \text{ simultaneously,}$$

having for consequences: $(V_\xi^2 + V_\eta^2)_{ij} = f_{1ij}(\eta,\zeta)$, and $(V_\xi^2 + V_\eta^2)_{ij} = f_{2ij}(\xi,\zeta)$, or globally: $(V_\xi^2 + V_\eta^2)_{ij} = f_{ij}(\zeta)$ function *of* ζ *only*), which can be expressed as:

$$\frac{1}{h_\xi^2}\left(\frac{\partial \Phi_{ij}}{d\xi}\right)^2 + \frac{1}{h_\eta^2}\left(\frac{\partial \Phi_{ij}}{d\eta}\right)^2 = f_{ij}(\zeta), \text{or in compact form:}$$

$$\left(\frac{\partial \Phi_{ij}}{d\lambda}\right)^2 + \left(\frac{\partial \Phi_{ij}}{d\mu}\right)^2 = f_{ij}(\nu) \quad , \quad \text{or more:}$$

$$\left[\left(\frac{\partial \Phi_{ij}}{d\lambda}\right)^2 + \left(\frac{\partial \Phi_{ij}}{d\mu}\right)^2\right]_{\nu=\nu_{0i};V=V_j} = f_{ij}(\nu) = \text{const.}(\nu) \quad ,$$

which must be also considered for solving the system (4), besides the boundary conditions of respective rotational flow, along an "ij" space line, lying on an isentropic "i" surface and an isothermal & isotachic "j" one.

7 Special forms of the 2-D velocity "quasi-potential" PDE along other space lines of a compressible fluid rotational flow

Intuitively, we can put the existence problem for a family of space curves passing through any point also, along which the velocity quasi-potential function Φ_{ij} respects another rule, e.g. the wave (vibrating string) PDE of the 2nd order (in one space dimension), instead of Laplace's one, that means being of a hyperbolic type, just in its canonical form: $\partial^2\Phi_{ij}(\lambda, \mu, \nu_{0i})/\partial\lambda\partial\mu + \cdots$ (lower order terms) $= 0$, *irrespective of the "pseudo-flow" character* (subsonic or supersonic), in the same $O\lambda\mu\nu$ smart intrinsic triorthogonal curvilinear coordinate system previously used. But we will treat this problem, for simplicity reasons, in the classical Cartesian system, at least for the beginning, to understand the new proposed mechanism for solving the above problem, this meaning to treat a true potential flow (throughout). In the case of a rotational flow, like previously, the searched for space curves must be obtained as being the intersection lines of two surface families: 1. the isentropic ($\mathbf{V}, \mathbf{\Omega}$) ones, allowing us to introduce a 2-D velocity quasi-potential Φ_i; and 2. the new special surfaces (like previous isotachic & isothermal ones) over which another condition must be satisfied. Let us write Steichen's equation in the form:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} - \frac{1}{a^2}\left[V_x^2\frac{\partial V_x}{\partial x} + V_y^2\frac{\partial V_y}{\partial y} + V_z^2\frac{\partial V_z}{\partial z}\right.$$
$$+ V_xV_y\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + V_yV_z\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$
$$\left. + V_zV_x\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right)\right] = 0, \text{ with } \mathbf{k}_xV_x + \mathbf{k}_yV_y + \mathbf{k}_zV_z = \mathbf{V};$$

$$a^2 = \frac{\gamma-1}{2}\left[W^2 - \left(V_x^2 + V_y^2 + V_z^2\right)\right] = \frac{\gamma-1}{2}\left(W^2 - V^2\right) \quad ;$$

$$V_x^2\frac{\partial V_x}{\partial x} + V_y^2\frac{\partial V_y}{\partial y} + V_z^2\frac{\partial V_z}{\partial z} = \frac{1}{3}\left(\frac{\partial V_x^3}{\partial x} + \frac{\partial V_y^3}{\partial y} + \frac{\partial V_z^3}{\partial z}\right)$$
$$= \frac{1}{3}\nabla\mathbf{V}_3 \quad , \quad \text{with} \quad \mathbf{V}_3 = \mathbf{k}_xV_x^3 + \mathbf{k}_yV_y^3 + \mathbf{k}_zV_z^3 \quad ,$$

thus obtaining a new special form for this equation:

$$\nabla\mathbf{V} - \frac{2}{\gamma-1}\frac{1}{W^2 - V^2}\left\{\frac{1}{3}\nabla\mathbf{V}_3 + \left[V_xV_y\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)\right.\right.$$
$$\left.\left. + V_yV_z\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right) + V_zV_x\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right)\right]\right\} = 0. \quad (5)$$

Now we impose (consider the possibility to satisfy):

$\nabla\mathbf{V} - \nabla\mathbf{V}_3/3a^2 = 0$, or $a^2\nabla\mathbf{V} = \nabla\mathbf{V}_3/3$, (6)

or in a more suggestive form: $\nabla\mathbf{V}_3/\nabla(3\mathbf{V}) = a^2(\mathbf{V})$ – the ratio of two divergences (in this case), condition giving the new searched for special surfaces, concurrently further satisfying Steichen's PDE (5) above:

$a^2\nabla\mathbf{V} = \mathbf{V}\nabla(V^2)/2$, or, equating with that given by Eq. (6):

$3\mathbf{V}\nabla(V^2) = 2\nabla\mathbf{V}_3 \quad (\neq 0)$, (7)

(not depending on a). Let us write Eq. (7) in a scalar form:

$$V_xV_y\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + V_yV_z\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$
$$+ V_zV_x\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right) = 0 \quad , \quad (7')$$

generally valid along the intersection lines of the isentropic "i" surfaces with the "j" special ones satisfying Eq. (6). These lines are solutions of the system formed by Eqs. (5) and (6). If $\nabla\mathbf{V}_3 = 0$, the problem reduces to that in section 6. Introducing the velocity potential Φ, $\mathbf{V} = \nabla\Phi$, and:

$V_x = \partial\Phi/\partial x$; $V_y = \partial\Phi/\partial y$; $V_z = \partial\Phi/\partial z$; $\nabla\mathbf{V} = \Delta\Phi$,

we obtain for the condition Eq. (6) the expanded equivalent forms below (the PDE of the new searched for surfaces):

$$\left(a^2 - V_x^2\right)\frac{\partial V_x}{\partial x} + \left(a^2 - V_y^2\right)\frac{\partial V_y}{\partial y} + \left(a^2 - V_z^2\right)\frac{\partial V_z}{\partial z} = 0 \quad ,$$

also expressing the orthogonality of the vectors $\mathbf{V}_2$ and $\boldsymbol{\delta}\mathbf{V}$, defined as follows:

$$\mathbf{V}_2 = \mathbf{k}_x\left(a^2 - V_x^2\right) + \mathbf{k}_y\left(a^2 - V_y^2\right) + \mathbf{k}_z\left(a^2 - V_z^2\right) \quad ;$$

$$\boldsymbol{\delta}\mathbf{V} = \mathbf{k}_x\frac{\partial V_x}{\partial x} + \mathbf{k}_y\frac{\partial V_y}{\partial y} + \mathbf{k}_z\frac{\partial V_z}{\partial z}; (\mathbf{V}_2 \cdot \boldsymbol{\delta}\mathbf{V} = 0); \text{or :}$$

$$a^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right) = \left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2}$$

$$\text{or:} \quad \left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{\partial\Phi}{\partial x}\right)^2 - \left(\frac{\partial\Phi}{\partial y}\right)^2 - \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\frac{\partial^2\Phi}{\partial x^2}$$
$$+ \left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{\partial\Phi}{\partial y}\right)^2 - \left(\frac{\partial\Phi}{\partial z}\right)^2 - \left(\frac{\partial\Phi}{\partial x}\right)^2\right]\frac{\partial^2\Phi}{\partial y^2}$$
$$+ \left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{\partial\Phi}{\partial z}\right)^2 - \left(\frac{\partial\Phi}{\partial x}\right)^2 - \left(\frac{\partial\Phi}{\partial y}\right)^2\right]\frac{\partial^2\Phi}{\partial z^2} = 0 \quad ,$$

and for previous Eq. (7′) (using Schwarz' theorem):

$$2\left(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial y\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial z\partial x}\right) = 0,$$

just the sum of until now negligible terms with respect to $a^2\Delta\Phi$ in the velocity potential PDE for small perturbations.

This result is very important and leads for the case of a 2-D plane motion ($V_z = \partial\Phi/\partial z = 0$) to the simplest equation of the potential Φ (a very particular situation):

$$\frac{\partial^2\Phi}{\partial x\partial y} = 0 \quad , \text{ (meaning: } \partial V_x/\partial y = \partial V_y/\partial x = 0 \text{ ,} \qquad (8)$$

leading to a special plane flow: $V_x = f_1(x)$; $V_y = f_2(y)$ and $\Phi = \int f_1(x)dx + \int f_2(y)dy = \Phi_1(x) + \Phi_2(y)$, like that in a channel with hyperbolic walls, or a right-angled corner), representing the canonical form of a 2nd order PDE of a hyperbolic type with constant coefficients. The condition Eq. (6) becomes:

$$\left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{\partial\Phi}{\partial x}\right)^2 - \left(\frac{\partial\Phi}{\partial y}\right)^2\right]\frac{d}{dx}\left(\frac{\partial\Phi}{\partial x}\right)$$
$$+\left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{\partial\Phi}{\partial y}\right)^2 - \left(\frac{\partial\Phi}{\partial x}\right)^2\right]\frac{d}{dy}\left(\frac{\partial\Phi}{\partial y}\right) = 0 \quad \text{, or :}$$

$$\left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{d\Phi_1}{dx}\right)^2 - \left(\frac{d\Phi_2}{dy}\right)^2\right]\frac{d}{dx}\left(\frac{d\Phi_1}{dx}\right)$$
$$+\left[W^2 - \frac{\gamma+1}{\gamma-1}\left(\frac{d\Phi_2}{dy}\right)^2 - \left(\frac{d\Phi_1}{dx}\right)^2\right]\frac{d}{dy}\left(\frac{d\Phi_2}{dy}\right) = 0, \text{or}$$

$$\left(W^2 - \frac{\gamma+1}{\gamma-1}V_x^2 - V_y^2\right)\frac{dV_x}{dx} + \left(W^2 - \frac{\gamma+1}{\gamma-1}V_y^2 - V_x^2\right)\frac{dV_y}{dy} = 0$$

– a first order ODE with two unknown functions.

The same results can be obtained for a 2-D axisymmetric motion (in a meridian plane, in cylindrical coordinates), leading to another special flow: $V_x = f_1(x)$; $V_r = f_2(r)$; $\Phi = \Phi_1(x) + \Phi_2(r)$, and (the most important), *for a rotational flow of a compressible fluid* (over the isentropic surfaces, in the smart intrinsic triorthogonal curvilinear coordinates), instead of velocity's potential Φ appearing its 2-D quasi-potential Φ_{ij}. In the last case the relation (8) is more intricate, due to versors direction variation (see the right-hand side of the last equation in section 2), but the essential is the same, it being satisfied only along the space curves – intersection lines of the "i" isentropic surfaces with the "j" special ones, *irrespective of the "pseudo-flow" character* (not depending on a).

Concluding, as regards the PDE of the velocity quasi-potential Φ_{ij}, through any point of a certain rotational flow two interesting space curves are always passing:

1. the quasi-Laplace line (an elliptic PDE) – see section 6;
2. the new special line leading to a hyperbolic PDE.

Analogously we can suppose the existence of a third family of space curves passing through any point of a certain rotational flow, along which the velocity quasi-potential Φ_{ij} respects another rule, corresponding this time to a PDE of the 2nd order of a parabolic type, just in its canonical form: $\partial^2\Phi_{ij}(\lambda, \mu, \nu_{0i})/\partial\mu^2 + \cdots$ (lower order terms) = 0, in the same $O\lambda\mu\nu$ smart intrinsic coordinate system, *irrespective of the "pseudo-flow" character* also. The analysis in sections 6 & 7 can be useful for other cases where the physical phenomena lead to PDEs of the second order of a mixed type.

8 A simple example: the 3-D conical flow

An interesting example of application was given in [8], establishing the general ODE (!) of the velocity quasi-potential $\Phi_i = R\mathcal{V}(\varphi, \chi_{0i}) = R\mathcal{V}_i(\varphi)$ – on every "i" isentropic sheet $\chi = \chi_{0i}$ for any (rotational, therefore non-isentropic) 3-D conical flow, writing and analyzing it in the smart intrinsic generalized spherical (conical) triorthogonal coordinate system (R, φ, χ):

$$(\gamma-1)\{\mathcal{V}_i'' + [\ln|f(\varphi)|]'\cdot\mathcal{V}_i' + 2\mathcal{V}_i\}(W^2 - \mathcal{V}_i^2 - \mathcal{V}_i'^2)$$
$$-2(\mathcal{V}_i'' + \mathcal{V}_i)\mathcal{V}_i'^2 = 0 \text{ , with } ' = d/d\varphi \text{ and } '' = d^2/d\varphi^2 \text{ ,}$$

instead of the usually written (approximate) PDE of the velocity potential $\Phi = R\mathcal{V}_p(\theta, \omega)$, with $\mathbf{V} = \mathbf{V}_p + \mathbf{V}_r$ (Helmholtz) where: $\mathbf{V}_p = \nabla\Phi$ and $\mathbf{V}_r = \nabla\times\mathbf{\Psi}$ or (see [9]) $\mathbf{V}_r = \Phi_1\nabla\Phi_2$

$$\left(1-\frac{V_{\theta p}^2}{a^2}\right)\frac{\partial V_{\theta p}}{\partial\theta} + \left(1-\frac{V_{\omega p}^2}{a^2}\right)\frac{1}{\sin\theta}\frac{\partial V_{\omega p}}{\partial\omega} + V_{Rp}\left(2+\frac{V_{\theta p}^2+V_{\omega p}^2}{a^2}\right)$$
$$-\frac{2V_{\theta p}V_{\omega p}}{a^2\sin\theta}\frac{\partial V_{\theta p}}{\partial\omega} + \cot\theta\left(1+\frac{V_{\omega p}^2}{a^2}\right)V_{\theta p} = 0; \text{ with } \Phi = R\cdot\mathcal{V}_p(\theta,\omega):$$

$$V_{Rp} = \frac{\partial\Phi}{\partial R} = \mathcal{V}_p; V_{\theta p} = \frac{1}{R}\frac{\partial\Phi}{\partial\theta} = \frac{\partial\mathcal{V}_p}{\partial\theta}; V_{\omega p} = \frac{1}{R\sin\theta}\frac{\partial\Phi}{\partial\omega} = \frac{1}{\sin\theta}\frac{\partial\mathcal{V}_p}{\partial\omega}$$

so obtaining:

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\mathcal{V}_p}{\partial\theta}\right)^2\right]\frac{\partial^2\mathcal{V}_p}{\partial\theta^2} + \left[1-\frac{1}{a^2\sin^2\theta}\left(\frac{\partial\mathcal{V}_p}{\partial\omega}\right)^2\right]\frac{1}{\sin^2\theta}\frac{\partial^2\mathcal{V}_p}{\partial\omega^2}$$
$$+\left\{2+\frac{1}{a^2}\left[\left(\frac{\partial\mathcal{V}_p}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial\mathcal{V}_p}{\partial\omega}\right)^2\right]\right\}\mathcal{V}_p$$
$$-\frac{2}{a^2\sin^2\theta}\frac{\partial\mathcal{V}_p}{\partial\theta}\frac{\partial\mathcal{V}_p}{\partial\omega}\frac{\partial^2\mathcal{V}_p}{\partial\theta\partial\omega} + \cot\theta\left[1+\frac{1}{a^2\sin^2\theta}\left(\frac{\partial\mathcal{V}_p}{\partial\omega}\right)^2\right]\frac{\partial\mathcal{V}_p}{\partial\theta} = 0,$$

$$\text{with: } a^2 = \frac{\gamma-1}{2}\left(W^2 - V_p^2 - V_r^2\right) \cong \frac{\gamma-1}{2}\left(W^2 - V_{Rp}^2 - V_{\theta p}^2 - V_{\omega p}^2\right)$$
$$= \frac{\gamma-1}{2}\left[W^2 - \mathcal{V}_p^2 - \left(\frac{\partial\mathcal{V}_p}{\partial\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{\partial\mathcal{V}_p}{\partial\omega}\right)^2\right], \quad \text{(see, e.g.: [10])}$$

so a nonlinear PDE of the 2nd order in two variables (the spherical coordinates θ, ω), obtained by throughout neglecting the $\mathbf{V}_r$ components (V_{Rr}, $V_{\theta r}$, $V_{\omega r}$), so being an *approximate* PDE, though being an *exact* one when it is written on the isentropic surfaces $(\mathbf{V}, \mathbf{\Omega})_i\ \zeta = \zeta_{0i}$ (containing streamlines & vortex lines) only, becoming now identical to the general ODE of the velocity quasi-potential; $\mathbf{V}_r$ – the rotational part of $\mathbf{V}$; $\mathcal{V}_i(\varphi) = V_{Ri}$ (the radial component of the velocity on the conical isentropic sheet "i"). The quasi-potential is: $\Phi_i(\varphi) = R_i\mathcal{V}_i$, also having: $R_i'/R_i = \mathcal{V}_i/\mathcal{V}_i'$ (see [11]), getting by integration: $R_i = R_i(\varphi)$, representing the polar equation of all flow streamlines contained in the isentropic surface "i" (all being homotethic space curves, their family being given by the polar equation: $R_i = R_0R_{1i}(\varphi)$, for various values of the positive constant R_0, with $R_{1i}(\varphi)$ – a non-dimensional quantity depending on φ only).

The generalized spherical coordinate φ is defined by:

$$\varphi = \int(d\theta^2 + \sin^2\theta d\omega^2)^{1/2} = \int\sin\theta[(d\ln|\tan(\theta/2)|)^2 + d\omega^2]^{1/2} \quad (9)$$

– the relation with the classical spherical coordinates, (R, θ, ω), due to supersonic flow's kinematics (see [11]):

$$\dot{\varphi}^2 = \dot{\theta}^2 + \sin^2\theta \cdot \dot{\omega}^2 . \qquad (10)$$

In the smart intrinsic orthogonal curvilinear coordinate system: $\beta = \int[d\varphi^2 + f^2(\varphi)d\chi^2]^{1/2}$, which on $\chi = \chi_{0i} = ct._i$ becomes: $\beta = \varphi$. The general ODE above is similar to the classical Taylor–Maccoll one ([12], [13]) for an axisymmetric conical (supersonic) irrotational flow, this one being a particular case obtained for: $\varphi = \theta$ and $f(\varphi) = f(\theta) = \sin\theta$, and so: $[\ln|f(\varphi)|]' = [\ln|\sin\theta|]' = \cot\theta$, the equation becoming:

$$(\gamma-1)\{V''+\cot\theta\cdot V'+2V\}(W^2-V^2-V'^2)-2(V''+V)V'^2=0$$

and the sliding condition [11]: $R'V' = RV$, with: $' = d/d\theta$ and $'' = d^2/d\theta^2$ and the solution $V(\theta) = V_R$ – valid throughout (the velocity quasi-potential $\Phi_i(\varphi)$ coincides in this case with the velocity potential $\Phi(\theta)$). Along the "vector radii" half-straight lines one gets a 2-D Laplace's PDE, with the solution $\Phi_i/R = V_i = V_{Ri} = \text{const.}_i(\theta)$. Through any flow's point an "ij" line is passing (the intersection of two conical surfaces: the isentropic "i" sheet $\chi = \chi_{0i} = \text{const.}_i$ and the isotachic & isothermal "j" one $\varphi = \varphi_{0j} = \text{const.}_j$). This means the searched "ij" line is a half-straight line passing through the flow apex and the considered flow's point. Along any such a half-straight line the searched for velocity quasi-potential $\Phi_{ij}(\varphi, \chi)$ becomes a harmonic function (the general non-linear Steichen's PDE of the 2nd order becoming a 2-D Laplace's PDE: $\Delta\Phi_{ij} = 0$ – with the solution: $\Phi_{ij}/R = V_{ij} = V_{Rij} = \text{const.}_{ij}(\varphi, \chi)$. The annulment of the radial component of the acceleration vector, $a_R = 0$ (the first equation of motion: $\ddot{R} - R\dot{\varphi}^2 = 0$), is identical to the "sliding condition", expressing the condition that the velocity **V** be tangent to a certain streamline of the conical flow – the scalar product $\mathbf{V}\cdot\mathbf{n} = 0$, where **n** is the unit vector of the normal to the streamline, having the components $-R/(R^2 + R'^2)^{1/2}$ and $R'/(R^2 + R'^2)^{1/2}$, respectively. It can be written in the forms: $R'V' = RV\ (= \Phi)$, or $R'/R = V/V'$; $' = d/d\varphi$ (representing the streamline's ODE), leading to:

$$\frac{d\ln R}{d\varphi}\frac{d\ln|\dot{R}_i|}{d\varphi} = 1, \quad \text{or:} \quad \frac{d\ln R}{d\varphi}\frac{d\ln|V_{Ri}|}{d\varphi} = 1.$$

In fact, the "velocity quasi-potential" function Φ_i must also be a conical one, which means to be in the form: $\Phi_i(R, \varphi, \chi_{0i}) = \Phi_i(R, \varphi) = R^n\cdot V(\varphi, \chi_{0i}) = R^n\cdot V_i(\varphi)$, this being a conical scalar "quasi-potential" of the n-th order ($n \in Z$); let choose its simplest form, that for the first order ($n = 1$), having also a physical significance: $\Phi_i(R, \varphi, \chi_{0i}) = \Phi_i(R, \varphi) = R\cdot V(\varphi, \chi_{0i}) = R\cdot V_i(\varphi)$;

$$\Phi_i(\varphi) = R\dot{R}_i = RV_{Ri}\ ; \Rightarrow \frac{d\ln R}{d\varphi} + \frac{d\ln|V_{Ri}|}{d\varphi} = \frac{d\ln|\Phi_i|}{d\varphi},$$

the trivial system (product and sum of the logarithmic derivatives of R and $|V_{Ri}|$ with respect to the generalized spherical coordinate φ), its solution and the analysis on the existence domains for the "(quasi-)potential" $\Phi_i(\varphi)$, both from the mathematical viewpoint and the physical one (also including fig. 1, here reproduced as fig. 2) – all from subsection 2.1 in [11], become valid again, this time for any 3-D conical flow (both potential and rotational). So, for an axisymmetric conical flow around a circular cone this domain is $\beta \in (\beta_0, \beta_L)$ with $\beta_0 = \pi/2 - \theta_s$ and $\beta_L = \pi/2 - \theta_c$ where: $\sin^2\theta_s \approx 1/M^2 + [(\gamma + 1)/2]\sin^2\theta_c$ (see [14], [15]), θ_c, θ_s being the solid cone and shock wave half-angles, resp., meaning that: $\sin\mu = 1/M$, $\sin\nu = [(\gamma + 1)/2]^{1/2}\sin\theta_c$ and $\sin\theta_s$ (all < 1) are usually quasi-Pythagorean numbers ($\sin^2\mu + \sin^2\nu \approx \sin^2\theta_s$), with $\sin^2\theta_c < [2/(\gamma + 1)]\cdot(1 - 1/M^2)$; M is the emergent supersonic stream Mach number. If is known or is given the conical flow velocity (quasi-) potential $\Phi_i(\beta)$, the system above admits the solution:

$$d\ln R/d\beta = d\ln\sqrt{|\Phi_i|}\big/d\beta - \sqrt{\left(d\ln\sqrt{|\Phi_i|}\big/d\beta\right)^2 - 1} \quad ;$$

$$d\ln|V_{Ri}|/d\beta = d\ln\sqrt{|\Phi_i|}\big/d\beta + \sqrt{\left(d\ln\sqrt{|\Phi_i|}\big/d\beta\right)^2 - 1} \quad ,$$

ODEs which can be easily integrated, so being found the solution ($\ln R$, $\ln|V_{Ri}|$), and further the (R, $|V_{Ri}|$) one. We introduce now the hyperbolic functions:

$$\cosh u = d\ln\sqrt{|\Phi_i|}\big/d\beta\ ; \sinh u = \sqrt{\left(d\ln\sqrt{|\Phi_i|}\big/d\beta\right)^2 - 1}\ ,$$

so expressing the solution in the simpler form below:

$$\frac{d\ln R}{d\beta} = e^{-u} = e^{-\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}} \ ; \ \ln R = \int e^{-\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta\ ;$$

$$\frac{d\ln|V_{Ri}|}{d\beta} = e^{u} = e^{\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}} \ ; \ \ln|V_{Ri}| = \int e^{\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta;$$

$$\ln\frac{R}{R_0} = \int_{\beta_0}^{\beta} e^{-\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta \ ; \ R = R_0 e^{\int_{\beta_0}^{\beta} e^{-\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta} ;$$

$$\ln\frac{|V_{Ri}|}{|V_{R_{0i}}|} = \int_{\beta_0}^{\beta} e^{\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta; |V_{Ri}| = |V_{R_{0i}}| e^{\int_{\beta_0}^{\beta} e^{\mathrm{argch}\frac{d\ln\sqrt{|\Phi_i|}}{d\beta}}d\beta} .$$

The existence conditions of the roots in first solution give: $|\Phi_i| \le \Phi_{i0}e^{-2\beta}$ and $|\Phi_i| \ge \Phi_{i0}e^{2\beta}$, with $\Phi_{i0} > 0$, relations which establish the existence domains of the velocity potential $\Phi_i(\beta)$ to obtain isentropic 2-D (plane, axisymmetric and general 3-D) conical flows. It can be noticed that for $\beta > 0$ cannot exist isentropic 2-D conical respectively, between their symmetric ones with respect to the β axis, as one can see in fig. 2. The equations of motion are getting the special forms (2.1.14), (2.2.1) and (2.2.2) in [11]. The current cones (C_2): $\chi = 1/C_2$ represent the isentropic sheets $\chi = c(S - c_0) = c(S_{0i} - c_0) = \chi_{0i} = 1/C_2$, in which is analyzed the conical motion, these sheets being a particular case of D. Bernoulli surfaces, namely for isoenergetic flows (rigid surfaces in the fluid flow); the velocity **V** deriving from a conical quasi-potential of the first order, on each "i" such a sheet $\chi = \chi_{0i}$ having: $\mathbf{V}_i = \nabla\Phi_i$; $\Phi_i = R\cdot V_i(\varphi)$ – a conical scalar quasi-potential, specific to the respective "i" sheet, the general flow being rotational ($\mathbf{\Omega} \ne 0$, but with $\Omega_\chi = 0$, so that $\mathbf{\Omega} = \mathbf{k}_R\Omega_R + \mathbf{k}_\varphi\Omega_\varphi$) – also see section 1 in [8].

In the classical spherical coordinate system (R, θ, ω) and in the smart intrinsic generalized spherical one (R, φ, χ), resp., the 1st equation of motion for a certain 3-D conical flow is:

$$a_R = \ddot{R} - R\dot{\theta}^2 - R\sin^2\theta\cdot\dot{\omega}^2 = \ddot{R} - R\dot{\varphi}^2 = -(\partial p/\partial R)/\rho = 0, \text{and}$$

$$a_R = \ddot{R} - R\dot{\varphi}^2 - Rf^2(\varphi)\cdot\dot{\chi}^2 = \ddot{R} - R\dot{\beta}^2 = \ddot{R} - R\dot{\varphi}^2 = 0; (\dot{\chi} = 0).$$

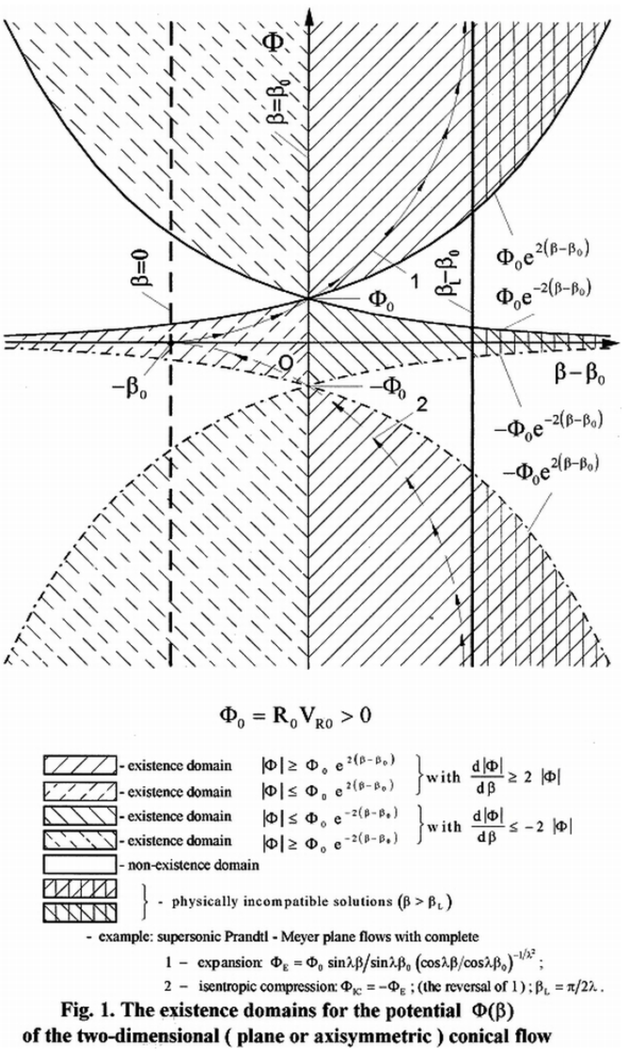
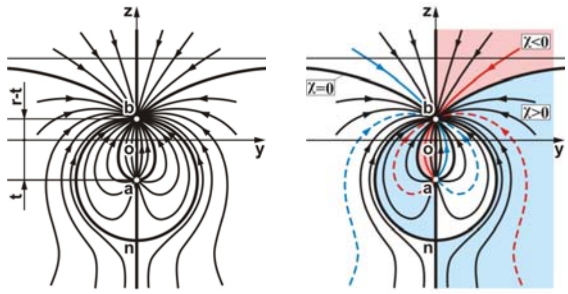


Fig. 2. Reproduction of fig. 1 from subsection 2.1 in [11]
(here we introduced the notation: $\lambda = [(\gamma - 1)/(\gamma + 1)]^{1/2}$)

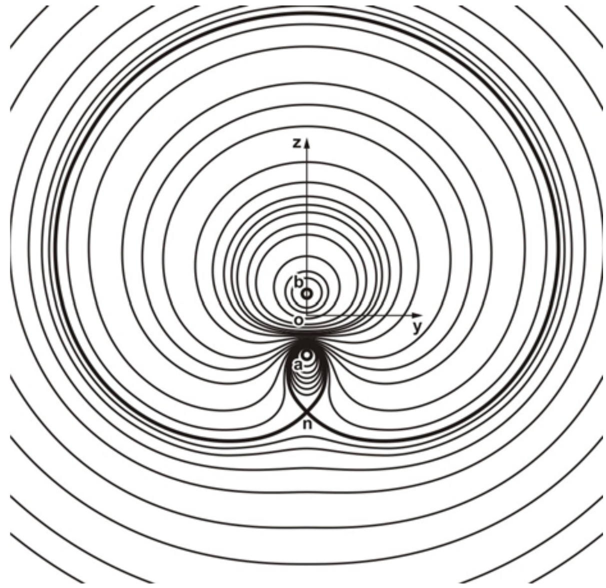
In the case of a plane conical flow: $\theta = \pi/2$; ⇒ $\sin\theta = 1$ and: $\dot{\theta} = 0; \dot{\varphi} = \dot{\omega}$. In the case of an axisymmetric conical flow, the streamline is contained in a meridian plane (phenomenon independent of ω): $\omega = \omega_0 =$ const.; ⇒ $\dot{\omega} = 0; \dot{\varphi} = \dot{\theta}$. In the case of a general 3-D conical flow, using the notation (10), one obtains the relation (9). There also are helicoidal conical flows (see [11]) – θ and ω bound by a certain DE. For all 2-D flows the first equation of motion takes the forms $\ddot{R} - R\dot{\beta}^2 = 0$, or: $(d\ln R/d\beta)\cdot(d\ln|\dot{R}|/d\beta) = 1$ – the product of a pair of logarithmic derivatives with respect to β equal to the unity (β is the angular coordinate: β = ω – for the plane conical flows (Prandtl–Meyer); β = θ and β = π/2 – θ – for the axisymmetric conical flows (Busemann–Taylor–Maccoll) inside and outside a circular cone, resp.; the angle of attack $\alpha \neq 0$; r is the cone local radius;

$\bar{r} = r/x$; $\bar{t} = t/x = \tan\alpha$; ⇒ $\alpha = \mathrm{atan}\,\bar{t}$;



3.a. $\bar{Z} = \mathrm{Im}\,F(x_1) = \mathrm{Im}\left\{\ln\frac{[x_1 - i(\bar{r} - \bar{t})]^2}{x_1 + i\bar{t}}\right\} = \tan^{(-1)}\chi = C_2'$;

$$\chi = \frac{\bar{y}^2(\bar{z} + \bar{t} - 2\bar{r}) + (\bar{z} + \bar{t} - \bar{r})^2(\bar{z} + \bar{t})}{\bar{y}[\bar{y}^2 + (\bar{z} + \bar{t})^2 - \bar{r}^2]} = \frac{1}{C_2} = \tan C_2';$$



3.b. $\bar{Y} = \mathrm{Re}\,F(x_1) = \mathrm{Re}\left\{\ln\frac{[x_1 - i(\bar{r} - \bar{t})]^2}{x_1 + i\bar{t}}\right\} = \ln\sqrt{\varphi} = C_1'$;

$$\varphi = \frac{[\bar{y}^2 + (\bar{z} + \bar{t} - \bar{r})^2]^2}{\bar{y}^2 + (\bar{z} + \bar{t})^2} = C_1 = e^{2C_1'} ;$$

$x_1 = \bar{y} + i\bar{z}$ is a complex variable; $\bar{y} = y/x$; $\bar{z} = z/x$; x, y, z – Cartesian coordinates, x – abscissa of cross section current plane

Fig. 3. Simple qualitative example of "generalized spherical" smart intrinsic coordinate surfaces for the case of a circular cone at a small angle of attack (cross section) – the incompressible approximation (slender body; no shock wave), giving the relations between the new spherical coordinates (R, φ, χ) and the Cartesian ones; the flow is due to two semi-infinite line sources along cone's axis (a) and back (b), replacing it; (a), (b), (n) – nodal and saddle lines; 3.a. the conical isentropic sheets $\chi = \chi_{0i} = c(S_{0i} - c_0) = 1/C_2$ (a smart intrinsic coordinate tied to S_{0i} – the local specific entropy value), having as remarkable directrices: the **oz** axis and a circle (the solid cone trace) centered on it (both for $C_2 = 0$), and a right strophoid ($\chi = 0$) centered on **oz** axis too (for $1/C_2 = 0$); c, S_{0i}, $c_0 > 0$; 3.b. the conical sheets $\varphi = \varphi_{0j} = C_1$ (smart intrinsic coordinate), orthogonal to the conical isentropic ones: $\nabla\bar{Y}(\bar{y},\bar{z})\cdot\nabla\bar{Z}(\bar{y},\bar{z}) = 0$, having as remarkable directrices: a Pascal's limaçon and the circle at infinity (both for $C_1 = 0$ and centered on the **oz** axis).

$\beta = \varphi -$ for any 3-D conical flow in the intrinsic system); so:
$\arctan(d\ln R/d\beta) + \arctan(d\ln|\dot{R}|/d\beta) = \pi/2$;
(the curves lnR and $\ln|\dot{R}|$ are symmetrically inclined with respect to first bisectrix of (β, Φ) axes for any value of β, like a pair of *characteristic lines* with respect to velocity direction, in both physical and hodographic plane). A simple example of 3-D conical flow is given in fig. 3 (see fig. 1 in [2] – [4]).

9 Other possible application: a supersonic axisymmetric rotational flow (quasi-conical), confluent to a potential one (tronconical)

In a series of papers (see [16] – [19]) a new shock-free axisymmetric configuration inspired by a Schlieren picture (fig. 260 – a forebody in supersonic flow with isentropic compression – achieved by isentropic way (smoothly) and not by shock) from the flow visualizations album [20] was proposed and studied. This flow was called *tronconical*, due to the fact that the simple compression waves are co-axial truncated cones (conical frusta) having as a common basis (directrix) the "foci" circle, along which they focus into an axisymmetric shock wave (with curved meridian line, and so with variable intensity) – see fig. 4.a, and various types of air intake were imagined: frontal, annular, frontal-annular, etc. (see figs. 3 – 7 in [18]), all with dynamic compression. The tronconical potential flow is associated (confluent) to a rotational flow 2 - 2' treated in [21], around an axisymmetric body without incidence, consisting of a cylinder of radius r_0 and a truncated cone of half-angle τ_c (see fig. 4.a), with a bow shock wave, assumed to be attached at the intersection circle CC' of the two axisymmetric surfaces. In order to determine the solution (the velocity field and implicitly that of static pressure, the shape of the shock wave and the entropy gradient), the author develops an extremely interesting analytic perturbations method, the disturbance affecting not as usual the incident parallel and uniform supersonic flow of Mach number M_1 from the upstream, but a potential flow very close to that rotational real, called by him a *quasi-conical motion*. Later, this problem was resumed in [22], also considering a supersonic combustion wave in the presence of a truncated cone, giving an analytic solution. The confluent flows are related to other simpler supersonic potential flows (the conical axisymmetric one and the plane one with shock wave or Prandtl–Meyer isentropic compression obtained as particular cases; for the last one see fig. 227 from [20]). All tronconical waves (truncated cones) are envelope surfaces of the Mach cones (the local simple waves) generated by the points of the given axisymmetric body surface, all situated in the same cross section plane. In the toroidal (tronconical) coordinate system (r_0, R, θ) the PDE of the velocity potential can be set in the forms:

$$\left(1-\frac{V'^2}{a^2}\right)(V''+V)+\frac{1}{1+r_0/(R\sin\theta)}(\cot\theta\cdot V'+V)=0\,,$$

(with $R\sin\theta \neq -r_0$), therefore a nonlinear ODE, V being the radial component of the velocity ($V_R = V$), and a the local speed of sound, with: $' = d/d\theta$ and $'' = d^2/d\theta^2$, or

$$\left(1+\frac{r_0}{R\sin\theta}\right)\left(1-\frac{V'^2}{a^2}\right)(V''+V)+\cot\theta\cdot V'+V=0\quad,$$

both useful for the presentation of the particular cases. Analyzing both equations one can see that in these cases the ODE of the velocity potential gets typical forms. Thus, for various values of the cylinder radius r_0, we have:

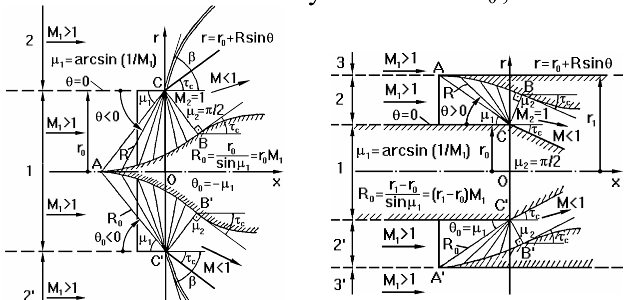


Fig. 4. Reproduction of figs. 3, 4 from [16] – [19] (oblate on Ox) The tronconical (shock-free compression) flow inside an axisymmetric supersonic inlet diffuser with (4.a) outer and resp. (4.b) inner compression (frontal and annular air intake): 4.a) 1 – homentropic cylindrical core of radius r_0 with tronconical flow – flow with complete (until $M_2 = 1$) isentropic compression; 2 - 2' – rotational ring – a tronconical flow too, but with expansion, downstream of an attached bow shock wave, weakened and canceled by the tronconical expansion waves (whose axisymmetric fan was not represented) originating on the "foci" circle CC', the same for both flows; 4.b) 1 – fuselage (cylinder-cone body); 2 - 2' – homentropic ring with tronconical flow inside a specially shaped annular channel; 3 - 3' – infinite ring with non-perturbed supersonic flow.

$r_0 \to \infty$; $[1 - (V'^2/a^2)](V'' + V) = 0$, hence $V' = a$,
the plane flow with Prandtl–Meyer isentropic compression;
$r_0 = 0$; $V'' + \cot\theta\cdot V' + 2V = (V'^2/a^2)(V'' + V)$,
the conical axisymmetric flow with isentropic compression inside Busemann's nozzle (upstream of a conical shock wave with the same tip as the conical simple waves fan), as well as that downstream of an attached conical shock wave (between the shock wave and an infinite solid cone). In the trivial case of a parallel and uniform supersonic stream ($R\sin\theta = kr_0$, with $k \geq -1$) the velocity potential ODE becomes:

$$(1 + 1/k)[1 - (V'^2/a^2)](V'' + V) + \cot\theta\cdot V' + V = 0 \quad .$$

Both Prandtl–Meyer (see [23] – [27]) and Taylor–Maccoll (see [12], [13]) ODEs (which describe the plane and respectively axisymmetric conical flows) are obtained as simple limit cases of the general ODE of the velocity potential for tronconical flows. From this equation it can be noticed that, unlike the conical flows cases, when the equation has an unknown function of a single variable – $V(\theta)$, in the tronconical flow case, the equation has the same unknown function V, but depending on two variables – θ and R. In this equation intervening only the function derivatives with respect to θ, the dependence on R can be considered weaker or even zero (this being called by the author *the tronconical approximation*: [16] – [18]).

Setting (also see fig. 1 from [20]): $\theta = \alpha$; $R = r$; $r_0 = b$;
$$\frac{1}{1 + r_0/(R\sin\theta)} = \frac{R\sin\theta}{r_0 + R\sin\theta} = \frac{r\sin\alpha}{b + r\sin\alpha} = m \quad ;$$
and $V = \varphi(r,\alpha)$; $V' = \varphi_\alpha$; $V'' = \varphi_{\alpha\alpha}$; $a = a_1$,
one obtains for the general differential equation the form:
$$\left(1 - \frac{\varphi_\alpha^2}{a_1^2}\right)\varphi_{\alpha\alpha} + m\,\varphi_\alpha \cot\alpha + \left(1 + m - \frac{\varphi_\alpha^2}{a_1^2}\right)\varphi = 0, \quad (11)$$
identical with Eq. (17) from [21] (but the last one having different boundary conditions), describing an axisymmetric *potential* flow – a component (namely just the *quasi-conical* one) of the rotational general supersonic one downstream of an attached axisymmetric bow shock wave. The boundary conditions above are given by the relations:
$$\alpha = \alpha_0; (\varphi_\alpha)_0 = 0, \text{ where } \alpha_0 = \tau_c \text{ (see fig. 3)} \quad (18\text{ a) in [21]}$$
$$\alpha = \alpha_s; \varphi_s = \cos\alpha_s; (\varphi_\alpha)_s = -\left(k + \frac{1-k}{K_s^2}\right)\sin\alpha_s \quad (18\text{ b) in [21]}$$
α_s is the half-angle of a certain tronconical expansion wave (unknown, function of r), weakening the bow shock;
$$k = \lambda^2 = \frac{\gamma - 1}{\gamma + 1}; \quad K = M_1 \sin\alpha_0 \ ; \quad K_s = M_1 \sin\alpha_s \ .$$
The quantity a_1 in Eq. (11) is given by the energy equation:
$$a_1^2 = a^2 = (\gamma - 1)(W^2 - V'^2 - V^2)/2 \quad .$$
The solution was obtained by analytic way, the general integral of Eq. (11) being given in [21] by the relations (21), (22), (24 a) and (24 b), for the linearized equation:
$$\varphi = (C_0 I + D_0)\cos\alpha, \quad (21) \text{ in } [21]$$
where C_0 and D_0 are functions of the variable r only;
$$C_0 = \frac{\sin 2\alpha_0}{2 + I_s \sin 2\alpha_0} \ , \quad D_0 = 1 - C_0 I_s \ , \quad (22) \text{ in } [21]$$
being used the following notations:
$$I(\alpha,r) = \int_{\alpha_0}^{\alpha} \frac{b + r\sin\alpha_0}{b + r\sin\tau}\frac{d\tau}{\cos^2\tau} = E(\alpha) - E(\alpha_0)$$
$$-\frac{2r^2}{b^2 - r^2}\left(1 + \frac{r}{b}\sin\alpha_0\right)\int_{z_0}^{z}\frac{b^2 d\tau}{(b\tau + r)^2 + b^2 - r^2};$$
$$I_s = I(\alpha_s, r) \quad ; \quad (24\text{a) in [21]}$$
and $E(\alpha) =$
$$\frac{b + r\sin\alpha_0}{b + r\sin\alpha}\left(\tan\alpha - \frac{2r^2}{b^2 - r^2}\frac{b + rz}{1 + z^2}\right); z = \tan\frac{\alpha}{2}; \ (24\text{b) in [21]}$$
τ being an integration variable. The last integral in (24 a) takes various forms, function of the ratio (r/b) – also see the relations (14) and (15) from [22].
The entire (complete) nonlinear equation of the flow in the downstream of an attached axisymmetric shock wave
$$\varphi_{\alpha\alpha} + m\varphi_\alpha \cot\alpha + (1 + m)\varphi = F(r,\alpha) + N(r,\alpha), \quad (31) \text{ in } [21]$$
having in its right-hand side the nonlinear terms F and N
$$F(r,\alpha) = \frac{\varphi_\alpha^2}{a_1^2}(\varphi_{\alpha\alpha} + \varphi) \quad , \quad (19) \text{ in } [21]$$
(just the nonlinear term of Eq. (11), neglected until now);
$$N(r,\alpha) = \left(\frac{\varphi_\alpha^2}{a_1^2} - m\right)r\varphi_r + 2\frac{\varphi_r\varphi_\alpha}{a_1^2}r\varphi_{r\alpha} - (2r\varphi_r + r^2\varphi_{rr})\left(1 - \frac{\varphi_r^2}{a_1^2}\right),$$
(16) in [21]
has the following general integral:
$$\varphi = \left[(C + H)I + (D - L)\right]\cos\alpha \quad ; \quad (32) \text{ in } [21]$$
C and D are functions of the variable r only, and are determined imposing the boundary conditions, like α_s ;
$$H = \int_{\alpha_0}^{\alpha}\frac{(b + r\sin\tau)[F(r,\tau) + N(r,\tau)]}{b + r\sin\alpha_0}\cos\tau\, d\tau \quad ,$$
$$L = \int_{\alpha_0}^{\alpha}\frac{(b + r\sin\tau)[F(r,\tau) + N(r,\tau)]I(r,\tau)}{b + r\sin\alpha_0}\cos\tau\, d\tau \quad ,$$
representing the relations (33 a) and (33 b) in [21], resp. The integral I is given by the relations (24 a), (24 b) and resp. (32), (33 a) and (33 b), for the entire nonlinear equation. In the last two relations, we must consider for the second nonlinear term, $N(r, \alpha)$, defined by the relation (16) from [21], the condition $N(r, \alpha) = 0$, this corresponding to our tronconical flow case. An improvement in our problem solution can be obtained by replacing the relation for the main boundary condition $\mathbf{V}\cdot\mathbf{n} = 0$ with the new (exact) streamline's equation:
$R'V' = RV$ $(= \Phi)$, or $r_\alpha\varphi_\alpha = r(\varphi + r\varphi_r)$, (12)
with: $r_\alpha = dr/d\alpha$, $\varphi_\alpha = \partial\varphi/\partial\alpha$ and $\varphi_r = \partial\varphi/\partial r$, where, taking into account the definition of the velocity potential – relation (14) in [21], assumed in the simple new form $\Phi = r\varphi(r, \alpha)$ (therefore not a tronconical one, $r\varphi(\alpha)$), were used for the velocity components the expressions
$$V_R = V_r = \frac{\partial\Phi}{\partial r} = \varphi + r\,\varphi_r \ ; \quad V_\theta = V_\alpha = \frac{1}{r}\frac{\partial\Phi}{\partial\alpha} = \varphi_\alpha \ .$$
On the other hand, in order to replace the expression a_1^2 in the general PDE of the tronconical flow – the former ODE (11), one must write a new (exact) form of the energy PDE:
$(a_1)^2 = (\gamma - 1)[W^2 - (\varphi + r\varphi_r)^2 - (\varphi_\alpha)^2]/2$. (13)
The new solution will be obtained by integrating the system of three PDEs (11) – (13), the subscripts α and r having now the significance of partial differentiation with respect to the respective variables), these equations ceasing to still represent the mathematical model of a tronconical flow, but of a new flow, related to that considered in [21]. This new system, by solving of which is obtained the improved solution, has three unknowns: $\varphi(r, \alpha)$, $a_1(r, \alpha)$, and $r(\alpha)$ – the streamline (body wall) equation, the functions φ and a_1 depending on α both directly and via r as well, so that one can write on a certain streamline: $d\varphi = \varphi_\alpha d\alpha + \varphi_r r_\alpha d\alpha = (\varphi_\alpha + \varphi_r r_\alpha)d\alpha$ and respectively: $da_1 = (a_{1\alpha} + a_{1r}r_\alpha)d\alpha$.
It is also to be noticed that in all the cases, the equation of the velocity potential admits the particular (trivial) solution $V = -U_\infty\cos\theta$ (a parallel and uniform stream).
In the case of absence of the specially shaped central body (see fig. 4.a), when only the cylinder-cone shell exists, the outer *quasi-conical motion* is no more associated (confluent) to the inner tronconical flow with isentropic compression, but to a non-tronconical flow with expansion, considered

in [28], called by the author *Prandtl-Meyer expansion in axisymmetric flow*. It must be mentioned that the general equation from the beginning of this section can describe any tronconical flow, not only those with isentropic compression. Thus, it can be imagined a *tronconical flow with expansion* in an axisymmetric nozzle similar to Busemann's one, but having inside a central cylinder-cone body and a pre-determined shape of the wall meridian line, corresponding to a strictly calculated design Mach number of the downstream parallel and uniform supersonic flow (also see fig. 4.b and fig. 173 from [29]). This flow can be easily obtained by *a simple reversal* of the flow from figure 4.b and extends itself only in the supersonic range. This is possible, because the inner flow is entirely shock-free (and thus isentropic). In the same manner, starting from the tronconical flow with isentropic compression represented in fig. 4.a one can obtain the tronconical flow with expansion given in fig. 4.1.a from [30]. Besides these cases, there also is that of the tronconical flow with expansion in the second region of the mixed flow (isentropic compression – expansion) around a specially shaped axisymmetric forebody, immediately followed by a cylinder (see fig. 7 in [18], here reproduced as fig. 5). This expansion flow succeeds to a tronconical flow with isentropic compression in the first region of the above flow and takes place just around the circular edge (having in the meridian plane the trace B) of the intersection of the two bodies, this edge being just the foci circle of the tronconical flow with expansion. Besides these methods, another way to obtain tronconical flows with expansion consists in *the generalization* of their corresponding axisymmetric conical flows (see [31] – [34]), instead of the conical flow's vertex appearing now the tronconical flow's circle of foci. The new flow's velocity potential equation (valid for tronconical flows with both compression and expansion) was established as a first step in finding the exact solution and a streamlining method for the axisymmetric bodies and channels was imagined.

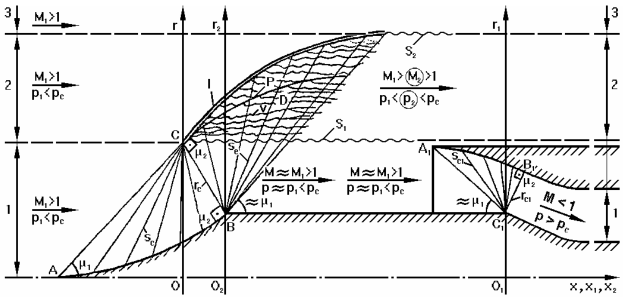


Fig. 5. Reproduction of fig. 7 from [18] (reduced caption) The confluent supersonic flows around an axisymmetric specially shaped configuration, consisting in a central forebody followed in the downstream by a co-axial inlet diffuser (annular air intake)

In order to unify the methods of solution for the PDEs of the confluent flows from figs. 4.a or 5 (potential – internal and rotational – external, both having a strong tronconical character) without dividing the external flow into a potential (quasi-conical) one, governed by the same PDE of the velocity potential (11), and a rotational perturbation, and then adding (superposing) these terms of solution, we can apply the new 2-D velocity quasi-potential theory on the whole axisymmetric flow's isentropic surfaces. Thus, even the external rotational flow can be treated as a potential one, if we can determine the equation of the axisymmetric isentropic surfaces, starting from the already known streamline's Eq. (12), for a given velocity field $r\varphi(r, \alpha)$.

10 First approach to extension of the new quasi-potential model to the rotational flow of a viscous Newtonian compressible fluid

Let start from the vector general form of the motion equation for a viscous Newtonian fluid flow (Navier–Stokes):

$$\frac{\partial \mathbf{V}}{\partial t}+\nabla\left(\frac{\mathbf{V}^2}{2}\right)+\mathbf{\Omega}\times\mathbf{V}=\mathbf{f}-\frac{\nabla p}{\rho}+\frac{\mu_1}{\rho}\Delta\mathbf{V}+\left(\frac{\mu_2}{\rho}+\frac{\mu_1}{3\rho}\right)\nabla(\nabla\mathbf{V})$$

(the Helmholtz ([35])–Gromeka–Lamb form, binding the acceleration and force density terms of a fluid particle); $\mathbf{f}$ – the mass force density (conservative – a gradient): $\mathbf{f} = \nabla(-gz) = -\nabla(gz)$; g – the acceleration of gravity; z – the geometrical height (height of the considered point above a reference horizontal plane xOy);
μ_1 – the dynamic viscosity of the fluid or *the coefficient of internal friction*; μ_1/ρ – *kinematic viscosity* of the fluid; μ_2 – *second*, or *bulk*, *viscosity* (μ_1, μ_2 assumed const.). For a steady motion and respectively for gases, we have: $\partial\mathbf{V}/\partial t = 0$ and $\mathbf{f} = 0$, thus remaining:

$$\nabla\left(\frac{\mathbf{V}^2}{2}\right)+\mathbf{\Omega}\times\mathbf{V}=-\frac{1}{\rho}\left[\nabla p-\mu_1\Delta\mathbf{V}-\left(\mu_2+\frac{\mu_1}{3}\right)\nabla(\nabla\mathbf{V})\right].$$

Searching for a 2-D velocity quasi-potential $\Phi_i(\xi, \eta)$ (therefore in *a new smart intrinsic coordinate system* $O\xi\eta\zeta$), we will perform first a scalar multiplication of this equation by a special virtual elementary displacement $d\mathbf{R} \in (\mathbf{V}, \mathbf{\Omega})$ plane (see section 2), so obtaining some surfaces similar to the isentropic ones for the inviscid compressible fluid flow case, envelope sheets of the **(V, Ω)** planes above. Over these surfaces we can write:

$$\mathbf{\Omega}=\nabla\times\mathbf{V}=\frac{1}{h_\xi h_\eta h_\zeta}\begin{vmatrix} h_\xi\mathbf{k}_\xi & h_\eta\mathbf{k}_\eta & h_\zeta\mathbf{k}_\zeta \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\zeta} \\ h_\xi V_\xi & h_\eta V_\eta & h_\zeta V_\zeta \end{vmatrix}=\mathbf{k}_\xi\Omega_\xi+\mathbf{k}_\eta\Omega_\eta+\mathbf{k}_\zeta\Omega_\zeta,$$

$$\text{with : } V_\xi = h_\xi\dot{\xi} \;\; ; \;\; V_\eta = h_\eta\dot{\eta} \;\; ; \;\; V_\zeta = h_\zeta\dot{\zeta} = 0 \;\; ,$$

where: $\mathbf{k}_i$ – a 3-D basis; h_i – Lamé's coefficients; the dotted variables are derivatives with respect to the time t, and so:

$$\Omega_\zeta=\frac{1}{h_\xi h_\eta}\left[\frac{\partial(h_\eta V_\eta)}{\partial\xi}-\frac{\partial(h_\xi V_\xi)}{\partial\eta}\right]=0 \quad .$$

Let introduce a scalar function $\Phi_i(M) = \Phi_i(\xi, \eta, \zeta_{0i})$, also called by the author a 2-D velocity "quasi-potential", whose partial derivatives along the directions of the elementary orthogonal arcs $h_\xi d\xi$ and $h_\eta d\eta$ on the "i" **(V, Ω)** sheet ($\zeta = \zeta_{0i}$) are just the components $V_{\xi i}$ and $V_{\eta i}$ of the velocity

$\mathbf{V}_i$ vector ($V_{\zeta i} = 0$). Let still define λ and μ as being two orthogonal arc lengths (also see sections 3, 5 – 7), so that: $d\lambda = h_\xi d\xi$ and $d\mu = h_\eta d\eta$, ($d\mathbf{R} = c_1\mathbf{V} + c_2\mathbf{\Omega} = \mathbf{k}_\xi h_\xi d\xi + \mathbf{k}_\eta h_\eta d\eta = \mathbf{k}_\xi d\lambda + \mathbf{k}_\eta d\mu$), on these "i" surfaces having:

$$V_{\xi i} = \frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial\Phi_i}{\partial\lambda}; V_{\eta i} = \frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial\Phi_i}{\partial\mu}; V_{\zeta i} = \frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta} = 0;$$

$$\Rightarrow\ h_\xi V_{\xi i} = \frac{\partial\Phi_i}{\partial\xi}\ \ ;\ \ h_\eta V_{\eta i} = \frac{\partial\Phi_i}{\partial\eta}\ \ ; \tag{14}$$

$$\frac{\partial(h_\eta V_{\eta i})}{\partial\xi} = \frac{\partial}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial^2\Phi_i}{\partial\xi\partial\eta}\ ;\ \ \frac{\partial(h_\xi V_{\xi i})}{\partial\eta} = \frac{\partial}{\partial\eta}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial^2\Phi_i}{\partial\eta\partial\xi}\ .$$

So the relation $\Omega_{\zeta i} = 0$ leads to: $\partial^2\Phi_i/\partial\xi\partial\eta - \partial^2\Phi_i/\partial\eta\partial\xi = 0$, this representing just Schwarz' theorem for the functions of two variables (the so-called theorem of "the equality of the mixed derivatives of the second order", they differing as to the order of differentiation only). This relation proves that $\Omega_{\zeta i} = 0$ and the existence of a 2-D velocity "quasi-potential" function Φ_i so that:

$V_{\xi i} = V_{\lambda i} = \dfrac{\partial\Phi_i}{\partial\lambda}$; $V_{\eta i} = V_{\mu i} = \dfrac{\partial\Phi_i}{\partial\mu}$ (also see [2] – [5]).

Applying the second law of thermodynamics, one can write the heat transport equation (called by this author the generalized Crocco's equation for roto-viscous fluids) as:

$$TdS = (\mathbf{\Omega}\times\mathbf{V})\cdot d\mathbf{R} + \frac{\mu_1}{\rho}\left[\left(\frac{\mu_2}{\mu_1} - \frac{2}{3}\right)I_1^2 + 2I_2\right]dt$$

$$= \left\{(\mathbf{\Omega}\times\mathbf{V})\cdot\mathbf{V}_v + \frac{\mu_1}{\rho}\left[\left(\frac{\mu_2}{\mu_1} - \frac{2}{3}\right)I_1^2 + 2I_2\right]\right\}dt$$

$= TdS_1\,(\mathbf{\Omega} \neq 0;\ \mu_1 = \mu_2 = 0) + TdS_2\,(\mathbf{\Omega} = 0;\ \mu_1 \neq \mu_2 \neq 0)$, (also see [2], [4], [36]) with the following notations:

$$d\mathbf{R} = \mathbf{V}_v dt\ \ ;\ \ I_1 = e^{i\cdot}_{\cdot i} = e^i_i = e_{11} + e_{22} + e_{33} = \nabla\mathbf{V}\,;$$

$$I_2 = e^{ij}e_{ij} = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{12}^2 - e_{23}^2 - e_{31}^2 \geq 0\,,$$

where: $\mathbf{V}_v$ is the virtual velocity vector (along the virtual displacement $d\mathbf{R}$); I_1, I_2 – first and second invariant of the tensor of the fluid particle deformation rate e ([37], [38]). The first three terms ($e_{ii}e_{jj}$; $j \neq i$) in the I_2 expression are given by the linear deformation, while the last three ones (e_{ij}^2 ; $j \neq i$) are given by the angular deformation: $e_{ii} = \partial V_{xi}/\partial x_i = v_{ii}$; $e_{ij} = (\partial V_{xi}/\partial x_j + \partial V_{xj}/\partial x_i)/2 = (v_{ij} + v_{ji})/2$; $e_{ji} = (\partial V_{xj}/\partial x_i + \partial V_{xi}/\partial x_j)/2 = (v_{ji} + v_{ij})/2 = e_{ij}$.

The new equation of the isentropic surfaces ($dS = 0$, needed to assure that $a^2 = (dp/d\rho)_{S = S_0}$ in Steichen's PDE) is:

$$TdS = TdS_1 + TdS_2 = 0\ \ ; \tag{15}$$

this is an equivalent form of *the "zero-work" condition*: $[\mathbf{\Omega}\times\mathbf{V} - \mu_1/\rho\cdot\Delta\mathbf{V} - (\mu_1 + \mu_2/3)/\rho\cdot\nabla(\nabla\mathbf{V})]\cdot d\mathbf{R} = 0$ *for the non-conservative rotational and viscous terms* (see [2], [4], [36]), having two main particular cases (independent annulment):

1. rotational flow of an inviscid fluid ($\mu_1 = \mu_2 = 0$), in the equation above missing the second term of the sum (dS_2), the motion equation admitting a D. Bernoulli first integral;
2. irrotational flow ($\mathbf{\Omega} = 0$) of a viscous fluid, in this case missing the first term of the sum (dS_1), obtaining:
1. the annulment of the mixed (scalar triple) product in the left-hand side (coplanarity of three vectors), so getting in the Cartesian system Oxyz the following PDE of $(\mathbf{V}, \mathbf{\Omega})$ sheets:

$$d\mathbf{R}\cdot(\mathbf{\Omega}\times\mathbf{V})/T = 0\ \ ;\ \ \frac{1}{T}\begin{vmatrix} dx & dy & dz \\ \Omega_x & \Omega_y & \Omega_z \\ V_x & V_y & V_z \end{vmatrix} = 0\ \ ,$$

or, expressing the vortex components Ω_x , Ω_y and Ω_z:

$$\frac{1}{T}\begin{vmatrix} dx & dy & dz \\ \dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z} & \dfrac{\partial V_x}{\partial z} - \dfrac{\partial V_z}{\partial x} & \dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y} \\ V_x & V_y & V_z \end{vmatrix} = 0\ \ ;$$

in the smart intrinsic system $O\xi\eta\zeta$ this equation is $\zeta = \zeta_{0i}$;
2. the annulment of the second factor of the second product in the left-hand side, $(\mu_2/\mu_1 - 2/3)I_1^2 + 2I_2$, written in the principal deformation rates ($e_{ii} = e_i$; $e_{ij} = e_{ji} = 0$):

$$(1/T)\cdot[(\mu_2/\mu_1 - 2/3)I_1^2 + 2I_2] = 0\ \ \Rightarrow\ \ e_1^2 + e_2^2 + e_3^2 -$$
$$- 2(\mu_2/\mu_1 + 1/3)/(2/3 - \mu_2/\mu_1)\cdot(e_1e_2 + e_2e_3 + e_3e_1) = 0\,;$$

– a pure viscous *isentropic* right-circular cone having as vertex the origin and as axis the straight line: $e_1 = e_2 = e_3$ (diagonal of the cube of equation: $e_1e_2e_3(e_1 - a)(e_2 - a)(e_3 - a) = 0$; $a > 0$; $e_1, e_2, e_3 \in [0, a]$). In the principal Cartesian system $O_1x_1y_1z_1$, replacing e_1, e_2, e_3 by $\partial V_{x1}/\partial x_1$, $\partial V_{y1}/\partial y_1$, $\partial V_{z1}/\partial z_1$, we get the PDE of the isentropic cone, having as unknown functions the velocity $\mathbf{V}$ components on the principal axes V_{x1}, V_{y1}, V_{z1}:

$$\frac{1}{T}\left[\left(\frac{\partial V_{x1}}{\partial x_1}\right)^2 + \left(\frac{\partial V_{y1}}{\partial y_1}\right)^2 + \left(\frac{\partial V_{z1}}{\partial z_1}\right)^2 - 2\cdot\frac{\mu_2/\mu_1 + 1/3}{2/3 - \mu_2/\mu_1}\times\right.$$
$$\left.\times\left(\frac{\partial V_{x1}}{\partial x_1}\cdot\frac{\partial V_{y1}}{\partial y_1} + \frac{\partial V_{y1}}{\partial y_1}\cdot\frac{\partial V_{z1}}{\partial z_1} + \frac{\partial V_{z1}}{\partial z_1}\cdot\frac{\partial V_{x1}}{\partial x_1}\right)\right] = 0\ \ .$$

In all these cases ($dS_i = 0$; $i = 1, 2$) the physical equation is: $p = K_i\rho^\gamma$, with: $K_i = p_{0i}/(\rho_{0i})^\gamma\cdot\exp[(S - S_{0i})/C_v] = p_{0i}/(\rho_{0i})^\gamma > 0$ (the isentropic constant of the respective particular "i" isentropic (zero-work) virtual surface, C_v being a constant – the isochoric specific heat of the respective ideal gas; p_{0i} and ρ_{0i} are stagnation values). Over the isentropic "i" surface the speed of sound is given by: $a^2 = (dp/d\rho)_{S = S_{0i}} = \gamma\mathcal{R}T$.

11 The velocity "quasi-potential" PDE for a viscous fluid flow and its validity domain; introducing a new Selescu's roto-viscous vector

Therefore along the intersection space lines of the particular isentropic surface families ($(\mathbf{V}, \mathbf{\Omega})_i$ sheets and "j" circular cones with $K_j = K_i$), the general rotational flow of a viscous Newtonian compressible fluid is governed by Steichen's PDE of the 2-D quasi-potential $\Phi_{ij}(M) = \Phi_{ij}(\xi, \eta, \zeta_{0i})$ (see sections 3 and 5), therefore at the same point:

$$\left[\frac{1}{h_\xi^2}\frac{\partial^2\Phi_{ij}}{\partial\xi^2} + \frac{1}{h_\eta^2}\frac{\partial^2\Phi_{ij}}{\partial\eta^2} + \frac{1}{h_\xi^2}\frac{\partial\ln(h_\eta h_\zeta)}{\partial\xi}\frac{\partial\Phi_{ij}}{\partial\xi} + \frac{1}{h_\eta^2}\frac{\partial\ln(h_\zeta h_\xi)}{\partial\eta}\frac{\partial\Phi_{ij}}{\partial\eta}\right]\times$$
$$\times\left[W^2 - \frac{1}{h_\xi^2}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2 - \frac{1}{h_\eta^2}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2\right]$$

$$=\frac{2}{\gamma-1}\left\{\frac{1}{h_\xi^4}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2\frac{\partial^2\Phi_{ij}}{\partial\xi^2}+\frac{1}{h_\eta^4}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2\frac{\partial^2\Phi_{ij}}{\partial\eta^2}\right.$$
$$+\frac{2}{h_\xi^2h_\eta^2}\frac{\partial\Phi_{ij}}{\partial\xi}\frac{\partial\Phi_{ij}}{\partial\eta}\frac{\partial^2\Phi_{ij}}{\partial\xi\partial\eta}-\left[\frac{1}{h_\xi^4}\frac{\partial \ln h_\xi}{\partial\xi}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^3+\frac{1}{h_\eta^4}\frac{\partial \ln h_\eta}{\partial\eta}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^3\right.$$
$$\left.\left.+\frac{1}{h_\xi^2h_\eta^2}\frac{\partial \ln h_\xi}{\partial\eta}\left(\frac{\partial\Phi_{ij}}{\partial\xi}\right)^2\frac{\partial\Phi_{ij}}{\partial\eta}+\frac{1}{h_\xi^2h_\eta^2}\frac{\partial \ln h_\eta}{\partial\xi}\frac{\partial\Phi_{ij}}{\partial\xi}\left(\frac{\partial\Phi_{ij}}{\partial\eta}\right)^2\right]\right\}. \quad (16)$$

These space lines coincide with those of intersection of the (**V**, **Ω**) surfaces and the isentropic ones, being defined by d**R** || **Ş** − Selescu's *roto-viscous* vector $\mathbf{Ş} \equiv (\mathbf{\Omega} \times \mathbf{V}) \times \nabla S$. But the most general case for Eq. (15) to represent the isentropic (zero-work) surfaces of the general steady rotational flow of a viscous Newtonian compressible fluid corresponds to the existence of a special virtual elementary displacement d**R** to satisfy this equation. So if both terms of the sum in the left-hand side have the same modulus and sign: $|(\mathbf{\Omega}\times\mathbf{V})\cdot d\mathbf{R}| = |(\mu_1/\rho)[(\mu_2/\mu_1-2/3)I_1^2+2I_2]dt|$, and $\mathrm{sign}[(\mathbf{\Omega}\times\mathbf{V})\cdot d\mathbf{R}] \equiv \mathrm{sign}\{(\mu_1/\rho)[(\mu_2/\mu_1-2/3)I_1^2+2I_2]dt\}$, in order to assure the annulment of this sum, one has to change the sense of d**R** (and so the sign of the first term of the sum), obtaining a mutual annulment of both terms, ($dS_2 = -dS_1$), therefore satisfying Eq. (15): TdS = 0 ($T\nabla S_1$ is always > 0, but the sign of TdS_1 depends on d**R**). This $d\mathbf{R}_i$ describes the searched for "i" virtual isentropic surface. But over these surfaces, even isentropic, the considered flow does not admit a velocity quasi-potential Φ_i (like over the (**V**, **Ω**) ones in the previous case). In subsection 1.4 of papers [2], [4] the dependence of gas particle specific entropy S on velocity "quasi-potential" Φ was established in the form below (see Eq. (5) in [2], [4]), valid assuming that the velocity vector **V** is derived from a velocity potential Φ_i only (also see Eq. (14)):

$$dS=-\frac{1}{2T}\frac{\partial}{\partial\nu}\left[\left(\frac{\partial\Phi}{\partial\lambda}\right)^2+\left(\frac{\partial\Phi}{\partial\mu}\right)^2\right]d\nu \quad ;$$

$$\frac{\gamma-1}{\gamma\mathcal{R}}S=\int\frac{\partial}{\partial\zeta}\ln\left[W^2-\left(\frac{\partial\Phi}{\partial\lambda}\right)^2-\left(\frac{\partial\Phi}{\partial\mu}\right)^2\right]d\zeta=2\int\frac{\partial \ln a}{\partial\zeta}d\zeta \quad ;$$

$$\Omega_\zeta=\frac{1}{h_\xi h_\eta}\left[\frac{\partial(h_\eta V_\eta)}{\partial\xi}-\frac{\partial(h_\xi V_\xi)}{\partial\eta}\right]=0\text{, meaning that}$$

$$V_\xi=\frac{\partial\Phi}{\partial\lambda}\ ;\quad V_\eta=\frac{\partial\Phi}{\partial\mu}\ ;\quad V_\zeta=\frac{\partial\Phi}{\partial\nu}=0\text{, (see Eq. (14))}$$

where λ and μ are the orthogonal arcs lengths in a smart intrinsic triorthogonal system Oλμν (or Oξηζ) tied to the (**V**, **Ω**) sheets, so that Eq. (16) of Φ_{ij} is the single possible 2-D velocity quasi-potential PDE to be written.

12 Model extension to the magnetically inviscid (with no magnetic viscosity) MHD

All this analysis can be extended to the magnetically inviscid MHD. The results obtained in studying the local physical phenomena in fluid mechanics & MHD (searching for their main hidden features) in a new smart intrinsic coordinate system were presented in [39], and the last researches performed in this field were given in [40], both as plenary lecture, representing *a real "physiology" of the fluid medium*, treating Crocco's, motion, continuity, flow rate, velocity potential and vorticity equations (and, by analogy, that of "magnetic induction" in MHD).

The equation of the isentropic surfaces is (two cases):

1. $TdS = [(\mathbf{\Omega} \times \mathbf{V}) - (1/4\pi\rho)\cdot(\mathbf{j} \times \mathbf{H})]\cdot d\mathbf{R} = 0$, for an inviscid fluid, **V** and **Ω** being now local mean (of the ionized fluid – plasma – component particles: neutral atoms, cations and anions of a single species) vectors (see subsection 2.2 in [2], sections 2 in [41], [42], and 3 in [43]):

$\mathbf{V} = (\rho_a\mathbf{V}_a + \rho_+\mathbf{V}_+ + \rho_-\mathbf{V}_-)/\rho$, where: $\mathbf{V}_a$, $\mathbf{V}_+$, $\mathbf{V}_-$ are the velocities of the components; ρ_a, ρ_+, ρ_- are the densities of the components: $\rho_a = n_am_a$; $\rho_+ = n_+m_+$; $\rho_- = n_-m_-$;

$\rho = n_am_a + n_+m_+ + n_-m_- = \rho_a + \rho_+ + \rho_-$ – plasma density (analogously to the case of a mixture of components);

m_a is the mass of a neutral atom ($m_a = m_+ + m_-$);

m_+ is the mass of the positive ion (a single species);

m_- is the mass of the negative ion or of electron;

n_a is plasma concentration in neutral atoms, and, respectively (for a three-component neutral plasma):

n_+, n_- are the concentrations in positive and negative particles (a single species of cations and anions; in the case of a quasi-neutral plasma: $n_+ \approx n_-$), all according to a simplified model proposed by the author (no collision).

Therefore the flow (mean) vorticity is given by:

$\mathbf{\Omega} = \nabla \times \mathbf{V} = \nabla \times [(\rho_a\mathbf{V}_a + \rho_+\mathbf{V}_+ + \rho_-\mathbf{V}_-)/(\rho_a + \rho_+ + \rho_-)]$;

H is the strength of the local magnetic field, using the same convention of equivalence (see for reference [38], [44] − [46]) to the magnetic induction **B** (as a rule variable in time), with $\nabla\mathbf{H} = 0$ ($\mathbf{H} = \nabla \times \mathbf{W}$ − a solenoidal field);

j is the density of the conduction electric current (see Maxwell's 2nd equation in [47]):

$\mathbf{j}/k = \nabla \times \mathbf{H} - 1/c\cdot\partial\mathbf{E}/\partial t = \nabla \times \mathbf{H} + 1/c^2\, \partial/\partial t(\mathbf{V} \times \mathbf{H})$; $k = c/4\pi$;

E is the intensity of the local induced electric field (also variable in time); *c* is the light speed in a vacuum. The low-frequency Ampère's law neglects displacement current, the density of conduction current becoming thus: $\mathbf{j} = k\nabla \times \mathbf{H}$;

2. $TdS = \{[(\mathbf{\Omega} \times \mathbf{V}) - (1/4\pi\rho)\cdot(\mathbf{j} \times \mathbf{H})]\cdot\mathbf{V}_v + \mu_1/\rho\cdot[(\mu_2/\mu_1 - 2/3)I_1^2 + 2I_2]\}\cdot dt = 0$, respectively, for a viscous fluid (see, e.g.: [2], [42]), but in the case of magnetically inviscid MHD (with *no magnetic viscosity*).

$\mathbf{V}_v$ is the virtual velocity of a neutral atom or of a small electroconducting fluid particle, given by: $d\mathbf{R} = \mathbf{V}_v dt$;

μ_1 is the *dynamic viscosity* of the fluid or *the coefficient of internal friction*; μ_1/ρ is the fluid *kinematic viscosity*;

μ_2 is the *second*, or *bulk*, *viscosity* of the fluid, assuming that μ_1 and μ_2 are constant (mean values);

I_1, I_2 are the mean values of first and second invariant of the tensor of the particle deformation rate (see section 10).

13 Conclusions and remarks

In this paper *the expanded form of Steichen's PDE of the velocity potential* in a certain triorthogonal curvilinear coordinate system *was established*, for both steady and unsteady motion of an inviscid compressible fluid.

Choosing a smart intrinsic system $O\xi\eta\zeta$ tied to the isentropic surfaces $(\mathbf{V}, \mathbf{\Omega})_i$, *some simpler forms of the equation were obtained, even for the rotational flows*, using a 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta)$. Over flow's polytropic surfaces the fluid has a quasi-barotropic behavior. So, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta)$ satisfying Steichen's PDE over some rigid virtual "i" isentropic surfaces, and even more, one can find a similar 2-D velocity "quasi-potential" $\Phi_{ij}(\xi, \eta)$ satisfying Laplace's PDE $\Delta\Phi_{ij} = 0$ (so a harmonic function Φ_{ij} along some rigid virtual "ij" isentropic and polytropic space curves (intersection lines of "i" isentropic surfaces with "j" polytropic ones) – a quasi-incompressible quasi-potential behavior in a *rotational pseudo-flow of a compressible fluid*. For any unsteady rotational motion of an inviscid compressible fluid one can find a new 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta, t)$ which satisfies Steichen's PDE over some time-deformable virtual "i" isentropic surfaces. Simpler forms of all above equations can be obtained using the orthogonal elementary arc lengths: $d\lambda = h_\xi d\xi$; $d\mu = h_\eta d\eta$ ($d\nu = h_\zeta d\zeta = 0$ – isentropic sheet), without any Lamé's coefficient at the denominators.

So, along the streamlines of a 3-D steady rotational flow of an inviscid compressible fluid the motion is given by an ODE of the 2nd order for the 1-D velocity quasi-potential $\Phi_i(\xi)$, while along the intersection lines of the isentropic virtual "i" surfaces with the isothermal & isotachic (polytropic) virtual "j" surfaces (the flow's Laplace lines) the motion is described by a Laplace's PDE for the 2-D velocity quasi-potential $\Phi_{ij}(\xi, \eta)$.

Therefore, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta)$ satisfying Steichen's PDE over some virtual "i" isentropic surfaces, and even more, one can find another 2-D velocity "quasi-potential" $\Phi_{ij}(\xi, \eta)$ satisfying Laplace's PDE (so a harmonic function $\Phi_{ij}(\xi, \eta)$) along some virtual "ij" isentropic and polytropic space curves (that is the intersection lines of the "i" isentropic surfaces with the "j" polytropic ones), *meaning an elliptic PDE* (its canonical form: $\Delta\Phi_{ij}(\lambda, \mu, \nu_{0i}) = 0$), *irrespective of the pseudo-flow character* (subsonic or supersonic).

Their direction in space is given by the local Selescu's isentropic & isotachic elementary virtual displacement vector ($d\mathbf{R}_{ij} = k_1 \cdot \nabla S|_i \times \nabla V|_j$ mainly). Through any point of the rotational flow of an inviscid compressible fluid always passes such an incompressible "Laplace" "ij" line (isentropic & isotachic), except for the case when both surfaces are identical, so appearing an undetermined solution.

Choosing the smart intrinsic coordinate ξ just along flow's streamlines (assumed as being known), Steichen's PDE becomes an ODE of the 2nd order, therefore simpler, and so the streamlines are *just the characteristic lines*.

More, one can find another 2-D velocity "quasi-potential" $\Phi_{ij}(\xi, \eta)$ satisfying the wave (vibrating string) PDE along other virtual "ij" isentropic and polytropic space curves (that is the intersection lines of the "i" isentropic surfaces with the "j" polytropic ones), *meaning a hyperbolic PDE* (its canonical form: $\partial^2\Phi_{ij}(\lambda, \mu, \nu_{0i})/\partial\lambda\partial\mu + \cdots$ (lower order terms) $= 0$), *irrespective of the pseudo-flow character*. Concluding: ***the analysis for a steady flow case*** in sections 6 and 7 ***is independent on the value of*** *a* (the speed of sound).

Along the intersection lines of the particular isentropic sheets ($(\mathbf{V}, \mathbf{\Omega})_i$ sheets with "j" circular cones; $dS_1 = dS_2 = 0$, and so $dS = 0$) only, the steady rotational flow of a viscous Newtonian compressible fluid is governed by Steichen's PDE of a 2-D velocity quasi-potential also $\Phi_{ij}(\xi, \eta, \zeta_{0i}) \equiv \Phi_{ij}(\lambda, \mu, \nu_{0i})$ – a line potential; $\zeta = \zeta_{0i}$ is a $(\mathbf{V}, \mathbf{\Omega})_i$ sheet, admitting this Φ_i.

***Remarks*:** 1. The difference between a potential function Φ and a 2-D "quasi-potential" Φ_i one consists in that the first function is valid everywhere (at any point in space), whereas the second one is valid over a family of some "i" surfaces $\zeta = \zeta_{0i}$ only. While $\mathbf{V} = \nabla\Phi$ is a gradient, $\mathbf{V}_i = \nabla\Phi_i$ is not. For the first one: $\mathbf{\Omega} = \nabla \times \mathbf{V} = \nabla \times \nabla\Phi = 0$, whereas for the 2nd one: $\mathbf{\Omega}_i = \nabla \times \mathbf{V}_i = \nabla \times \nabla\Phi_i = \mathbf{k}_\xi\Omega_{\xi i} + \mathbf{k}_\eta\Omega_{\eta i} \neq 0$, being contained in the plane tangent to the "i" surface $\zeta = \zeta_{0i}$, only its component directed along the normal to this surface becoming zero: $\Omega_{\zeta i} = (1/h_\xi h_\eta)\cdot(\partial^2\Phi_i/\partial\xi\partial\eta - \partial^2\Phi_i/\partial\eta\partial\xi) = 0$.
2. Sections 6, 7, 10 and 11 are dedicated to the analysis of some interesting particular forms of the 2-D velocity quasi-potential PDE of a certain rotational "flow", forms taken along some *space lines* different from the streamlines, but *having a physical-mathematical property kept along them*, so that a more appropriate term would be "*pseudo-flow*".
3. If a PDE has coefficients that are not constant, it is possible that it will not belong to any of the categories: elliptic, hyperbolic, parabolic, but rather be of *mixed type*. Besides Steichen's PDE, a simple but important example is the Euler–Tricomi equation, used in the investigation of transonic flow: $\partial^2u/\partial x^2 = x\partial^2u/\partial y^2$, which is called *elliptic-hyperbolic* because it is elliptic in region $x < 0$, hyperbolic in region $x > 0$, and degenerate parabolic on the line $x = 0$.
4. All results here obtained substantiate the existence of some 2-D quasi-potentials (velocity's Φ and magnetic Ξ) even for rotational flow, and more, for viscous fluid flow. Not all isentropic surfaces allow introducing such a quasi-potential, but the envelope sheets of the local planes $(\mathbf{V}, \mathbf{\Omega})$ for Φ, and $(\mathbf{H}, \mathbf{j})$ for Ξ (using the low-frequency Ampère's law $\mathbf{j}/k = \nabla \times \mathbf{H}$, neglecting the displacement current) only. So we can write: $\mathbf{V} = \nabla\Phi$ and $\mathbf{H} = \nabla\Xi$. Another 2-D quasi-potential (vorticial X – the Greek "Chi") was introduced ($\mathbf{\Omega} = \nabla X$), over the $(\mathbf{\Omega}, \mathbf{\Theta})$ sheets ($\mathbf{\Theta} = \nabla \times \mathbf{\Omega}$), to study the vorticity equation for a viscous incompressible fluid ([5], [7]).

All quasi-potentials Φ satisfy Steichen's PDE. The validity domains for these PDEs – the "ij" space lines (intersection lines of two particular isentropic sheet families (section 11) and along Selescu's *magnetohydrodynamic* vector lines: $d\mathbf{R} \parallel \mathbf{\$}$, with $\mathbf{\$} \equiv (\mathbf{V}, \mathbf{\Omega}) \cap (\mathbf{H}, \mathbf{j}) \equiv (\mathbf{\Omega} \times \mathbf{V}) \times (\mathbf{j} \times \mathbf{H})$ in inviscid MHD (see [2], [41], [48]), allowing to introduce both 2-D quasi-potentials Φ and Ξ), and the "i" surfaces – were theoretically established (their existence was firmly predicted). Finding them effectively is a very difficult job and is not the subject matter of this work. The solid and solidifiable (like in the case of confluent flows) boundaries of the flow domain are always $(\mathbf{V}, \mathbf{\Omega})$ surfaces. There also is a 2-D "quasi-stream function" Ψ_c defined in section 2 of [49], satisfying a similar PDE as the 2-D "quasi-potential" Φ does, the interdependence $\Phi_i \leftrightarrow \Psi_{ci}$ being given in section 3 of [49], on the $(\mathbf{V}, \mathbf{\Omega})_i$ (isentropic) sheets: $V_{\xi i} = \partial\Phi_i/\partial\lambda = (\partial\Psi_{ci}/\partial\mu)/\rho$; $V_{\eta i} = \partial\Phi_i/\partial\mu = -(\partial\Psi_{ci}/\partial\lambda)/\rho$. So, their gradient vector lines are orthogonal each other: $\nabla\Phi_i \cdot \nabla\Psi_{ci} = (\partial\Phi_i/\partial\lambda)(\partial\Psi_{ci}/\partial\lambda) + (\partial\Phi_i/\partial\mu)(\partial\Psi_{ci}/\partial\mu) = 0$. Finally we can cite a remarkable sentence from [50]: "*There is nothing more practical than a good theory*".

Notes

This paper (the last in a series dedicated to the intrinsic analytic study of the basic equations in compressible fluid mechanics) is fully original, however having as starting point another one with almost the same title (see [51]). It addresses to researchers in higher mathematics and fluid dynamics, MHD included. All authors use further Steichen's PDE of the velocity potential for true potential flows only.

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On December 16, 2010, the Romanian Academy (proposal from Ioan Anton, the only specialist in Fluid mechanics and MHD in the Section of Technical Sciences) awarded the "Traian Vuia" prize for year 2008 to this author, for a group of 7 papers (entitled "*Intrinsic analytic study on the isoenergetic flow of a compressible fluid*"); also see the hyperlinks: http://www.acad.ro/com2010/pag_com10_1216.htm; http://www.acad.ro/com2010/doc/2010ListaPremi2008.doc; http://www.incas.ro/images/stories/events_2010/2010ListaPremi2008.doc; http://www.incas.ro/index.php?option=com_content&view=article&id=189&Itemid=75 and http://www.incas.ro/images/stories/events_2010/Premiul%20Traian%20Vuia_Richard%20Selescu.jpg.

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