The Velocity Potential PDE in an Orthogonal Curvilinear Coordinate System

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Abstract: This work studies and clarifies some local physical phenomena in fluid mechanics, in the form of an intrinsic analytic study, regarding the PDEs of the velocity potential and (especially) 2-D “quasi-potential” (their simpler and special forms), over the “isentropic” or the 3-D (V, Ω) surfaces and along the “isentropic & isotachic” space curves, written for any potential and even rotational flow of an inviscid compressible fluid for both steady and unsteady motions. It continues a series of works presented at some conferences and a congress during 2006 – 2012, representing a real deep insight into the still hidden theory of the isoenergetic flow. Applying the advantages offered by the special virtual surfaces (“isentropic” and “polytropic”) and space curves (intersection lines of these surfaces) introduced in the previous works, a simpler PDE of the 2nd order in only two variables, and more, a Laplace’s PDE (for any rotational “pseudo-flow”, using a new smart intrinsic coordinate system), instead of the general PDE (Steichen, 1909, for plane potential supersonic flows only) of the 2nd order in three variables. So far, this equation was known as being written for potential flows only. A model extension for rotational flows of a viscous compressible fluid was given.

Key-Words: rotational flows; steady and unsteady flows; inviscid and viscous fluids; compressible fluids; isentropic and polytropic surfaces; Selescu’s isentropic & isotachic vector (dR_i), quasi-Laplace lines (quasi-isothermal quasi-potential)

1 Steichen’s vector equation; nomenclature

Joining the continuity and physical equations to the motion equation (Euler) for an inviscid compressible fluid steady irrotational (isentropic) flow (of a small fluid particle), and using the local speed of sound a definition, one gets:

\[ \nabla V = \frac{V}{2a^2} \nabla \left( V^2 \right) \], with \( V = \nabla \Phi \), so obtaining (see [1])

\[ \Delta \Phi = \frac{2a^2}{2a^2} \nabla \left( \nabla \Phi \right) = \Delta \Phi \] (Steichen), where, symbolical ly:

\[ \Delta = \frac{1}{3} \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial}{\partial x_i} \left( \left( \frac{3}{h_i} \right) \frac{1}{h_i} \frac{\partial}{\partial x_i} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \], and

\[ V = \sum_{i=1}^{3} \frac{k_i}{h_i} \frac{\partial}{\partial x_i} = k_i \frac{\partial}{\partial x_i} + k_i \frac{\partial}{\partial y_i} + k_i \frac{\partial}{\partial z_i} - \text{del (nabl)} \] (Laplace’s and Hamilton’s operators, respectively), in a triorthogonal system of curvilinear coordinates \( x_i \); \( k_i - 3-D \) basis (versions of \( Ox_i \) axes); \( h_i - \text{Lamé’s coefficients}; \ V - \text{the local velocity of translation (of the fluid particle)} - \text{intensity of the local fluid field}; \ V = |V|; \ \Phi - \text{the velocity potential}; \ \Omega = \frac{dp}{d\rho} = \gamma \mathcal{R} T, \text{with } \gamma, \mathcal{R} - \text{constants} (\gamma, \mathcal{R}, p, \rho \text{ are defined later}); \ S - \text{the specific entropy of the fluid particle}; \ T - \text{the static temperature (absolute) of the fluid particle.} \)

For a general (rotational) inviscid flow we also introduce: \( \Omega = \nabla \times V = 2 \omega \) – the vorticity, with: \( \omega \) – the local velocity of rotation (of the small fluid particle); \( i_0 = i + V^2/2 \) – the total specific enthalpy.

2 Steichen’s PDE of the velocity potential in an orthogonal curvilinear coordinate system

So far, to the best of the author’s knowledge, nowhere in the world literature, except for some particular cases (like the cylindrical and spherical coordinate systems), this equation was expressed. This author established Steichen’s PDE for a steady (and then unsteady) flow’s velocity potential in its general expanded form ([2] – [5]), considering a curvilinear coordinate system \( Ox_\xi \) (with \( h_\xi, h_\eta, h_\zeta - \text{Lamé’s coefficients} ). The vector form of this equation (usually written for irrotational flows only) can be expanded as follows (now for rotational flows also):

\[ \nabla V_i = \frac{V_i}{2a^2} \nabla \left( V_i^2 \right), \text{ with } V_i = \nabla \Phi \text{, but now } \Omega \neq 0; \]

\[ \Rightarrow \Delta \Phi = \frac{2a^2}{2a^2} \nabla \left( \nabla \Phi \right), \text{ or } \Delta \Phi = \frac{1}{\alpha} \left( \nabla \Phi \cdot V \right) \]

where \( \Delta \Phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \ \text{and, symbolically} \)

\[ \left( \nabla \Phi \cdot V \right) = \left( \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} \right) \]

\[ = \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \]

\[ + 2 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} + 2 \frac{\partial \Phi}{\partial y} \frac{\partial \Phi}{\partial z} + 2 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial z} \]
of this new PDE surfaces are analogous to D. Bernoulli’s (Lamb’s) ones ([2] – [5]) and section 10). The speed of sound \(a_i\) is given by
\[
a_i^2 = \frac{1}{h_i h_j} \left[ \frac{\partial}{\partial y_i} \left( \frac{h_j h_k}{h_i} \frac{\partial \Phi}{\partial y_k} \right) + \frac{\partial}{\partial y_j} \left( \frac{h_i h_k}{h_j} \frac{\partial \Phi}{\partial y_k} \right) + \frac{\partial}{\partial y_k} \left( \frac{h_i h_j}{h_k} \frac{\partial \Phi}{\partial y_k} \right) \right],
\]
where \(p\) – the fluid static pressure; \(\rho\) – the fluid density; \(\gamma\) – the adiabatic exponent (ratio of specific heats, \(C_p/C_v\)); \(W\) – the gas maximum speed (corresponding to the expansion to Crocco’s equation for steady isoenergetic (\(i_0 = \text{const.}; S = S_0\)) allowing to introduce a new scalar \(h\).

To express the right-hand side of Eqs. (1), (2) in a more compact form, we must prepare the scalar product in the right-hand side of Steichen’s equation, so having:
\[
\mathbf{V} \cdot \nabla (\mathbf{V}) = \mathbf{V} \cdot \nabla (\mathbf{V}^2) = \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \frac{\partial \Phi}{\partial y_i} + \frac{1}{2} \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \frac{\partial \Phi}{\partial y_j} + \frac{1}{2} \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \frac{\partial \Phi}{\partial y_k} - \frac{1}{2} \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \frac{\partial \Phi}{\partial y_i}.
\]

So, Steichen’s vector equation in section 1 becomes:
\[
\mathbf{V} = \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \mathbf{y}_i - \nabla \Phi = \sum_{i=1}^{3} \frac{\partial \Phi}{\partial y_i} \mathbf{y}_i - \left[ \frac{\partial}{\partial y_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial y_2} \right) + \frac{\partial}{\partial y_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial \Phi}{\partial y_3} \right) + \frac{\partial}{\partial y_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \Phi}{\partial y_1} \right) \right]
\]

3 The 2-D velocity “quasi-potential” PDE (for a rotational flow) in a smart intrinsic orthogonal curvilinear coordinate system

In a system of 3-orthogonal intrinsic coordinates \(O\xi\eta\zeta\) tied to the isentropic (\(V, \Omega\)) surfaces (or \(O\mu\nu\), with \(\lambda, \mu, \nu\) – lengths of the orthogonal arcs, with \(\lambda, \mu, \nu\) contained in the local plane tangent to (\(V, \Omega\)), and \(V\) directed along the normal), Laplace’s and Hamilton’s operators (\(\Delta\) and \(\nabla\)), as well as the speed of sound \(a_i\) are given by the general expressions below (\(\Phi\) depends on \(\xi, \eta, \zeta\), or on \(\lambda, \mu, \nu\), where: \(d\lambda = h_\lambda d\xi; d\mu = h_\mu d\eta\) and \(dv = h_\nu d\zeta – \) the elementary arc lengths):
\[
\Delta \Phi = \frac{1}{h_{i}h_{j}} \left[ \frac{\partial}{\partial \xi} \left( \frac{h_{i} \Phi_{j}}{\xi} \right) \right] + \frac{1}{h_{i}^{2}} \left( \frac{h_{i} \Phi_{j}}{\xi} \right) + \frac{1}{h_{i}^{2}} \left( \frac{h_{i} \Phi_{j}}{\xi} \right)
\]

\[
= \left( \prod_{k=1}^{3} h_{k} \right) \sum_{j=1}^{3} \frac{\partial}{\partial \xi} \left( \frac{h_{i} \Phi_{j}}{\xi} \right)
\]

\[
\nabla \Phi_{i} = k_{i} \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + k_{i} \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

\[
\nabla \Phi_{i} = \frac{1}{h_{i}^{2}} \sum_{j=1}^{3} \frac{\partial}{\partial \xi} \left( \frac{h_{i} \Phi_{j}}{\xi} \right) + \frac{1}{h_{i}^{2}} \sum_{j=1}^{3} \frac{\partial}{\partial \eta} \left( \frac{h_{i} \Phi_{j}}{\xi} \right)
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

4 The case of the unsteady flow (see [5])

For the unsteady flow of a compressible fluid, the velocity potential equation has the vector form below (t – the time):

\[
\nabla \Phi - \frac{V}{2a^{2}} \nabla (V^{2}) = \frac{1}{a} \left[ \frac{\partial}{\partial t} \left( \frac{\partial \Phi}{\partial t} \right) + \frac{\partial}{\partial \xi} \left( \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial \Phi}{\partial \eta} \right) \right]
\]

\[
\Delta \Phi - \frac{V}{2a^{2}} \nabla \left[ (\nabla \Phi)^{2} \right] = \frac{1}{a} \left[ \frac{\partial^{2} \Phi}{\partial t^{2}} + \frac{\partial}{\partial \xi} \left( \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial \Phi}{\partial \eta} \right) \right]
\]

\[
\nabla \Phi_{i} = k_{i} \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + k_{i} \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta} + k_{i} \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

Analogously, in Steichen’s equation – a nonlinear PDE of the order in three variables – ξ, η, and ζ (written for a rotational flow – Ω ≠ 0, but on the “i” isentropic surface ζ = ζi) all the terms containing the partial derivative with respect to ζ of the quasi-potential function Φi disappear (dΦi/dζ = 0) and its derivatives with respect to ξ, η disappear also, thus a nonlinear PDE of the second order in only two variables – ξ and η – being obtained (a simpler form for the 2-D velocity quasi-potential PDE):

\[
\nabla \Phi_{i} = \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

\[
\nabla \Phi_{i} = \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

\[
\nabla \Phi_{i} = \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]

which reduces on a certain (V, Ω) surface ζ = ζi to a simpler form for the 2-D velocity quasi-potential PDE:

\[
\nabla \Phi_{i} = \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \eta} + \frac{1}{h_{i}^{2}} \frac{\partial \Phi_{i}}{\partial \xi}
\]

\[
\nabla (V_{i}^{2}) = \nabla \left[ (\nabla \Phi_{i} )^{2} \right]
\]

\[
a^{2}_{i} = \frac{\gamma - 1}{2} \left[ W_{i}^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \xi} \right)^{2} - \frac{1}{h_{i}^{2}} \left( \frac{\partial \Phi_{i}}{\partial \eta} \right)^{2} \right]
\]
\[ \frac{2}{\gamma - 1} \left( \frac{\partial^2 \Phi_1}{\partial \xi^2} + \frac{\partial^2 \Phi_1}{\partial \eta^2} + \frac{\partial \ln h}{\partial \xi} \left( \frac{\partial \Phi_1}{\partial \xi} \right)^2 + \frac{\partial \ln h}{\partial \eta} \left( \frac{\partial \Phi_1}{\partial \eta} \right)^2 + 2 \frac{\partial \Phi_1}{\partial \xi} \frac{\partial \Phi_1}{\partial \eta} \right) \times \left( W^2 - \frac{1}{\gamma h^2} \left( \frac{\partial \Phi_1}{\partial \xi} \right)^2 - \left( \frac{\partial \Phi_1}{\partial \eta} \right)^2 \right) \]

5 Simpler forms of the 2-D velocity “quasi-potential” PDE over the \((V, \Omega)_i\) surfaces

On a certain \(v = v_0\) isentropic surface \((V, \Omega)_i\), for a steady flow, using the elementary arc lengths \(d\xi\) and \(d\mu\), one gets:

\[ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial \ln h}{\partial \xi} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \frac{\partial \ln h}{\partial \mu} \left( \frac{\partial \Phi}{\partial \mu} \right)^2 + 2 \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \mu} \left( \frac{\partial \ln h}{\partial \xi} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h}{\partial \mu} \frac{\partial \Phi}{\partial \mu} \right) \times \left[ W^2 - \frac{1}{\gamma h^2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 - \left( \frac{\partial \Phi}{\partial \mu} \right)^2 \right] \]

and the corresponding PDE of the 2-D velocity quasi-potential for the unsteady flow case becomes simpler too:

\[ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial \ln h}{\partial \xi} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \frac{\partial \ln h}{\partial \mu} \left( \frac{\partial \Phi}{\partial \mu} \right)^2 + 2 \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \mu} \left( \frac{\partial \ln h}{\partial \xi} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h}{\partial \mu} \frac{\partial \Phi}{\partial \mu} \right) \times \left[ 2C_i(t) - \frac{1}{\gamma h^2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 - \left( \frac{\partial \Phi}{\partial \mu} \right)^2 \right] \]

\[ = \frac{2}{\gamma - 1} \left( \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial \ln h}{\partial \xi} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \frac{\partial \ln h}{\partial \mu} \left( \frac{\partial \Phi}{\partial \mu} \right)^2 + 2 \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \mu} \left( \frac{\partial \ln h}{\partial \xi} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h}{\partial \mu} \frac{\partial \Phi}{\partial \mu} \right) \right) \times \left[ \right. \]

Choosing the \(\xi\) intrinsic coordinate along the streamlines direction (assumed as being known), the new PDE becomes an ODE \((\partial \Phi / \partial \xi = d \Phi / d \xi)\) of the 2nd order:

\[ \frac{d^2 \Phi}{d \xi^2} + \frac{d \Phi}{d \xi} \left( \frac{\partial \ln h}{\partial \xi} \right) + \frac{d^2 \Phi}{d \xi^2} \left( \frac{\partial \ln h}{\partial \mu} \right) + \frac{d \Phi}{d \xi} \left( \frac{\partial \ln h}{\partial \xi} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h}{\partial \mu} \frac{\partial \Phi}{\partial \mu} \right) \times \left[ W^2 - \frac{1}{\gamma h^2} \left( \frac{d \Phi}{d \xi} \right)^2 \right] \]

\[ = \frac{2}{\gamma - 1} \left( \frac{d^2 \Phi}{d \xi^2} + \frac{d \Phi}{d \xi} \left( \frac{\partial \ln h}{\partial \xi} \right) + \frac{d^2 \Phi}{d \xi^2} \left( \frac{\partial \ln h}{\partial \mu} \right) + \frac{d \Phi}{d \xi} \left( \frac{\partial \ln h}{\partial \xi} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h}{\partial \mu} \frac{\partial \Phi}{\partial \mu} \right) \right) \times \left[ W^2 - \frac{1}{\gamma h^2} \left( \frac{d \Phi}{d \xi} \right)^2 \right] \]

6 Special forms of the 2-D velocity “quasi-potential” PDE along the quasi-Laplace lines of a compressible fluid rotational flow

On a certain virtual “\(i\)” surface, for an inviscid fluid flow we have: \(\nabla(V^2/2) = (\Omega \times V)_i = -(\nabla \Phi)_i\). Performing a scalar multiplication of this relation by a virtual elementary displacement \(dR_i = (V, \Omega)_i\) (therefore an isentropic virtual surface), we obtain: \(d(V^2/2)_i = -(\partial \Phi / \partial \mu)_i\), with: \(p = K \rho^p\), so a first integrable form. On the other hand, on any polytropic virtual surface we have: \((p \rho^n)_i = (\partial \Phi / \partial \mu)_i\) or \(p = const_1 \rho^n\), and so \(\nabla \Phi = const_1 \nabla \rho^n\). Performing its scalar multiplication by a virtual elementary displacement \(dR_i\) contained in the plane tangent to the polytropic “\(j\)” surface, we obtain: \(dp_j = const_2 \rho^n \partial \Phi_j\). So, along the intersection “\(ij\)” lines (with: \(dR_i = k \nabla S_i \times \nabla (p \rho^n)_i\)) of the two surface families we have: \(d(V^2/2)_j = -(\partial \Phi_j / \partial \mu)_j = -const_3 \rho^n \partial \Phi_j\). Analogously we have: \(p = const_4 T_{(i)}^{(i+1)}\) and so: \(\nabla V = const_5 \partial (T_{(i)}^{(i+1)}) \nabla T_{(i)}^{(i+1)}\), and: \(dp_j = const_6 (T_{(i)}^{(i+1)} \partial T_{(i)}^{(i+1)})\). Taking into account that through any intersection “\(ij\)” line of the two surface families above there is a “star (pencil) of sheets” passing (for various parti-
cular values of the polytropic exponent “n”, e.g.: isentropic, isothermal & isotachic, isobaric, isochoric and general polytropic one), we can write along these lines: $dS_{ij} = d(V^2/2)_{ij} = -(dp/\rho)_{ij} = -const_1$, $d\Phi = d\Gamma = dp = 0$, or simpler: $dS_{ij} = dV_{ij} = d\Phi_{ij} = d\Gamma_{ij} = dp_{ij} = 0$, so having: $d\Phi = kSVx \times SVy = kSVx \times SVz = kSVx \times SV\Gamma = kSVx \times SV/p$. Through any point of this rotational flow, such of “star of integral sheets” is passing.

Applying a scalar-multiplication of Steichen’s-PDE of the compressible 2-D velocity “quasi-potential” $\Phi_{ij}, \nabla V_{ij}$, $V_{ij} = V_x(V_x) + d\rho/dV_{ij}$ with $V_x = \nabla \Phi_x$ by the $dR_{ij}$ above, one gets $\nabla V_{ij} = d\Phi_{ij} = 0$, and so $V_{ij} = 0$ (because $dR_{ij} \neq 0$). Introducing the scalar function $\Phi_{ij}(\xi, \eta)$ one obtains a PDE identical to Laplace’s one: $\Delta \Phi_{ij} = 0$, so $\Phi_{ij}(\xi, \eta)$ being a harmonic function. This simpler PDE is valid for a certain rotational flow ($V_{ij} = V_x \Phi_x; \Omega_{ij} = V \times V_{ij} \neq 0; (p/\rho)_{ij} = \text{const})$ – a “quasi-incompressible” fluid behavior along the intersection lines of any isotropic virtual sheet with any polytropic one. Over flow’s isothermal & isotachic virtual surfaces we have: $V = V_x = \text{const}, or \nabla V \times dR = dV = 0$ (the fluid has a “quasi-uniform” behavior). The vector equation $V \cdot V(\nabla^2/V) = 0$ (a zero-scalar product in $V$), $V \cdot V(\nabla^2/V) = 0$ (along the intersection “ij” lines above), $V \cdot V(\nabla^2/V) = 0$ (Vx = Vy = Vz = 0), representing a trivial solution: $V_x = \text{const}$, with $V_x = \text{const}$, $V_x = \text{const}$, (a hyperbolic variation law). Their directions are tangent to the concentric circles having the center in the origin, the well-known models of plane and axisymmetric flows of a viscous fluid, like Couette and Hagen–Poiseuille ones; for an extension to the viscous fluid flow see sections 10, 11).

Fig. 1. The compressible fluid flow pattern (the velocity distribution) due to a straight infinite vortex filament – the free (irrotational) vortex plane flow

The vector field given in figure 1 corresponds to the velocity field due to a straight infinite vortex filament directed along the z-axis (normal to the x, y-plane):

$$V(x, y, z) = \frac{\Gamma}{2\pi} \left[ -k_x, \frac{y}{x^2 + y^2} + k_y, \frac{x}{x^2 + y^2} + k_z \cdot 0 \right].$$

$\Gamma$ is the vortex constant intensity (the velocity circulation). The lengths of the velocity vectors induced by the vortex filament are proportional to their intensities (moduli) $|V(x, y, z)| = V = \Gamma/[2\pi \cdot (x^2 + y^2)^{1/2}] = \Gamma/(2\pi R)$ (a hyperbolic variation law). Their directions are tangent to the concentric circles having the center in the origin, and their senses are represented by arrowheads, corresponding to a left-hand screw (counter-clockwise). Along a certain circle (streamline) $R = (x^2 + y^2)^{1/2} \cdot \text{const.}$, the velocity vector has a constant modulus. So, these circles are the flow isothachs (isotachic lines). The critical speed $c$ (for which $V = u$) is reached along a circle of radius $R_c = \Gamma'/2\pi = \Gamma'/2\pi [(\gamma + 1)/(\gamma)_{2}(\rho u)^{1/2}]$, and the maximum

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62
speed $W$ (for which $V = W$) is reached along another circle (the wall of the solid cylindrical nucleus – the no-motion region), of minimum radius $R_w = \Gamma/2\pi W = \Gamma/2\pi [(\gamma - 1)/(\gamma + 1)]^{1/2} R_e$, so that

for the region $R_e < R < R_w$ the plane flow is supersonic ($V$ decreases from $W$ to $c$) and for the semi-infinite region $R > R_w$ the flow is subsonic ($V$ decreases from $c$ to $0$). This field includes a vortex at its center (singularity), so it is rotational. However, any simply-connected subset that excludes the vortex line will have zero curl, $\Omega = 0$ (no vorticity), the fluid particles performing circular translations only. We give below a detailed explanation of solutions 2 and 3. Intersecting the isotropic $(\nabla \times \mathbf{r} = \mathbf{0})$ virtual surfaces $S = S_0i = \text{const.} (\zeta = \zeta_0)$ with the special isothetic ones ($V = V = \text{const.})$ one obtains a family (net) of space curves along which Steichen’s PDE of the compressible 2-D velocity “quasi-potential” $\Phi_i(\zeta, \eta)$ becomes again simpler, identical to Laplace’s one: $\Delta \Phi_\eta = 0, \Phi_i(\zeta, \eta)$ being now a harmonic function.

So, the solution $V_{ij}^\eta$ of the general Steichen’s PDE $\nabla^2 V_{ij}^\eta = V_{ij}^\eta \nabla^2 (V_{ij}^\eta)/2a^2_{ij}$, written for a special rotational flow ($V_{ij}^\eta = \nabla \Phi_i; \Omega = \nabla \times V_{ij}^\eta \neq 0 ; V_{ij} = V_{ij}^\eta = \text{const.}_{ij}^\eta$), with $V_{ij}^\eta (V_{ij}^\eta(j) = 0)$, is now the solution of the system:

$$
\begin{align*}
\nabla V_{ij}^\eta &= V_{ij}^\eta \nabla^2 (V_{ij}^\eta)/2a^2_{ij} \quad \text{with} \quad V_{ij}^\eta = \nabla \Phi_i(\zeta, \eta); \\
V_{ij} \cdot \nabla (V_{ij}^\eta) &\cdot \mathbf{d} \mathbf{r}_i = 0, \quad \text{or, mainly:} \\
\Delta \Phi_i &= 0; \\
(\nabla V_{ij}^\eta)^2 &= 0,
\end{align*}
$$

this leading to $V_{ij}^\eta = 0$ and so obtaining the simpler form: $\nabla V_{ij}^\eta = \Delta \Phi_i = 0$; with $V_{ij}^\eta = \nabla \Phi_i = (\mathbf{V} \times \Psi)^G/\rho; \Psi_i^G \neq \nabla G$ – the 3-D stream function vector (Selescu, see section 7 in [7]), for the mass flux density vector $\rho \mathbf{V}_{ij}$):

$$
\begin{align*}
\Delta \Phi_i &= \frac{1}{h_i h_j h_k} \left[ \frac{\partial}{\partial \zeta} \left( \frac{h_i h_j \partial \Phi_i}{h_k \partial \zeta} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_i h_k \partial \Phi_i}{h_j \partial \eta} \right) \right]_{(\zeta = \zeta_0)} = 0, \\
or \quad &\left[ \frac{\partial}{\partial \zeta} \left( \frac{h_i h_j \partial \Phi_i}{h_k \partial \zeta} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_i h_k \partial \Phi_i}{h_j \partial \eta} \right) \right]_{(\zeta = \zeta_0)} = 0, \quad \text{or more} \\
&\left[ \frac{1}{h_i h_j h_k} \left[ \frac{\partial}{\partial \zeta} \left( \frac{h_i h_j \partial \Phi_i}{h_k \partial \zeta} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_i h_k \partial \Phi_i}{h_j \partial \eta} \right) \right] + \left[ \frac{1}{h_i h_j h_k} \left[ \frac{\partial}{\partial \zeta} \left( \frac{h_i h_j \partial \Phi_i}{h_k \partial \zeta} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_i h_k \partial \Phi_i}{h_j \partial \eta} \right) \right] \right]_{(\zeta = \zeta_0)} = 0.
\end{align*}
$$

for a steady flow, therefore an elliptic PDE, irrespective of the “pseudo-flow” character (subsonic or supersonic),

corresponding to the annulment of the first factor (inside the first square brackets) in the left-hand side of PDE (3).

For an unsteady flow Steichen’s equation takes the form

$$
\begin{align*}
\nabla V_{ij}^\eta \cdot \nabla (V_{ij}^\eta) &= 0, \\
\nabla \Phi_i \cdot (\nabla V_{ij}^\eta) &= 0,
\end{align*}
$$

so obtaining along the space curves ($S = S_0i; V = V_{ij}^\eta$, i.e. coupling the cases 4 and 5 in subsection 1.2 in [7]), for the function $\Phi_i(\zeta, \eta, t)$ the following PDE:

$$
\begin{align*}
&\Delta \Phi_i = 0; \\
&\nabla (\nabla V_{ij}^\eta)^2 = 0,
\end{align*}
$$

More, over flow’s isothermal & isothetic virtual surfaces ($V = 0$) one can write $V = \mathbf{V} \times \Psi^G$, this time $\Psi^G (\neq \nabla G)$ being a true 3-D stream function vector due to the fact that over these surfaces the fluid has an incompressible behavior.

Until now, we used only the first equation of system (4): $\Delta \Phi_i = 0$. The second equation ($d V_{ij}^\eta = 0$, or $V_{ij}^\eta = \text{const.)}$ leads to the following vector condition ($V$ lies on the isentropic “i” surface $\zeta = \zeta_0$, so being: $V = k_i V_i + k_i V_i^G$):

$$
\begin{align*}
&\frac{k_i}{h_i} \frac{\partial}{\partial \zeta} \left( V_{ij}^2 + V_{ij}^2 \right) + \frac{h_i}{\eta} \frac{\partial}{\partial \eta} \left( V_{ij}^2 + V_{ij}^2 \right) = 0, \\
&\text{and to:} \\
&\frac{\partial}{\partial \zeta} (V_{ij}^2 + V_{ij}^2) = 0 \text{ and } \frac{\partial}{\partial \eta} (V_{ij}^2 + V_{ij}^2) = 0 
\end{align*}
$$

having for consequences: ($V_{ij}^2 + V_{ij}^2 = \text{f}_{ij}^G(\zeta, \xi)$, and ($V_{ij}^2 + V_{ij}^2 = \text{f}_{ij}^G(\zeta, \xi)$), or globally: ($V_{ij}^2 + V_{ij}^2 = \text{f}_{ij}^G(\zeta)$ function of $\zeta$ only), which can be expressed as:
\[
\frac{1}{h_z^2} \left( \frac{\partial \Phi_y}{\partial x} \right)^2 + \frac{1}{h_z^2} \left( \frac{\partial \Phi_y}{\partial y} \right)^2 = f_y(\zeta), \text{ or in compact form:}
\]
\[
\left( \frac{\partial \Phi_y}{\partial x} \right)^2 + \left( \frac{\partial \Phi_y}{\partial y} \right)^2 = f_y(v), \text{ or more:}
\]
\[
\left[ \frac{\partial \Phi_y}{\partial x} \right]^2 + \left[ \frac{\partial \Phi_y}{\partial y} \right]^2 = f_y(v) = \text{const.}(v),
\]
which must be also considered for solving the system (4), besides the boundary conditions of respective rotational flow, along an “\text{ii}” space line, lying on an isentropic “\text{i}” surface and an isothermal & isotachic “\text{ji}” one.

7 Special forms of the 2-D velocity “quasi-potential” PDE along other space lines of a compressible fluid rotational flow

Intuitively, we can put the existence problem for a family of space curves passing through any point also, along which the velocity quasi-potential function \( \Phi_q \) respects another rule, e.g. the wave (vibrating string) PDE of the 2nd order (in one space dimension), instead of Laplace’s one, that means being of a hyperbolic type, just in its canonical form: \( \frac{\partial^2 \Phi_q}{\partial x^2} + \frac{\partial^2 \Phi_q}{\partial y^2} + \cdots \) (lower order terms) = 0, irrespective of the “pseudo-flow” character (subsonic or supersonic), in the same O\( \bar{\text{m}} \mu \nu \) smart intrinsic triorthogonal curvilinear coordinate system previously used. But we will treat this problem, for simplicity reasons, in the classical Cartesian system, at least for the beginning, to understand the new proposed mechanism for solving the above problem, this meaning to treat a true potential flow (throughout). In the case of a rotational flow, like previously, the searched for space curves must be obtained as being the intersection lines of two surface families: 1. the isentropic \((V, \Omega)\) ones, allowing us to introduce a 2-D velocity quasi-potential \( \Phi_q \); and 2. the new special surfaces (like previous isotachic & isothermal ones) over which another condition must be satisfied. Let us write Steichen’s equation in the form:

\[
\nabla V_x \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} - \frac{1}{a^2} \left[ V_x^2 \frac{\partial V_x}{\partial y} + V_y^2 \frac{\partial V_y}{\partial y} + V_z^2 \frac{\partial V_z}{\partial y} + V_x \frac{\partial V_y}{\partial z} + V_y \frac{\partial V_y}{\partial z} + V_z \frac{\partial V_z}{\partial z} \right] = 0,
\]

\[
\nabla V_x \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0,
\]

\[
\partial^2 V_x \partial^2 V_y + \partial^2 V_y \partial^2 V_z + \partial^2 V_z \partial^2 V_x = \frac{1}{3} \left( \frac{\partial^3 V_x}{\partial x^2 \partial y} + \frac{\partial^3 V_y}{\partial y^2 \partial z} + \frac{\partial^3 V_z}{\partial z^2 \partial x} \right)
\]

\[
= \frac{1}{3} \nabla V^3, \text{ with } V_x = k_3 V_x^3 + k_2 V_x^3 + k_1 V_x^3,
\]

thus obtaining a new special form for this equation:

\[
\nabla V - \frac{2}{\gamma - 1} \frac{1}{W^2 - V^2} \left[ \frac{1}{3} \nabla V^3 + V_x V_y \left( \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) \right] = 0, \quad (5)
\]

Now we impose (consider the possibility to satisfy): \( \nabla V = V \Omega / 3a^2 = 0 \), or \( a^2 \nabla V = V \Omega / 3 \), \( (6) \)
or in a more suggestive form: \( \nabla V y / (3V) = a^2 (V) \) – the ratio of two divergences (in this case), condition giving the new searched for special surfaces, concurrently further satisfying Steichen’s PDE (5) above:

\[
a^2 \nabla V = V \Omega (V^2 / 2), \text{ or, with that given by Eq. (6):}
\]

\[
3 \nabla V (V^2) = 2 V \Omega (\neq 0), \quad (7)
\]

(not depending on \( a \)). Let us write Eq. (7) in a scalar form:

\[
V_x V_y \left( \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) + V_y V_z \left( \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \right) + V_z V_x \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \right) = 0, \quad (7')
\]

generally valid along the intersection lines of the isentropic “\text{ii}” surfaces with the “\text{ji}” special ones satisfying Eq. (6). These lines are solutions of the system formed by Eqs. (5) and (6). If \( \nabla V = 0 \), the problem reduces to that in section 6. Introducing the velocity potential \( \Phi = V \Omega \), and \( \nabla V = \Phi \frac{\partial \Phi}{\partial x} \), \( \nabla V = \Phi \frac{\partial \Phi}{\partial y} \), \( \nabla V = \Phi \frac{\partial \Phi}{\partial z} \), \( \nabla V = \Phi \frac{\partial \Phi}{\partial x} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial y} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial z} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial x} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial y} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial z} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial x} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial y} \); \( \nabla V = \Phi \frac{\partial \Phi}{\partial z} \);

\[
a^2 \left( \frac{\partial \Phi}{\partial x} \right)^2 + a^2 \left( \frac{\partial \Phi}{\partial y} \right)^2 + a^2 \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0,
\]

e also expressing the orthogonality of the vectors \( V_2 \) and \( \delta V \), defined as follows:

\[
V_2 = k_1 (a^2 - V_x^2) + k_2 (a^2 - V_y^2) + k_3 (a^2 - V_z^2); \quad \delta V = k_1 \frac{\partial V_x}{\partial x} + k_2 \frac{\partial V_y}{\partial y} + k_3 \frac{\partial V_z}{\partial z}; \quad (V_2 \cdot \delta V = 0); \text{ or:}
\]

\[
a^2 \left( \frac{\partial \Phi}{\partial x} \right)^2 + a^2 \left( \frac{\partial \Phi}{\partial y} \right)^2 + a^2 \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0,
\]

or:

\[
W^2 - \gamma + 1 \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 - \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0,
\]

\[
W^2 - \gamma + 1 \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 - \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0,
\]

\[
W^2 - \gamma + 1 \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 - \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0,
\]

and for previous Eq. (7') (using Schwarz’ theorem):

\[
2 \left[ \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial \Phi}{\partial z} \right] = 0,
\]

just the sum of until now negligible terms with respect to \( a^2 \Delta \Phi \) in the velocity potential PDE for small perturbations.
This result is very important and leads for the case of a 2-D plane motion \((V_x = \partial \Phi / \partial z = 0)\) to the simplest equation of the potential \(\Phi\) (a very particular situation):

\[
\frac{\partial^2 \Phi}{\partial x \partial y} = 0 , \quad \text{(meaning: } \partial V_x / \partial y = \partial V_y / \partial x = 0 , \quad (8)
\]

leading to a special plane flow: \(V_x = f_1(x); V_y = f_2(y)\) and \(\Phi = \int f_1(x) \, dx + \int f_2(y) \, dy = \Phi_1(x) + \Phi_2(y)\), like that in a channel with hyperbolic walls, or a right-angled corner, representing the canonical form of a 2nd order PDE of a hyperbolic type with constant coefficients. The condition Eq. (6) becomes:

\[
\begin{aligned}
W^2 - \frac{\gamma + 1}{\gamma - 1} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \frac{2}{\gamma - 1} \left( \frac{\partial \Phi}{\partial y} \right)^2 & \frac{d \left( \frac{\partial \Phi}{\partial x} \right)}{dx} + \frac{d \left( \frac{\partial \Phi}{\partial y} \right)}{dy} = 0 , \quad \text{or :} \\
W^2 - \frac{\gamma + 1}{\gamma - 1} \left( \frac{d \Phi_1}{d x} \right)^2 & + \frac{1}{\gamma - 1} \left( \frac{d \Phi_2}{d y} \right)^2 = 0 , \quad \text{or :}
\end{aligned}
\]

\[
\begin{aligned}
W^2 - \frac{\gamma + 1}{\gamma - 1} \left( V_x^2 - V_y^2 \right) & \frac{dV_x}{dx} + \left( W^2 - \frac{\gamma + 1}{\gamma - 1} V_y^2 - V_x^2 \right) \frac{dV_y}{dy} = 0
\end{aligned}
\]

– a first order ODE with two unknown functions.

The same results can be obtained for a 2-D axisymmetric motion (in a meridian plane, in cylindrical coordinates), leading to another special flow: \(V_r = f_1(r); V_\theta = f_2(\theta)\) and (the most important), for a rotational flow of a compressible fluid (over the isentropic surfaces, in the smart intrinsic triorthogonal curvilinear coordinates), instead of velocity’s potential \(\Phi\) appearing its 2-D quasi-potential \(\Phi_1\). In the last case the relation (8) is more intricate, due to versors direction variation (see the right-hand side of the last equation in section 2), but the essential is the same, it being satisfied only along the space curves – intersection lines of the “s” isentropic surfaces with the “t” special ones, irrespective of the “pseudo-flow” character (not depending on \(\alpha\)).

Concluding, as regards the PDE of the velocity quasi-potential \(\Phi_1\), through any point of a certain rotational flow two interesting space curves are always passing:

1. the quasi-Laplace line (an elliptic PDE) – see section 6; 2. the new special line leading to a hyperbolic PDE. Analogously we can suppose the existence of a third family of space curves passing through any point of a certain rotational flow, along which the velocity quasi-potential \(\Phi_1\) respects another rule, corresponding this time to a PDE of the 2nd order of a parabolic type, just in its canonical form:

\[
\frac{\partial \Phi_1}{\partial x_1 \partial x_2} \partial t^2 + \cdots \text{ (lower order terms) } = 0 , \quad \text{in the same } \Omega x y v \text{ smart intrinsic coordinate system, irrespective of the “pseudo-flow” character also.
}\]

The analysis in sections 6 & 7 can be useful for other cases where the physical phenomena lead to PDEs of the second order of a mixed type.

### 8 A simple example: the 3-D conical flow

An interesting example of application was given in [8], establishing the general ODE (I) of the velocity quasi-potential \(\Phi\) = \(R V(\varphi, \psi)\) = \(R V(\varphi)\) – on every “s” isentropic sheet \(\chi = \text{const}\) for any (rotational, therefore non-isentropic) 3-D conical flow, writing and analyzing it in the smart intrinsic generalized spherical (conical) triorthogonal curvilinear coordinate system \((R, \varphi, \chi)\):

\[
(\gamma - 1) \left[ V_{\varphi}^2 + [\ln \left( f(\varphi) \right)] \cdot V_{\gamma}^2 + 2V_r V_{\varphi} \right] (W^2 - V_r^2 - V_{\varphi}^2) - 2(V_r + V_{\varphi}) V_{\gamma}^2 = 0 , \quad \text{with } \gamma = d/\varphi \text{ and } \gamma' = d^2/\varphi^2 \text{, instead of the usually written (approximate) PDE of the velocity potential } \Phi = RV(\vartheta(\theta), \omega) \text{ with } V = V_r + V_{\gamma} (Helmholtz where: } V_r = \Phi_{\varphi} \text{ and } V_{\gamma} = \Phi_{\gamma} \text{ (see [9]), V}_\gamma = \Phi_1 \text{.}
\]

so obtaining:

\[
\left[ 1 - \frac{1}{a^2} \left( \frac{\partial V_r}{\partial \varphi} \right)^2 \right] \frac{\partial V_r}{\partial \varphi} + \left[ 1 - \frac{1}{a^2} \sin^2 \theta \right] \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \theta} + \left[ 1 - \frac{1}{a^2} \sin^2 \theta \right] \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \theta} + \left[ 1 - \frac{1}{a^2} \sin^2 \theta \right] \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \theta} + \left[ 1 - \frac{1}{a^2} \sin^2 \theta \right] \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \theta} = 0 , \quad \text{with: } \gamma = \frac{1}{2} \left( W^2 - V_r^2 - V_{\gamma}^2 \right) \approx \frac{1 - \frac{1}{2}}{2} \left( W^2 - V_r^2 - V_{\gamma}^2 \right) = \frac{1 - \frac{1}{2}}{2} \left( W^2 - V_r^2 \right) \frac{dV_r}{dx} - \frac{1}{\sin \theta} \frac{dV_r}{d\theta} , \quad \text{(see,e.g.:[10])}
\]

so a nonlinear PDE of the 2nd order in two variables (the spherical coordinates \(\vartheta, \omega\)), obtained by throughput neglecting the \(V_r\) components \((V_{\varphi}, V_{\omega}, V_{\varpi})\), so being an approximate PDE, though being an exact one when it is written on the isentropic surfaces \((V, \Omega, \zeta) = \text{const}\) (containing streamlines & vortex lines) only, becoming now identical to the general ODE of the velocity quasi-potential: \(V_\gamma\) – the rotational part of \(V\); \(V_r(\varphi) = V_\varphi\) (the radial component of the velocity on the conical isentropic sheet “\(\gamma\)”). The quasi-potential is: \(\Phi_1(\varphi) = R V_r\), also having: \(R_1/R_\omega = V_{\varphi}/V_r\) (see [11]), getting by integration: \(R_1 = R(\varphi)\), representing the polar equation of all flow streamlines contained in the isentropic surface “\(\gamma\)” (all being homothetic space curves, their family being given by the polar equation: \(R_1 = R_1(\varphi)\), for various values of the positive constant \(R_0\), with \(R_1(\varphi)\) – a non-dimensional quantity depending on \(\varphi\) only).

The generalized spherical coordinate \(\varphi\) is defined by:

\[
\varphi = \sqrt{2} \left( \frac{d\varphi + \sin \theta \, d\omega}{\sin \theta} \right)^2 = \left[ \sin \theta \left( dtan(\theta/2) \right)^2 + \sin \theta \right]^{1/2} \quad (9) \quad \text{– the relation with the classical spherical coordinates,}
\]
In the smart intrinsic orthogonal curvilinear coordinate system: 

$$\dot{\phi}^2 = \theta^2 + \sin^2 \theta \cdot \hat{o} \hat{\theta}^2.$$

(10)

In the smart intrinsic orthogonal curvilinear coordinate system: 

$$\beta = \left[ \left( \phi^2 + \hat{f}(\theta) d\theta^2 \right)^{1/2} \right]$$

which on \( \chi = \chi_{0i} = \text{const} \), becomes: \( \beta = \phi \).

The general ODE above is similar to the classical Taylor–Maccoll one ([12], [13]) for an axisymmetric conical (supersonic) rotational flow, this being a particular case obtained for: \( \phi = 0 \) and \( \hat{f}(\theta) = \theta = 0 \), and so: 

$$[\ln(\Phi)]" = [\ln(\sin \theta)]" = 0 \text{, the equation becoming:}$$

$$\left( \gamma - 1 \right) \left( \theta^2 + \mu^2 \right) \left( \Phi^{\alpha} \right) + 2 \theta^2 \left( \theta^2 - \Phi^{\alpha} \right) - 2 \left( \theta^2 + \mu^2 \right) = 0$$

and the sliding condition [11]: \( R \theta^* = \theta V \), with: \( \theta = d \theta \) and the solution \( \theta(i) = R - \theta e \) valid throughout the flow domain (the velocity quasi-potential \( \Phi(\phi) \) coincides in this case with the velocity potential \( \Phi(\theta) \)).

Along the “vector radii” half-straight lines one gets a 2-D Laplace’s PDE, with the first order \( n = 1 \), having also a physical significance: 

$$\Phi = \text{const} \text{, this being a conical scalar “quasi-potential” of the n-th order (} \chi = \chi_{0i} \text{, becomes:}$$

$$\phi = \phi = \text{const}.$$ 

$$\Phi = \Phi = \text{const}.$$ 

This means the searched “ij” line is a half-straight line passing through the point of flow’s axis and the considered point of flow’s axis.

Along any such a half-straight line searched for velocity quasi-potential \( \Phi_{ij}(\phi, \chi) \) becomes a harmonic function (the general non-linear Steichen’s PDE of the 2nd order becoming a 2-D Laplace’s PDE: \( \Delta \Phi_{ij} = 0 \) – with the solution: 

$$\Phi_{ij}/R = V_{ij} = \text{const}_{ij}(\phi, \chi).$$

The annulment of the radial component of the acceleration vector, \( a_R = 0 \) (the first equation of motion: \( \dot{R} - R \dot{\phi}^2 = 0 \), is equivalent to the “sliding condition”, expressing the condition that the velocity \( V \) be tangent to a certain streamline of the conical flow – the scalar product \( V \cdot n = 0 \), where \( n \) is the unit vector of the normal to the streamline, having the components \( R/(R^2 + R^2)^{1/2} \) and \( R/(R^2 + R^2)^{1/2} \), respectively. It can be written in the forms: 

$$R^2 V' = R' V = \Phi(\phi), \quad R' V' = V'$$

' = d \theta \text{, representing the streamline’s ODE, leading to:}$$

$$\frac{d \ln R}{d \phi} + \frac{R}{R} = 0.$$ 

In fact, the “velocity quasi-potential” function \( \Psi \) must also be a conical one, which means to be in the form: 

$$\Phi_{ij}(\phi, \chi_{0i}) = \Phi_{ij}(\phi, \chi) = \Phi(\phi), \quad \chi_{0i} = \text{const},$$

\( R \) \( R \) \( R^2 V' 

' = d \theta \text{, representing the streamline’s ODE, leading to:}$$

$$\frac{d \ln R}{d \phi} + \frac{R}{R} = 0.$$ 

The existence conditions of the roots in first solution give: \( |\Phi| = |\Phi| e^{-2\theta/2} \geq |\Phi| > 0 \), \( \Phi > 0 \), relations which establish the existence domains of the velocity potential \( \Phi(\beta) \) to obtain isentropic 2-D (plane, axisymmetric and general 3-D) conical flows. It can be noticed that for \( \beta > 0 \) cannot exist isentropic 2-D conical respectively, between their symmetric ones with respect to the \( \beta \) axis, as one can see in fig. 2. The equations of motion are getting the special forms (2.1.14), (2.2.1) and (2.2.2) in [11].

The current cones (\( C \)): \( C = C_2 \) represent the isentropic sheets \( \chi = \chi_{0i} \text{, having:} \Phi = V \Omega, \quad \Omega = R V' \) (\( \Phi \) – a conical scalar quasi-potential, specific to the respective “ij” sheet, the general flow being rotational \( \Omega \neq 0 \), with \( \Omega \neq 0 \), so that \( \Omega_i = \kappa_0 \Omega \) – see also section 1 in [8].
In the classical spherical coordinate system \((R, \theta, \phi)\) and in
the smart intrinsic generalized spherical one \((R, \phi, \chi)\), resp.,
the 1st equation of motion for a certain 3-D conical flow is:
\[
a_R = - \frac{\partial}{\partial R} (\hat{R} R \phi^2 - \hat{R} \theta \phi^2) = - \frac{\partial (\hat{R} / R)}{\partial \phi} = 0, \text{and}
\]
\[
a_R = - \frac{\partial}{\partial R} (\hat{R} \phi^2 - \hat{R} \theta \phi^2) = - \frac{\partial (\hat{R} / R)}{\partial \phi} = 0, (\chi = 0).
\]
In the case of a plane conical flow:
\[
\theta = \frac{\pi}{2}; \quad \Rightarrow \sin \theta = 1 \quad \text{and:}
\]
\[
\omega = 0; \quad \hat{\omega} = 0.
\]
In the case of an axisymmetric conical flow, the streamline is contained in a meridian plane (phenomenon independent of \(\omega\)):
\[
\phi = \phi_0 = \text{const.;} \quad \Rightarrow \phi = \phi_0; \quad \hat{\phi} = 0.
\]
In the case of a general 3-D conical flow, using the notation (10), one obtains the relation (9). There also are helicoidal conical flows (see [11]) –
3.a. the conical isentropic sheets
\[
\chi = \chi_0 i = c(S_{0i} - c_0) = 1/C_2 \quad \text{(a smart intrinsic coordinate tied to} \ S_{0i} \text{ – the local specific entropy value),}
\]
\[
\text{having as remarkable directrices: the} \ oz \text{ axis and a circle (the solid cone trace) centered on it (both for} \ C_2 = 0; \text{and a right strophoid} (\chi = 0) \text{centered on} \ az \text{ axis too (for} \ C_2 = 0; \text{c}, S_{0i}, c_0 > 0; \text{).}
\]
3.b. the conical sheets
\[
\phi = \phi_0 j = C_1 \quad \text{(smart intrinsic coordinate),}
\]
\[
\text{orthogonal to the conical isentropic ones:} \ x, y, z \text{ – Cartesian coordinates,} \ x \text{ – abscissa of cross section current plane}
\]
\[
\chi = \chi_0 i + i \bar{z} \text{ is a complex variable;} \quad \bar{z} = y \bar{x}; \quad \bar{z} = z \bar{x}; \text{ y, z – Cartesian coordinates, x – abscissa of cross section current plane}
\]
3.c. the conical isentropic sheets
\[
\chi = \chi_0 i = c(S_{0i} - c_0) = 1/C_2 \quad \text{(a smart intrinsic coordinate tied to} \ S_{0i} \text{ – the local specific entropy value),}
\]
\[
\text{having as remarkable directrices: a Pascal’s limaçon and the circle at infinity (both for} \ C_2 = 0 \text{ and centered on the} \ az \text{ axis).}
\]
gradient), the author develops an extremely interesting the solution (the velocity field and implicitly that of static of Mach number \( M_1 \) from the upstream, but a potential flow as usual the incident parallel and uniform supersonic flow.

The confluent flows are related to other simpler supersonic flows (the conical axisymmetric one and the plane flow with Prandtl–Meyer isentropic compression; \( 2 - 2' \) – rotational ring – a conical flow too, but with expansion, downstream of an attached bow shock wave, weakened and canceled by the tronconical expansion waves (whose axisymmetric fan was not represented) originating on the “foci” circle CC’, the same for both flows; \( 4.b \) – fuselage (cylinder-cone body); \( 2' - 2 \) – homentropic ring with tronconical flow inside a specially shaped annular channel; \( 3 - 3' \) – infinite ring with non-perturbed supersonic flow.

\[
\left. \frac{\text{d} \theta}{\text{d} \ln \beta} \right|_{R=\infty} = \left. \frac{\text{d} \beta}{\text{d} \ln \theta} \right|_{R=\infty} = \frac{\pi}{2}; \quad \theta \neq \frac{\beta}{2} - \text{r}_0, \quad \text{hence } \theta = \frac{\beta}{2}, \quad \theta = \frac{\beta}{2} + \text{r}_0.
\]

In a series of papers (see [16] – [19]) a new shock-free axisymmetric configuration inspired by a Schlieren picture (fig. 260 – a forebody in supersonic flow with isentropic compression – achieved by isentropic way (smoothly) and not by shock) from the flow visualizations album [20] was proposed and studied. This flow was called tronconical, due to the fact that the simple compression waves are co-axial truncated cones (conical frusta) having as a common basis (directrix) the “foci” circle, along which they focus into an axisymmetric shock wave (with curved meridian line, and so with variable intensity) – see fig. 4.a, and various types of air intake were imagined: frontal, annular, frontal-annular, etc. (see figs. 3 – 7 in [18]), all with dynamic compression. The tronconical potential flow is associated (confluent) to a rotational flow \( 2 - 2' \) treated in [21], around an axisymmetric body without incidence, consisting of a cylinder of radius \( r_0 \) and a truncated cone of half-angle \( \tau \) (see fig. 4.a), with a bow shock wave, assumed to be attached at the intersection circle CC’ of the two axisymmetric surfaces. In order to determine the solution (the velocity field and implicitly that of static pressure, the shape of the shock wave and the entropy gradient), the author develops an extremely interesting analytic perturbations method, the disturbance affecting not as usual the incident parallel and uniform supersonic flow of Mach number \( M_1 \) from the upstream, but a potential flow very close to that rotational real, called by him a quasi-conical motion. Later, this problem was resumed in [22], also considering a supersonic convection wave in the presence of a truncated cone, giving an analytic solution. The confluent flows are related to other similar supersonic potential flows (the conical axisymmetric one and the plane one with shock wave or Prandl–Meyer isentropic compression obtained as particular cases; for the last one see fig. 227 from [20]). All tronconical waves (truncated cones) are envelope surfaces of the Mach cones (the local simple waves) generated by the points of the given axisymmetric body surface, all situated in the same cross section plane. In the toroidal (tronconical) coordinate system \((r_0, R, \theta)\) the PDE of the velocity potential can be set in the forms:

\[
\left(1 - \frac{V''}{a^2}\right)(V' + V) + \frac{1}{1 + \frac{r_0}{(R \sin \theta)}} (\cot \theta \cdot V' + V) = 0,
\]

(with \( R \sin \theta \neq -r_0 \)), therefore a nonlinear ODE, \( V' \) being the radial component of the velocity \( (V_R = V) \), and \( a \) the local speed of sound, with: \( \gamma = \frac{dV}{d\theta} \) and \( \gamma = \frac{d^2V}{d\theta^2} \), or

\[
\left(1 + \frac{r_0}{R \sin \theta}\right)
\begin{bmatrix}
\frac{1}{\gamma} & 0 & 0 \\
0 & \frac{1}{\gamma} & 0 \\
0 & 0 & \frac{1}{V'' + V}
\end{bmatrix}
\begin{bmatrix}
\frac{d^2V}{d\theta^2} \\
\frac{dV}{d\theta} \\
1
\end{bmatrix}
+ \cot \theta \cdot V' + V = 0.
\]

Fig. 4. Reproduction of figs. 3, 4 from [16] – [19] (oblate on Ox).

The tronconical (shock-free compression) flow inside an axisymmetric supersonic inlet diffuser with (4.a) outer and resp. (4.b) inner compression (frontal and annular air intake): 4.a) – homentropic cylindrical core of radius \( r_0 \) with tronconical flow – flow with complete (until \( M_2 = 1 \) isentropic compression; \( 2 - 2' \) – rotational ring – a conical flow too, but with expansion, downstream of an attached bow shock wave, weakened and canceled by the tronconical extension waves (whose axisymmetric fan was not represented) originating on the “foci” circle CC’, the same for both flows; \( 4.b \) – fuselage (cylinder-cone body); \( 2' - 2 \) – homentropic ring with tronconical flow inside a specially shaped annular channel; \( 3 - 3' \) – infinite ring with non-perturbed supersonic flow.

\[
r_0 \to \infty; \quad \left[1 - (V''/a^2)\right](V'' + V) = 0, \quad \text{hence } V'' = a \quad \text{plane flow with Prandl–Meyer isentropic compression; } r_0 = 0; \quad V'' + \cot \theta \cdot V' + 2V' = (V''/a^2)(V'' + V), \quad \text{the conical axisymmetric flow with isentropic compression inside Busemann’s nozzle (upstream of a conical shock wave with the same tip as the conical simple waves fan), as well as that downstream of an attached conical shock wave (between the shock wave and an infinite solid cone). In the trivial case of a parallel and uniform supersonic stream \( (R \sin \theta = k \text{r}_0) \) the velocity potential ODE becomes: \( (1 + 1/k)[1 - (V''/a^2)](V'' + V) + \cot \theta \cdot V' + V = 0 \).
\]

Both Prandl–Meyer (see [23] – [27]) and Taylor–Maccoll (see [12], [13]) ODEs (which describe the plane and respectively axisymmetric conical flows) are obtained as simple limit cases of the general ODE of the velocity potential for tronconical flows. From this equation it can be noticed that, unlike the conical flows cases, when the equation has an unknown function of a single variable \( V(\theta) \), in the tronconical flow case, the equation has the same unknown function \( V \), but depending on two variables \( \theta \) and \( R \). In this equation introducing only the function derivatives with respect to \( \theta \), the dependence on \( R \) can be considered weaker or even zero (this being called by the author the tronconical approximation: [16] – [18]).
Setting (also see fig. 1 from [20]): \( \theta = \alpha ; R = r; r_0 = b; \frac{1}{1 + r_0 / (R \sin \theta)} = \frac{1}{r_0 + R \sin \theta} = \frac{r \sin \alpha}{b + r \sin \alpha} = m ; \)
and \( V = \varphi(r, \alpha) ; \) \( V' = \varphi_\alpha \); \( V^* = \varphi_{\alpha \alpha} \); \( a = a_1 \) ,
one obtains for the general differential equation the form:
\[
\left(1 - \frac{\varphi_{\alpha \alpha}}{a_1^2}\right) \varphi_{\alpha \alpha} + m \varphi_{\alpha} \cot \alpha + \left(1 + m - \frac{\varphi_{\alpha \alpha}}{a_1^2}\right) \varphi = 0 , \quad (11)
\]
identical with Eq. (17) from [21] (but the last one having different boundary conditions), describing an axisymmetric potential flow – a component (namely just the quasi-conical one) of the rotational general supersonic one downstream of an attached axisymmetric bow shock wave. The boundary conditions above are given by the relations:
\[ \alpha = a_0_0 \varphi_{\alpha}(r_0) = 0 , \quad \text{where } a_0 = \tau \text{ (see fig. 3)} \] (18 a) in [21]
\[ \alpha = \alpha_1 ; \varphi_0 = \cos \alpha \varphi_{\alpha}(r_\alpha) = \left(k + \frac{1 - k}{K_2^2}\right) \sin \alpha_1 \] (18 b) in [21]
\( a_1 \) is the half-angle of a certain tronconical expansion wave (unknown, function of \( r \)), weakening the bow shock;
\[ k = \lambda_\alpha = \frac{\gamma - 1}{\gamma + 1} ; \quad K = M_1 \sin \alpha_0 ; \quad K_1 = M_1 \sin \alpha_\alpha . \]
The quantity \( a_1 \) in Eq. (11) is given by the energy equation:
\[ a_1^2 = a^2 = (\gamma - 1)(W^2 - V^2)^2 / 2 . \]
The solution was obtained by analytic way, the general integral of Eq. (11) being given in [21] by the relations
\[ (21), (22), (24) \text{ a) and } (24) \text{ b), for the linearized equation:} \]
\[ \varphi = \left(C_0 + D_0 \cos \alpha \right) \alpha_1 , \quad (21) \text{ in [21]}
\]
where \( C_0 \) and \( D_0 \) are functions of the variable \( r \) only;
\[ C_0 = - \frac{\sin 2\alpha_0}{2} + I_1 \sin 2\alpha_0 , \quad D_0 = 1 - C_0 I_1 , \quad (22) \text{ in [21]}
\]
being used the following notations:
\[ I_1(\alpha, r) = \int_0^r \frac{b + r \sin \alpha_0}{b + r \sin \alpha} \, d \tau = E(\alpha) - E(\alpha_0) \]
\[ = \frac{2 \varphi_0}{b^2 - r^2} \left(1 + \frac{r \sin \alpha_0}{(b \tau + r)^2 + b^2 - r^2}\right) - \frac{b^2 - r^2}{b^2 - r^2} \left(1 + \frac{r \sin \alpha_0}{(b \tau + r)^2 + b^2 - r^2}\right) \tau \]
\[ I_1 = I(\alpha, r) \quad ; \quad (24a) \text{ in [21]}
\]
and \( E(\alpha) = \frac{b + r \sin \alpha_0}{b + r \sin \alpha} \left(\tan \alpha - \frac{2 \varphi_0}{b^2 - r^2 + 1 + z^2}\right) z = \tan \alpha_1 / 2 \) (24b) in [21]
\( r \) being an integration variable. The last integral in
\[ (24 \text{ a}) \text{ takes various forms, function of the ratio } (r/b) \]
– also see the relations (14) and (15) from [22].

The entire (complete) nonlinear equation of the flow in the downstream of an attached axisymmetric shock wave \[ \varphi_{\alpha \alpha} = m \varphi_\alpha \cos \alpha + (1 + m) \varphi = F(r, \alpha) + N(r, \alpha) , \quad (31) \text{ in [21]}
\]
having in its right-hand side the nonlinear terms \( F \) and \( N \)
\[ F(r, \alpha) = \frac{\varphi_{\alpha \alpha}^2}{a_1^2} (\varphi_{\alpha \alpha} + \varphi) , \quad (19) \text{ in [21]}
\]
(just the nonlinear term of Eq. (11), neglected until now); has the following general integral:
\[ \varphi = \frac{1}{C + H} I + (D - L) \cos \alpha \quad ; \quad (32) \text{ in [21]}
\]
\( C \) and \( D \) are functions of the variable \( r \) only, and are determined imposing the boundary conditions, like \( a_\alpha \);
\[ H = \int_{r_0}^{r} \frac{b + r \sin \tau}{b + r \sin \alpha} \left[F(r, \tau) + N(r, \tau)\right] \cos \tau \, d \tau , \]
\[ L = \int_{r_0}^{r} \frac{b + r \sin \tau}{b + r \sin \alpha} \left[F(r, \tau) + N(r, \tau)\right] \sin \tau \, d \tau , \]
representing the relations (33 a) and (33 b) in [21], resp.
The integral \( I \) is given by the relations (24 a), (24 b) and resp. (32), (33 a) and (33 b), for the entire nonlinear equation.
In the last two relations, we must consider for the second nonlinear term, \( N(r, \alpha) \), defined by the relation (16) from [21],
the condition \( N(r, \alpha) = 0 \), this corresponding to our tronconical flow case. An improvement in our problem solution can be obtained by replacing the relation for the main boundary condition \( \nabla_0 = 0 \) with the new (exact) streamline’s equation:
\[ R'V' = RV = \varphi(\alpha) , \quad \text{or } r \varphi_\alpha = r(\varphi + r \varphi_\alpha) , \quad (12) \]
with:
\[ r_\alpha = \frac{dr}{d\alpha} , \quad \varphi_0 = \frac{d \varphi_\alpha}{d \alpha} \text{ and } \varphi_\phi = \frac{d \varphi}{d \phi} , \]
where, taking into account the definition of the velocity potential – relation (14) in [21], assumed in the simple new form \( \varphi = r \varphi(r, \alpha) \) (therefore not a tronconical one, \( r \varphi_\alpha(\alpha) \)), were used for the velocity components the expressions
\[ V_r = V_\alpha = R_\alpha = \frac{\partial \varphi}{\partial r} = \varphi + r \varphi_\alpha ; \quad V_\alpha = V_\phi = \frac{1}{r} \frac{\partial \varphi}{\partial \phi} = \varphi_\alpha . \]

On the other hand, in order to replace the expression \( a_1^2 \) in the general PDE of the tronconical flow – the former ODE (11), one must write a new (exact) form of the energy PDE:
\[ (a_\alpha)^2 = (\gamma - 1) \left[(W^2 - \varphi)^2 - (\varphi_\alpha)^2\right] / 2 . \]
(13) The new solution will be obtained by integrating the system of three PDEs (11) – (13), the subscripts \( \alpha \) and \( r \) having now the significance of partial differentiation with respect to the respective variables), these equations ceasing to still represent the mathematical model of a tronconical flow, but of a new flow, related to that considered in [21]. This new system, by solving of which is obtained the improved solution, has three unknowns: \( \varphi(r, \alpha), a_\alpha(r, \alpha) \), and \( r \alpha \) – the streamline (body wall) equation, the functions \( \varphi \) and \( a_\alpha \) depending on \( \alpha \) both directly and via \( r \) as well, so that one can write on a certain streamline: \( d \varphi = \frac{1}{a_\alpha} d \alpha + r \varphi_\alpha d \alpha = (\varphi_\alpha + \varphi_\alpha r) d \alpha \) and respectively: \( d \alpha = (a_\alpha + a_\alpha r) d \alpha \).
It is also to be noticed in that all the cases, the equation of the velocity potential admits the particular (trivial) solution \( V = - U_\alpha \cos \theta \) (a parallel and uniform stream).
In the case of absence of the specially shaped central body (see fig. 4 a), when only the cylinder-cone shell exists, the outer quasi-conical motion is no more associated (confluent) to the inner tronconical flow with isentropic compression, but to a non-tronconical flow with expansion, considered
in [28], called by the author Prandtl-Meyer expansion in axisymmetric flow. It must be mentioned that the general equation from the beginning of this section can describe any tronconical flow, not only those with isentropic compression. Thus, it can be imagined a tronconical flow with expansion in an axisymmetric nozzle similar to Busemann’s one, but having inside a central cylinder-cone body and a pre-determined shape of the wall meridian line, corresponding to a strictly calculated design Mach number of the downstream parallel and uniform supersonic flow (also see fig. 4.b and fig. 173 from [29]). This flow can be easily obtained by a simple reversal of the flow from figure 4.b and extends itself only in the supersonic range. This is possible, because the inner flow is entirely shock-free (and thus isentropic). In the same manner, starting from the tronconical flow with isentropic compression represented in fig. 4.a one can obtain the tronconical flow with expansion given in fig. 4.1.a from [30]. Besides these cases, there also is that of the tronconical flow with expansion in the second region of the mixed flow (isentropic compression – expansion) around a specially shaped axisymmetric forebody, immediately followed by a cylinder (see fig. 7 in [18], here reproduced as fig. 5). This expansion flow succeeds to a tronconical flow with isentropic compression in the first region of the above flow and takes place just around the circular edge (having in the meridian plane the trace B) of the intersection of the two bodies, this edge being just the foci circle of the tronconical flow with expansion. Besides these methods, another way to obtain tronconical flows with expansion consists in the generalization of their corresponding axisymmetric conical flows (see [31] – [34]), instead of the conical flow’s vertex appearing now the tronconical flow’s circle of foci. The new flow’s velocity potential equation (valid for tronconical flows with both compression and expansion) was established as a first step in finding the exact solution and a streamlining method for the axisymmetric bodies and channels was imagined.

10 First approach to extension of the new quasi-potential model to the rotational flow of a viscous Newtonian compressible fluid

Let start from the vector general form of the motion equation for a viscous Newtonian fluid flow (Navier–Stokes):

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{\nabla p}{\rho} + \frac{1}{\rho} \Delta \mathbf{V} + \left( \frac{\mu_1}{\rho} + \frac{\mu_2}{3\rho} \right) \nabla (\mathbf{V} \cdot \nabla) \mathbf{V}
\]

(\text{the Helmholtz ([35])–Gromeka–Lamb form, binding the acceleration and force density terms of a fluid particle});
\[ f \] – the mass force density (conservative – a gradient);
\[ f = \mathbf{V} \cdot (\mathbf{g} - \mathbf{V} \cdot \mathbf{g}) = -\mathbf{V} \cdot (\mathbf{g} - \mathbf{V} \cdot \mathbf{g}) \] – the acceleration of gravity;
\[ z \] – the geometrical height (height of the considered point above a reference horizontal plane \( xOy \));
\[ \mu_1 \] – the dynamic viscosity of the fluid or the coefficient of internal friction; \( \mu_1 / \rho \) – kinematic viscosity of the fluid; \( \mu_2 \) – second, or bulk, viscosity (\( \mu_1, \mu_2 \) assumed const.).

For a steady motion and respectively for gases, we have:

\[ \frac{\partial \mathbf{V} \cdot \nabla}{\partial t} = 0 \quad \text{and} \quad \mathbf{f} = 0 \] , thus remaining:

\[
\nabla \left( \frac{\mathbf{V} \cdot \nabla}{2} \right) + \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p - \mu_1 \Delta \mathbf{V} - \left( \frac{\mu_1}{\rho} + \frac{\mu_2}{3\rho} \right) \nabla (\mathbf{V} \cdot \nabla) \mathbf{V}
\]

Searching for a 2-D velocity quasi-potential \( \Phi_i (\zeta, \eta) \) (therefore in a new smart intrinsic coordinate system \( O \zeta \eta \zeta \zeta \)), we will perform first a scalar multiplication of this equation by a special virtual elementary displacement \( d\mathbf{R} = (\mathbf{V}, \Omega) \) plane (see section 2), so obtaining some surfaces similar to the isentropic ones for the inviscid compressible fluid flow case, envelope sheets of the \((\mathbf{V}, \Omega)\) planes above. Over these surfaces we can write:

\[
\Omega = \nabla \times \mathbf{V} = \frac{1}{h_i h_j h_k} \begin{vmatrix} h_i & h_j & h_k \\ \frac{\partial}{\partial \zeta} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \zeta} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \xi} \end{vmatrix} = k_{ij} \Omega_i + k_{ij} \Omega_j + k_{ij} \Omega_k
\]

with:
\[ V_\zeta = h_\zeta \xi \] ; \[ V_\eta = h_\eta \eta \] ; \[ V_\xi = h_\xi \xi = 0 \]

where:
\( k_i \) - a 3-D basis; \( h_i \) - Lamé’s coefficients; the dotted variables are derivatives with respect to the time \( t \), and so:

\[
\Omega_i = \frac{1}{h_i h_\eta h_\xi} \left[ \frac{\partial (h_\xi V_\eta)}{\partial \zeta} - \frac{\partial (h_\eta V_\xi)}{\partial \zeta} \right] = 0
\]

Let introduce a scalar function \( \Phi_i (\mathbf{M}) = \Phi_i (\zeta, \eta, \zeta_0) \), also called by the author a 2-D velocity “quasi-potential”, whose partial derivatives along the directions of the elementary orthogonal axes \( h_\zeta \xi \) and \( h_\eta \eta \) on the “i” \((\mathbf{V}, \Omega)\) sheet \((\zeta = \zeta_0)\) are just the components \( V_\zeta \) and \( V_\eta \) of the velocity...
The first three terms (\(\eta_1, \eta_2, \eta_3\)) in the \(L_2\) expression are given by the linear deformation, while the last three terms (\(\eta_4, \eta_5, \eta_6\)) are given by the angular deformation: 
\[
\eta_1 = \partial V_i / \partial x_i = V_i; \quad \eta_2 = (\partial V_i / \partial x_i + \partial V_j / \partial x_j) / 2 = (V_i + V_j) / 2; \\
\eta_3 = (\partial V_i / \partial x_i + \partial V_j / \partial x_j) / 2 = (V_i + V_j) / 2 = e_{ij}.
\]

The new equation of the isentropic surfaces (dS = 0, needed to assure that \(a^2 = (dp/ds)s - s_0 = \gamma \kappa T\)).

\[dS = (\Omega \times V) \cdot dR + \frac{1}{\rho} \left[ \left( \frac{\mu_1}{\mu_2} - \frac{2}{3} \right) I_1^2 + 2I_2 \right] dt = 0 .\]

11. The velocity “quasi-potential” PDE for a viscous fluid flow and its validity domain; introducing a new Selescu’s roto-viscous vector

Therefore along the intersection space lines of the particular isentropic surfaces \((\mathbf{V}, \Omega)\) sheets and “j” circular cones with \(K_j = K_j\), the general rotational flow of a viscous Newtonian compressible fluid is governed by Steichen’s PDE of the 2-D quasi-potential \(\Phi_i(M) = \Phi_i(\xi, \eta, \zeta_0)\) (see sections 3 and 5), therefore at the same point:

\[
C_2 \left[ \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta^2 + \frac{1}{h^2} \partial^2 \Phi_i / \partial \eta^2 + \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \eta + \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \zeta \right] + \frac{\partial \Phi_i / \partial \zeta}{h^2} \partial^2 \Phi_i / \partial \zeta^2 + \frac{\partial \Phi_i / \partial \eta}{h^2} \partial^2 \Phi_i / \partial \eta^2 + \frac{\partial \Phi_i / \partial \zeta}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \eta + \frac{\partial \Phi_i / \partial \zeta}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \zeta \right] \times \]

\[
W \left[ \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta^2 + \frac{1}{h^2} \partial^2 \Phi_i / \partial \eta^2 + \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \eta + \frac{1}{h^2} \partial^2 \Phi_i / \partial \zeta \partial \zeta \right] \]
The equation of the isentropic surfaces is (two cases):

1. \( \text{TdS} = [(\Omega \times \mathbf{V}) - (1/4\pi\rho)(\mathbf{j} \times \mathbf{H})] \cdot \text{dR} = 0 \), for an inviscid fluid, \( \mathbf{V} \) and \( \Omega \) being now local mean (of the ionized fluid – plasma – component particles: neutral atoms, cations and anions of a single species) vectors (see subsection 2.2 in [2], sections 2 in [41], [42], and 3 in [43]):

\[
\mathbf{V} = (\rho_a \mathbf{V}_a + \rho_1 \mathbf{V}_1 + \rho \mathbf{V}_2)/(\rho_a + \rho_1 + \rho) ;
\]

\( \mathbf{H} \) is the strength of the local magnetic field, using the same convention of equivalence (see for reference [38], [44] – [46]) to the magnetic induction \( \mathbf{B} \) (as a rule variable in time), with \( \nabla \times \mathbf{H} = 0 \) (a solenoidal field); \( \mathbf{j} \) is the density of the conduction electric current (see Maxwell’s 2nd equation in [47]):

\[
\mathbf{j} = \nabla \times \mathbf{H} - 1/c\epsilon \mathbf{E} \times \mathbf{E} = \nabla \times \mathbf{H} + 1/c^2 \epsilon \partial \partial (\mathbf{V} \times \mathbf{H}) ;
\]

\( c \) is the light speed in a vacuum. The low-frequency Ampère’s law neglects displacement current, the density of conduction current becoming thus: \( \mathbf{j} = k \nabla \times \mathbf{H} ; \)

2. \( \text{TdS} = [((\Omega \times \mathbf{V}) - (1/4\pi\rho)(\mathbf{j} \times \mathbf{H})) \cdot \text{dR} = 0 \), respectively, for a viscous fluid (see, e.g. [2], [42]), but in the case of magnetically inviscid MHD (with no magnetic viscosity), \( \mathbf{V}_c \) is the virtual velocity of a neutral atom or of a small electroconducting fluid particle, given by: \( \text{dR} = \mathbf{V}_c \cdot \text{dt} \), \( \mu_1/\rho \) is the fluid kinematic viscosity; \( \mu_2 \) is the second, or bulk, viscosity of the fluid, assuming that \( \mu_1 \) and \( \mu_2 \) are constant (mean values); \( I_1, I_2 \) are the mean values of first and second invariant of the tensor of the particle deformation rate (see section 10).
13 Conclusions and remarks

In this paper the expanded form of Steichen’s PDE of the velocity potential in a certain triorthogonal curvilinear coordinate system was established, for both steady and unsteady motion of an inviscid compressible fluid. Choosing a smart intrinsic system Oξηζ tied to the isentropic surfaces (V, Ω), some simpler forms of the equation were obtained, even for the rotational flows, using a 2-D velocity “quasi-potential” Φ1(ξ, η). Over flow’s polytropic surfaces the fluid has a quasi-barotropic behavior. So, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity “quasi-potential” Φ1(ξ, η) satisfying Steichen’s PDE over some rigid virtual “i” isentropic surfaces, and even more, one can find a similar 2-D velocity “quasi-potential” Φ2(ξ, η) satisfying Laplace’s PDE ΔΦ2 = 0 (so a harmonic function Φ2 along some rigid virtual “ij” isentropic and polytropic space curves (intersection lines of “i” isentropic surfaces with “j”) polytropic ones) – a quasi-incompressible quasi-potential behavior in a rotational pseudo-flow of a compressible fluid. For any unsteady rotational motion of an inviscid compressible fluid one can find a new 2-D velocity “quasi-potential” Φ1(ξ, η, t) which satisfies Steichen’s PDE over some time-deformable virtual “i” isentropic surfaces. Simpler forms of all above equations can be obtained using the orthogonal elementary arc lengths: dξ = hξdξ; dη = hηdη (dv = hξdξ = 0 – isentropic sheet), without any Lame’s coefficient at the denominators. So, along the streamlines of a 3-D steady rotational flow of an inviscid compressible fluid the motion is given by an ODE of the 2nd order for the 1-D velocity quasi-potential Φ1(ξ, η), while along the intersection lines of the isentropical virtual “i” surfaces with the isothermal & isotachic (polytropic) virtual “j” surfaces (the flow’s Laplace lines) the motion is described by a Laplace’s PDE for the 2-D velocity quasi-potential Φ1(ξ, η).

Therefore, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity “quasi-potential” Φ1(ξ, η) satisfying Steichen’s PDE over some virtual “i” isentropic surfaces, and even more, one can find another 2-D velocity “quasi-potential” Φ1(ξ, η) satisfying Laplace’s PDE (so a harmonic function Φ2(ξ, η)) along some virtual “ij” isentropic and polytropic space curves (that is the intersection lines of the “i” isentropic surfaces with the “j” polytropic ones), meaning an elliptic PDE (its canonical form: ΔΦ1(ξ, η, v0) = 0), irrespective of the pseudo-flow character (subsonic or supersonic). Their direction in space is given by the local Selescuv’s isentropic & isotachic elementary virtual displacement vector (dRk = k5Vξ ∇ξ Vη mainly). Through any point of the rotational flow of an inviscid compressible fluid always passes such an incompressible “Laplace” “ij” line (isentropic & isotachic), except for the case when both surfaces are identical, so appearing an undetermined solution.

Choosing the smart intrinsic coordinate ζ just along flow’s streamlines (assumed as being known), Steichen’s PDE becomes an ODE of the 2nd order, therefore simpler, and so the streamlines are just the characteristic lines. More, one can find another 2-D velocity “quasi-potential” Φ1(ξ, η) satisfying the wave (vibrating string) PDE along other virtual “ij” isentropic and polytropic space curves (that is the intersection lines of the “i” isentropic surfaces with the “j” polytropic ones), meaning a hyperbolic PDE (its canonical form: c2d^2Φ1(ξ, η, v0) / dξ2 + ••• (lower order terms) = 0), irrespective of the pseudo-flow character. Concluding: the analysis for a steady flow case in sections 6 and 7 is independent on the value of (a) (the speed of sound).

Along the intersection lines of the particular isentropic sheets (V, Ω), sheets with “j” circular cones; dS1 = dS2 = 0, and so dS = 0 only, the steady rotational flow of a viscous Newtonian compressible fluid is governed by Steichen’s PDE of a 2-D velocity quasi-potential also Φ1(ξ, η, v0) = Φ1(λ, μ, v0) – a line potential; ζ = ζ0 is a (V, Ω) sheet, admitting this Φ1.

Remarks: 1. The difference between a potential function Φ and a 2-D “quasi-potential” Φ1 one consists in that the first function is valid everywhere (at any point in space), whereas the second one is valid over a family of some “i” surfaces ζ = ζ0 only. While V = ∇Φ is a gradient, V1 = ∇Φ1 is not. For the first one: Ω = ∇ × V = ∇ × Φ = 0, whereas for the 2nd one: Ω1 = ∇ × V1 = ∇ × Φ1 = k0Ω0 + k0Ω0 ≠ 0, being contained in the plane tangent to the “i” surface ζ = ζ0, only its component directed along the normal to this surface becoming zero: Ω2 = (1/ h2ξ) (c2dΦ1/∂ξ2 – c2dΦ1/∂η2) = 0.

2. Sections 6, 7, 10 and 11 are dedicated to the analysis of some interesting particular forms of the 2-D velocity quasi-potential PDE of a certain rotational “flow”, forms taken along some space lines different from the streamlines, but having a physical-mathematical property kept along them, so that a more appropriate term would be “pseudo-flow”.

3. If a PDE has coefficients that are not constant, it is possible that it will not belong to any of the categories: elliptic, hyperbolic, parabolic, but rather be of mixed type. Besides Steichen’s PDE, a simple but important example is the Euler–Tricomi equation, used in the investigation of transonic flow: c2w/∂ξ2 = x02w/∂η2, which is called elliptic-hyperbolic because it is elliptic in region x < 0, hyperbolic in region x > 0, and degenerate parabolic on the line x = 0.

4. All results here obtained substantiate the existence of some 2-D quasi-potentials (velocity’s Φ and magnetic Ξ) even for rotational flow, and more, for viscous fluid flow. Not all isentropic surfaces allow introducing such a quasi-potential, but the envelope sheets of the local planes (V, Ω) for Φ, and (H, j) for Ξ (using the low-frequency Ampère’s law jk = V × H, neglecting the displacement current) only. So we can write: V = ∇Φ and H = ∇Ξ. Another 2-D quasi-potential (vorticial ξ – the Greek “Ch”) was introduced (Ξ = VX), over the (Ω, Φ) sheets (Θ = V × Ω), to study the vorticity equation for a viscous incompressible fluid ([5], [7]).
All quasi-potentials $\Phi$ satisfy Steichen’s PDE. The validity domains for these PDEs – the “ij” space lines (intersection lines of two particular isentropic sheet families (section 11) and along Selescu’s magnetohydrodynamic vector lines: $d\mathbf{R} \parallel S$, with $S \equiv (V, \Omega) \cap (H, j) \equiv (\Omega \times V) \times (j \times H)$ in inviscid MHD (see [2], [41], [48]), allowing to introduce both 2-D quasi-potentials $\Phi$ and $\Xi$, and the “ii” surfaces – were theoretically established (their existence was firmly predicted). Finding them effectively is a very difficult job and is not the subject matter of this work. The solid and solidifiable (like in the case of confluent flows) boundaries of the flow domain are always $(V, \Omega)$ surfaces. There also is a 2-D “quasi-stream function” $\Psi$ potential” $PDE$ of the velocity potential for true potential flows only. 

Finally we can cite a remarkable sentence from [50]:

Lorentz, Michael Faraday, James Clerk Maxwell, Oliver Prescott Joule, Heinrich Lenz, Edwin Hall, Hendrik Lorentz, Michael Faraday, James Clerk Maxwell, Oliver Heaviside, Heinrich Hertz, Joseph Larmor and Hannes Alfvén. The author is very grateful to Milton Van Dyke for his book (album) [20], which presents an impressive collection of about 400 selected black-and-white photographs of flow visualization in experiments, received – on his request – from researchers all over the world, these photos inspiring many interesting theoretical works.


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Notes

This paper (the last in a series dedicated to the intrinsic analytic study of the basic equations in compressible fluid mechanics) is fully original, however having as starting point another one with almost the same title (see [51]). It addresses to researchers in higher mathematics and fluid dynamics, MHD included. All authors use further Steichen’s PDE of the velocity potential for true potential flows only.

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