New First Integrals for the Continuity, Vorticity and Related Equations

RICHARD SELESCU
Flow Physics Department, Experimental Aerodynamics Compartment, Trisonic Wind Tunnel Laboratory
"Elie Carafoli" National Institute for Aerospace Research – INCAS (under the Aegis of the Romanian Academy)
Bucharest, Sector 6, Bd. Iuliu Maniu, No. 220, Code 061126
ROMANIA
e-mail: rselescu@gmail.com web page: http://www.incas.ro

Abstract: This work studies and clarifies some local phenomena in fluid mechanics, in the form of an intrinsic analytic study, regarding the continuity equation, its first integral (the flow rate equation), for inviscid compressible fluids, and the vorticity equation, for viscous incompressible fluids, finding new first integrals. It continues a series of works presented at some conferences and a congress during 2006 – 2012, representing a real deep insight into the still hidden theory of the isoenergetic rotational flow. Several new functions, surfaces and vectors were introduced: the 2-D "quasi-stream" function on the 3-D (**V**, **Ω**) surfaces, for the continuity equation; the surfaces of *iso*-normal mass flux *density* (over which the continuity equation for the steady flow in a thick stream tube admits the same first integral as for the flow in a thin one, and whose envelope sheets are the sections of uniform flow), for the flow rate equation; the 3-D stream function vector, allowing new local and global forms of continuity equation (the global one similar to Helmholtz' 2nd theorem about vortices in an ideal fluid); Selescu's incompressible roto-viscous vector and the zero-work surfaces (for some non-conservative vectors), for the vorticity equation. The dependence "2-D velocity quasi-potential ↔ 2-D quasi-stream function" was established. The unsteady flow's continuity equation was analytically integrated.

Key-Words: rotational flows; steady and unsteady flows; inviscid and viscous fluids; compressible fluids; isentropic or (**V**, **Ω**) surfaces; Selescu's 3-D stream function vector, surfaces of *iso*-normal mass flux *density*, roto-viscous vector and zero-work surfaces

1 Introduction, nomenclature and the first approach to the new steady model

The continuity equation is the mathematical expression of the law of conservation of mass in fluid mechanics. For a certain 3-D unsteady motion, if there are no sources or losses of fluid within the considered flow subdomain, the general form of this equation (called its local form) is:

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{V})=0\ \text{(Euler), or}\ \frac{d\rho}{dt}+\rho\nabla\mathbf{V}=0\text{, where}$$

$$\nabla=\sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial}{\partial x_i}=\mathbf{k}_x\frac{\partial}{\partial x}+\mathbf{k}_y\frac{\partial}{\partial y}+\mathbf{k}_z\frac{\partial}{\partial z}\quad\text{– nabla}$$

(Hamilton's operator), in a triorthogonal system of curvilinear coordinates x_i; $\mathbf{k}_i$ – a 3-D basis; h_i – Lamé's coefficients; $\mathbf{V}$ – the local instantaneous velocity of translation (of a small fluid particle) – the intensity of the local fluid field; $V = |\mathbf{V}|$; $\mathbf{\Omega} = \nabla \times \mathbf{V} = 2\boldsymbol{\omega}$ – the vorticity, with: $\boldsymbol{\omega}$ – the local instantaneous velocity of rotation (of the particle); Oxyz – the Cartesian orthogonal coordinate system; ρ – the fluid local density; t – the time. Additionally: *a* – the speed of sound; *c* – the critical speed (V = *a*).

For a certain 3-D steady motion, studied in a certain triorthogonal curvilinear q_i (i = 1, 3) coordinate system (like the previously used Oξηζ or Oλμν – both intrinsic, with: $d\lambda = h_\xi d\xi$; $d\mu = h_\eta d\eta$ and $d\nu = h_\zeta d\zeta$ – the elementary arc lengths), the continuity equation yields the forms: $\nabla(\rho\mathbf{V}) = 0$ (with $\mathbf{V}=\mathbf{k}_{q_1}V_{q_1}+\mathbf{k}_{q_2}V_{q_2}+\mathbf{k}_{q_3}V_{q_3}$), or:

$$\frac{1}{\prod_{j=1}^{3}h_j}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\rho\left(\prod_{j=1}^{3}h_j\right)\frac{V_{q_i}}{h_i}\right]=0\quad,\quad\text{or more}$$

$$\frac{1}{\prod_{j=1}^{3}h_j}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\rho\left(\prod_{j=1}^{3}h_j\right)\dot{q}_i\right]=0\quad,\tag{1}$$

meaning the mass flux density ρ**V** is a solenoidal vector field ($\rho\mathbf{V} = \nabla \times \mathbf{\Psi}_c$ with $\mathbf{\Psi}_c \neq \nabla G$; if $\mathbf{\Psi}_c = \nabla G$ one can define the flow's stagnation lines – see next section); h_i are the corresponding Lamé's coefficients (h_ξ, h_η, h_ζ):

$$h_i=\sqrt{\sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i}\right)^2}\quad;\quad(i=\overline{1,3})\quad;\quad\text{with:}$$

$$\begin{cases}x_j=x,\,y,\,z-\text{Cartesian coordinates;}\\ q_i=\xi,\,\eta,\,\zeta-\text{orthogonal curvilinear coordinates.}\end{cases}$$

Eq. (1) may be written explicitly in the forms below:

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho h_\xi h_\eta h_\zeta\frac{V_\xi}{h_\xi}\right)+\frac{\partial}{\partial\eta}\left(\rho h_\xi h_\eta h_\zeta\frac{V_\eta}{h_\eta}\right)+\frac{\partial}{\partial\zeta}\left(\rho h_\xi h_\eta h_\zeta\frac{V_\zeta}{h_\zeta}\right)\right]=0\quad,\quad\text{or}$$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho h_\eta h_\zeta V_\xi\right)+\frac{\partial}{\partial\eta}\left(\rho h_\xi h_\zeta V_\eta\right)+\frac{\partial}{\partial\zeta}\left(\rho h_\xi h_\eta V_\zeta\right)\right]=0\ ,\quad\text{or with: } V_\xi=h_\xi\dot{\xi}\ ;\quad V_\eta=h_\eta\dot{\eta}\ ;\quad V_\zeta=h_\zeta\dot{\zeta}\ ;$$

(all the dotted q_i variables in Eq. (1) and in the next ones represent derivatives with respect to the time t);

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho h_\xi h_\eta h_\zeta\dot{\xi}\right)+\frac{\partial}{\partial\eta}\left(\rho h_\xi h_\eta h_\zeta\dot{\eta}\right)+\frac{\partial}{\partial\zeta}\left(\rho h_\xi h_\eta h_\zeta\dot{\zeta}\right)\right]=0 \quad , \quad \text{or more}$$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\left(\dot{\xi}\frac{\partial}{\partial\xi}+\dot{\eta}\frac{\partial}{\partial\eta}+\dot{\zeta}\frac{\partial}{\partial\zeta}\right)\left(\rho h_\xi h_\eta h_\zeta\right)+\rho h_\xi h_\eta h_\zeta\left(\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\right)\right]=0 \quad ;$$

$$\rho\left[\frac{1}{\rho h_\xi h_\eta h_\zeta}\left(\frac{d\xi}{dt}\frac{\partial}{\partial\xi}+\frac{d\eta}{dt}\frac{\partial}{\partial\eta}+\frac{d\zeta}{dt}\frac{\partial}{\partial\zeta}\right)\left(\rho h_\xi h_\eta h_\zeta\right)+\left(\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\right)\right]=0 \quad , \quad (\rho\neq 0)\text{, this resulting in}$$

$$\frac{1}{\rho h_\xi h_\eta h_\zeta}\frac{d}{dt}\left(\rho h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0 \quad , \quad \text{or}$$

$$\frac{d}{dt}\ln\left(\rho h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0 \quad , \quad \text{or more}$$

$$\left[\ln\left(\rho h_\xi h_\eta h_\zeta\right)\right]^{\bullet}+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0 \quad , \qquad (2)$$

or, in an expanded form

$$\frac{\dot{\rho}}{\rho}+\frac{\dot{h}_\xi}{h_\xi}+\frac{\dot{h}_\eta}{h_\eta}+\frac{\dot{h}_\zeta}{h_\zeta}+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0 \quad , \qquad (3)$$

or, taking into account that on any "i" (**V**, **Ω**) surface ($\zeta = \zeta_{0i}$) we have: $V_\zeta = h_\zeta\dot{\zeta} = \dot{v} = 0$ and so $\partial(\rho h_\xi h_\eta V_\zeta)/\partial\zeta = 0$, (where $\mathbf{\Omega} = \nabla\times\mathbf{V} = 2\boldsymbol{\omega}$ is the local vorticity, curl **V**):

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho h_\eta h_\zeta V_\xi\right)+\frac{\partial}{\partial\eta}\left(\rho h_\xi h_\zeta V_\eta\right)\right]=0 \quad , \quad \text{or}$$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho h_\xi h_\eta h_\zeta\dot{\xi}\right)+\frac{\partial}{\partial\eta}\left(\rho h_\xi h_\eta h_\zeta\dot{\eta}\right)\right]=0 \quad . \ (4)$$

Let us introduce on any (**V**, **Ω**) surface ($\zeta = \zeta_{0i}$) the 2-D velocity "quasi-potential" $\Phi_i(\xi, \eta)$ (see [1] – [4] and section 3):

$$V_{\xi i}=V_{\lambda i}=\frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}=\frac{\partial\Phi_i}{\partial\lambda}\,;\; V_{\eta i}=V_{\mu i}=\frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}=\frac{\partial\Phi_i}{\partial\mu}\,;$$

$$V_{\zeta i}=V_{v i}=\frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta}=0\,; \qquad \mathbf{V}_i=\nabla\Phi_i \quad .$$

The mass flux density $(\rho\mathbf{V})_i = (\rho\nabla\Phi)_i$ having also two components only, Eqs. (2) and (4) become respectively:

$$\left[\ln\left(\rho h_\xi h_\eta h_\zeta\right)\right]^{\bullet}+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}=0\,, \quad \text{and}$$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\rho\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\rho\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right]=0\,,$$

$$\text{or}\quad \frac{\partial}{\partial\xi}\left(\rho\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right)=\frac{\partial}{\partial\eta}\left(-\rho\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right) \quad .$$

2 Introducing the 2-D "quasi-stream function" on the (V, Ω) surfaces and Selescu's 3-D "stream function" vector (both for the mass flux density vector)

The last PDE is possible if and only if there is a scalar function $F_i(\xi, \eta, \zeta_{0i})$ (a 2-D "quasi-stream function" for the mass flux density vector $\mathbf{j} = \rho\mathbf{V}$ on the surface $\zeta = \zeta_{0i}$) so that the conditions below are fulfilled:

$$\left(\rho\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right)_i=\frac{\partial F_i}{\partial\eta}\,; \quad \left(\rho\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right)_i=-\frac{\partial F_i}{\partial\xi}\,, \quad \text{or:}$$

$$\left(\rho\frac{h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right)_i=\frac{\partial F_i}{h_\eta\partial\eta}\,; \quad \left(\rho\frac{h_\zeta}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right)_i=-\frac{\partial F_i}{h_\xi\partial\xi} \quad ,$$

to satisfy Schwarz' theorem for the mixed derivatives of the second order ($\partial^2F_i/\partial\xi\partial\eta = \partial^2F_i/\partial\eta\partial\xi$). So we can write:

$$dF_i=\left(\rho\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right)_i d\eta-\left(\rho\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right)_i d\xi \quad ,$$

a total differential admitting along flow's "quasi-streamlines" $dF_i = 0$ ("mass flux density" streamlines) the first integral: $F_i(\xi, \eta, \zeta_{0i}) = \text{const.}$

With the elementary orthogonal arcs: $d\lambda = h_\xi d\xi$ and $d\mu = h_\eta d\eta$, the conditions above become:

$$\frac{\partial F_i(\lambda,\mu,v_{0i})}{\partial\lambda_i}=-\left(\rho h_\zeta\frac{\partial\Phi}{\partial\mu}\right)_i;\;\frac{\partial F_i(\lambda,\mu,v_{0i})}{\partial\mu_i}=\left(\rho h_\zeta\frac{\partial\Phi}{\partial\lambda}\right)_i;$$

$$\frac{\partial F_i(\xi,\eta,\zeta_{0i})}{(h_\xi\partial\xi)_i}=-\left(\rho\frac{h_\zeta}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right)_i;\;\frac{\partial F_i(\xi,\eta,\zeta_{0i})}{(h_\eta\partial\eta)_i}=\left(\rho\frac{h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right)_i,$$

$$\text{or more:}\quad \begin{cases}\left(\dfrac{\partial\Phi}{\partial\xi}\right)_i=\left[\dfrac{1}{\rho}\dfrac{h_\xi}{h_\zeta}\dfrac{\partial F(\xi,\eta,\zeta)}{h_\eta\partial\eta}\right]_{\zeta=\zeta_{0i}}; \\[2ex] \left(\dfrac{\partial\Phi}{\partial\eta}\right)_i=-\left[\dfrac{1}{\rho}\dfrac{h_\eta}{h_\zeta}\dfrac{\partial F(\xi,\eta,\zeta)}{h_\xi\partial\xi}\right]_{\zeta=\zeta_{0i}}.\end{cases}$$

For a compressible fluid flow in a certain triorthogonal curvilinear coordinate system Oξηζ (of elementary orthogonal arcs $d\lambda = h_\xi d\xi$, $d\mu = h_\eta d\eta$ and $dv = h_\zeta d\zeta$), we can introduce (by definition) a new 3-D "stream function" $\mathbf{\Psi}_c$ (Selescu's vector), this being given by a relation involving the analytic expression below of the mass flux density vector $\mathbf{j} = \rho\mathbf{V}$ in the 3-D intrinsic basis $\mathbf{k}_i$ (so being obtained *a new local form for the continuity equation*):

$$\mathbf{j}=\rho\mathbf{V}=\nabla\times\mathbf{\Psi}_c=\frac{1}{h_\xi h_\eta h_\zeta}\begin{vmatrix} h_\xi\mathbf{k}_\xi & h_\eta\mathbf{k}_\eta & h_\zeta\mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\xi} & \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\zeta} \\ h_\xi\Psi_{c\xi} & h_\eta\Psi_{c\eta} & h_\zeta\Psi_{c\zeta}\end{vmatrix}$$

$$=\begin{vmatrix} \mathbf{k}_\xi & \mathbf{k}_\eta & \mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\lambda} & \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial v} \\ \Psi_{c\xi} & \Psi_{c\eta} & \Psi_{c\zeta}\end{vmatrix}=\mathbf{k}_\xi\rho V_\xi+\mathbf{k}_\eta\rho V_\eta+\mathbf{k}_\zeta\rho V_\zeta \quad ,$$

with $\mathbf{\Psi}_c \neq \nabla G$ (G being a certain scalar function), except for some space curves – singular lines of a saddle point type (bifurcation lines), along which $\mathbf{\Psi}_c = \nabla G_{ij}$, this meaning the existence of steady flow's "ij" stagnation lines, they being 0-vorticity lines too. The subscripts "i" and "j" denote two families of *orthogonal* surfaces, by whose intersection the "ij" curves are obtained; so, along these "ij" space curves the mass flux density vector $\mathbf{j} = \rho\mathbf{V} = 0$. The equation $\mathbf{\Psi}_c = \nabla G_{ij}$ is the definition of flow's stagnation lines. If $\mathbf{\Psi}_c$ is just $k\rho\mathbf{V}$, so having $\mathbf{j} = \rho\mathbf{V} = k\nabla\times\rho\mathbf{V} = k\nabla\times\mathbf{j}$, one gets a special compressible helicoidal (screw) motion – a special Beltrami flow. On any "i" isentropic ($\mathbf{V}, \mathbf{\Omega}$) surface ($\zeta = \zeta_{0i}$, see [1] – [4]) of elementary orthogonal arcs $d\lambda = h_\xi d\xi$ and $d\mu = h_\eta d\eta$, we have for this compressible flow: $\mathbf{\Psi}_{ci} = \mathbf{k}_\zeta \Psi_{ci}$ with $\Psi_{ci} = \Psi_{c\zeta i}(\xi, \eta, \zeta_{0i}) = \Psi_{c\zeta i}(\lambda, \mu, \nu_{0i})$ ($\Psi_{c\xi i} = \Psi_{c\eta i} = 0$) and respectively:

$$\mathbf{j}_i = \rho\mathbf{V}_i = \operatorname{curl}\mathbf{\Psi}_{ci} = \nabla\times\mathbf{\Psi}_{ci} = \frac{1}{h_\xi h_\eta h_\zeta}\begin{vmatrix} h_\xi\mathbf{k}_\xi & h_\eta\mathbf{k}_\eta & h_\zeta\mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\xi} & \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\zeta} \\ 0 & 0 & h_\zeta\Psi_{ci} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{k}_\xi & \mathbf{k}_\eta & \mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\lambda} & \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} \\ 0 & 0 & \Psi_{ci} \end{vmatrix} = \mathbf{k}_\xi j_{\xi i} + \mathbf{k}_\eta j_{\eta i} = \mathbf{k}_\xi \rho V_{\xi i} + \mathbf{k}_\eta \rho V_{\eta i}\ ;\mathbf{\Psi}_{ci} \neq \nabla G_i\ ,$$

like for a local quasi-plane 2-D fluid motion (in the $\xi O\eta$ plane tangent to the "i" isentropic surface), this leading to:

$$j_{\xi i} = \rho V_{\xi i} = \frac{1}{h_\eta}\frac{\partial\Psi_{ci}}{\partial\eta} = \frac{\partial\Psi_{ci}}{\partial\mu}; j_{\eta i} = \rho V_{\eta i} = -\frac{1}{h_\xi}\frac{\partial\Psi_{ci}}{\partial\xi} = -\frac{\partial\Psi_{ci}}{\partial\lambda};$$

($j_{\zeta i} = \rho V_{\zeta i} = 0$), the newly introduced vector $\mathbf{\Psi}_c$ "becoming" now a 2-D "quasi-stream" scalar function Ψ_{ci}.

3 Introducing the 2-D velocity "quasi-potential" over the isentropic surfaces

The vector $\mathbf{V}$ has now two components only (like the vector $\mathbf{\Omega}$, both lying in the plane tangent to an isentropic sheet). Let be Oxyz the Cartesian system. In a triorthogonal curvilinear coordinate system $O\xi\eta\zeta$ tied to this isentropic surface (having $\xi O\eta$ as tangent plane) – therefore *a smart intrinsic coordinate system*, the vorticity component normal to the isentropic sheet ($\zeta = \zeta_0$) must be: $\Omega_\zeta = 0$. The analytic expression of the vector $\mathbf{\Omega}$ is:

$$\mathbf{\Omega} = \nabla\times\mathbf{V} = \frac{1}{h_\xi h_\eta h_\zeta}\begin{vmatrix} h_\xi\mathbf{k}_\xi & h_\eta\mathbf{k}_\eta & h_\zeta\mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\xi} & \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\zeta} \\ h_\xi V_\xi & h_\eta V_\eta & h_\zeta V_\zeta \end{vmatrix} = \mathbf{k}_\xi\Omega_\xi + \mathbf{k}_\eta\Omega_\eta + \mathbf{k}_\zeta\Omega_\zeta,$$

with : $V_\xi = h_\xi\dot{\xi}$; $V_\eta = h_\eta\dot{\eta}$; $V_\zeta = h_\zeta\dot{\zeta} = 0$,

and equating to zero the expression of component Ω_ζ:

$$\Omega_\zeta = \frac{1}{h_\xi h_\eta}\left[\frac{\partial(h_\eta V_\eta)}{\partial\xi} - \frac{\partial(h_\xi V_\xi)}{\partial\eta}\right] = 0 \quad , \quad \text{or}$$

$$\frac{\partial(h_\eta V_\eta)}{\partial\xi} - \frac{\partial(h_\xi V_\xi)}{\partial\eta} = 0 \quad .$$

Between two isentropic sheets the specific entropy S can vary continuously (monotonously) or discontinuously (by jump, like for the cases of supersonic plane flow with direct and Mach reflected shocks, and of axisymmetric confluent flows). Even if varying the index "i", $S = S_{0i}(i)$ is not always a strictly increasing function to be accepted as a ζ coordinate (like for symmetric plane flows), the monotony of S on some intervals of "i" may be considered, thus needing delimiters. Let introduce a scalar function $\Phi_i(M) = \Phi_i(\xi, \eta, \zeta_{0i})$, called by the author "quasi-potential", whose partial derivatives along the directions of the elementary orthogonal arcs $h_\xi d\xi$ and $h_\eta d\eta$ on the "i" isentropic surface ($\zeta = \zeta_{0i}$) are just the components $V_{\xi i}$ and $V_{\eta i}$ of the velocity $\mathbf{V}_i$ vector ($V_{\zeta i} = 0$). Let be $d\mathbf{R}$ an elementary virtual displacement vector in the plane tangent to an isentropic sheet (coplanar to $\mathbf{V}$ and $\mathbf{\Omega}$): $d\mathbf{R} = c_1\mathbf{V} + c_2\mathbf{\Omega} = \mathbf{k}_\xi h_\xi d\xi + \mathbf{k}_\eta h_\eta d\eta = \mathbf{k}_\xi d\lambda + \mathbf{k}_\eta d\mu$, the elementary arc length ds (= $|d\mathbf{R}|$) on this surface being given by

$$ds^2 = dx^2 + dy^2 + dz^2 = Ed\xi^2 + 2Fd\xi d\eta + Gd\eta^2$$
$$= h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 = d\lambda^2 + d\mu^2 \quad , \quad \text{with}$$

$$E = h_\xi^2 \ ; \ G = h_\eta^2 \ ; \ F = \frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta} + \frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta} + \frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} = 0$$

– due to the orthogonality condition, where:

$$h_i = \sqrt{\sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i}\right)^2}\ ;\ (i = \overline{1, 3})\ ;\ \begin{cases} x_j = x, y, z \\ q_i = \xi, \eta, \zeta_0 \end{cases} ,\text{having:}$$

$$h_\xi = \left.\sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\xi\right)^2}\right|_{\zeta=\zeta_{0i}} = f_1(\xi, \eta, \zeta_{0i}) \quad ;$$

$$h_\eta = \left.\sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\eta\right)^2}\right|_{\zeta=\zeta_{0i}} = f_2(\xi, \eta, \zeta_{0i}) \quad ;$$

$$h_\zeta = \left.\sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\zeta\right)^2}\right|_{\zeta=\zeta_{0i}} = f_3(\xi, \eta, \zeta_{0i}) \quad ,$$

(with: $J = \dfrac{D(x, y, z)}{D(\xi, \eta, \zeta)} \neq 0$; J – the Jacobian determinant of this change of variables; $J = 0$ – gives the space curves representing the entropy singularities), so resulting on the respective "i" isentropic surface:

$$V_{\xi i} = \frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial\Phi_i}{\partial\lambda}; V_{\eta i} = \frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial\Phi_i}{\partial\mu}; V_{\zeta i} = \frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta} = 0;$$

$$\Rightarrow\ h_\xi V_{\xi i} = \frac{\partial\Phi_i}{\partial\xi} \ ; \ h_\eta V_{\eta i} = \frac{\partial\Phi_i}{\partial\eta} \ ;$$

$$\frac{\partial(h_\eta V_{\eta i})}{\partial\xi} = \frac{\partial}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial^2\Phi_i}{\partial\xi\partial\eta} \ ; \ \frac{\partial(h_\xi V_{\xi i})}{\partial\eta} = \frac{\partial}{\partial\eta}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial^2\Phi_i}{\partial\eta\partial\xi} \ .$$

So the relation $\Omega_{\zeta i} = 0$ leads to: $\dfrac{\partial^2\Phi_i}{\partial\xi\partial\eta} - \dfrac{\partial^2\Phi_i}{\partial\eta\partial\xi} = 0$,

representing a true relation – Schwarz' theorem for the functions of two variables (the so-called theorem of "the equality of the mixed derivatives of the second order", they differing as to the order of differentiation). This relation proves that $\Omega_{\zeta i} = 0$ and the existence of a 2-D "quasi-potential" function Φ_i so that:

$$V_{\xi i} = V_{\lambda i} = \frac{\partial \Phi_i}{\partial \lambda} \quad ; \quad V_{\eta i} = V_{\mu i} = \frac{\partial \Phi_i}{\partial \mu} \quad ,$$

the entropy gradient vector ∇S_i being normal to the new introduced isentropic surfaces $\zeta = \zeta_{0i}$. This is evidently, taking into account Crocco's equation for an *isoenergetic* non-isentropic (rotational) steady gas flow: $\mathbf{\Omega} \times \mathbf{V} = T\nabla S$ (see [5]), where T is the absolute local static temperature. Thus, introducing the scalar quasi-potential function $\Phi_i(M) = \Phi_i(x, y, z)$: $\mathbf{V}_i = \nabla\Phi_i$, the vector ODE of motion, joined to the continuity and the physical ($p/\rho^\gamma = K_i$) ones (and taking into consideration the local speed of sound a_i definition), enables to determine the velocity vector $\mathbf{V}_i$ from Steichen's equation (see [6]), usually a PDE (improperly called now the "velocity potential equation", taking into account that there is a vector $\mathbf{\Omega}_i \neq 0$; the flow being rotational, more appropriate would be the term "*velocity quasi-potential equation*"). In the physical equation K_i is the constant of the respective "i" isentropic surface.

4 The velocity "quasi-potential" equation for any steady flow of a compressible fluid

This vector equation may be written in a symbolic form:

$$\nabla \mathbf{V}_i = \frac{\mathbf{V}_i}{2a_i^2}\nabla\left(V_i^2\right), \text{ with } \mathbf{V}_i = \nabla\Phi_i, \text{ but } \mathbf{\Omega}_i \neq 0;\ V_i = |\mathbf{V}_i|;$$

$$\Rightarrow \Delta\Phi_i = \frac{\nabla\Phi_i}{2a_i^2}\nabla\left[(\nabla\Phi_i)^2\right] \text{ or } \Delta\Phi_i = \frac{1}{a_i^2}\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i \ ,$$

where, in a Cartesian coordinate system Oxyz:

$$\Delta\Phi_i = \frac{\partial^2\Phi_i}{\partial x^2} + \frac{\partial^2\Phi_i}{\partial y^2} + \frac{\partial^2\Phi_i}{\partial z^2} \quad , \text{ and, symbolically:}$$

$$\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i = \left(\frac{\partial\Phi_i}{\partial x}\cdot\frac{\partial}{\partial x} + \frac{\partial\Phi_i}{\partial y}\cdot\frac{\partial}{\partial y} + \frac{\partial\Phi_i}{\partial z}\cdot\frac{\partial}{\partial z}\right)^{(2)}\Phi_i \quad ,$$

and expanding the symbolic expression in the brackets:

$$\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i = \left[\left(\frac{\partial\Phi_i}{\partial x}\right)^2\frac{\partial^2}{\partial x^2} + \left(\frac{\partial\Phi_i}{\partial y}\right)^2\frac{\partial^2}{\partial y^2} + \left(\frac{\partial\Phi_i}{\partial z}\right)^2\frac{\partial^2}{\partial z^2}\right.$$
$$\left. + 2\frac{\partial\Phi_i}{\partial x}\frac{\partial\Phi_i}{\partial y}\frac{\partial^2}{\partial x\partial y} + 2\frac{\partial\Phi_i}{\partial y}\frac{\partial\Phi_i}{\partial z}\frac{\partial^2}{\partial y\partial z} + 2\frac{\partial\Phi_i}{\partial z}\frac{\partial\Phi_i}{\partial x}\frac{\partial^2}{\partial z\partial x}\right]\Phi_i \quad ,$$

the speed of sound a_i being given by the energy equation:

$$a_i^2 = \left(\frac{dp}{d\rho}\right)_{S=S_{0i}} = \frac{\gamma-1}{2}\left[W^2 - \left(\frac{\partial\Phi_i}{\partial x}\right)^2 - \left(\frac{\partial\Phi_i}{\partial y}\right)^2 - \left(\frac{\partial\Phi_i}{\partial z}\right)^2\right],$$

all the points at which is satisfied the previous PDE of the velocity potential Φ_i *belonging to a certain* "i" *isentropic surface*. This new equation is identical to the velocity potential equation (see [6]), written for a potential flow only. In a triorthogonal smart intrinsic coordinate system Oξηζ tied to these surfaces (or Oλμν, with λ, μ, ν – lengths of orthogonal arcs, with λ and μ contained in the local tangent plane and ν directed upon the normal) Laplace's operator Δ is given by the general expression below (the function Φ_i depending on ξ, η and ζ_{0i}, or on λ, μ and ν_{0i}):

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right.$$
$$\left. + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta}\right)\right] \quad ; \quad \text{but} \quad V_{\zeta i} = 0 \quad \Rightarrow \quad \frac{\partial\Phi_i}{\partial\zeta} = 0 \quad ;$$

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right]\Bigg|_{\zeta=\zeta_{0i}} \quad , \text{ or:}$$

$$\Delta\Phi_i = \frac{1}{h_\zeta}\left[\frac{1}{h_\eta}\frac{\partial}{\partial\lambda}\left(h_\eta h_\zeta\frac{\partial\Phi_i}{\partial\lambda}\right) + \frac{1}{h_\xi}\frac{\partial}{\partial\mu}\left(h_\zeta h_\xi\frac{\partial\Phi_i}{\partial\mu}\right)\right]\Bigg|_{\nu=\nu_{0i}} \quad ,$$

e.g.: the λ arcs taken along the streamlines and the μ arcs – along the equi(iso)-"quasi-potential" lines ≡ intersection lines of the isentropic surfaces with the iso-quasi-potential ones, these being normal to the local velocity $\mathbf{V}$. Similarly, in Steichen's equation – a nonlinear PDE of the 2nd order in three variables – ξ, η, ζ (now written for any rotational flow – $\mathbf{\Omega} \neq 0$, but *on the* "i" *isentropic surfaces* $\zeta = \zeta_{0i}$) disappear all the terms containing the partial derivative about ζ of the potential function Φ_i, $(\partial\Phi_i/\partial\zeta)$, and also its derivatives with respect to ξ, η and ζ, thus being obtained a nonlinear PDE of the second order in only two variables – ξ, η – the "velocity quasi-potential equation" (see [1] – [3]), which was thoroughly treated in a recent paper ([7]).

5 The interdependence between the 2-D "quasi-potential" function and the 2-D "quasi-stream" scalar function (Selescu); the orthogonality of their gradient lines

Taking into account the previously established relations (also see subsection 1.2 in [1], [3] and section 2 in [2]):

$$V_{\xi i} = \frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial\Phi_i}{\partial\lambda} \quad ; \quad V_{\eta i} = \frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial\Phi_i}{\partial\mu} \quad ;$$

$$V_{\zeta i} = \frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta} = 0 \text{ and that } \mathbf{V}_i = \nabla\Phi_i = \frac{1}{\rho}\nabla\times\mathbf{\Psi}_{ci} \quad ,$$

one gets *the searched for interdependence* $\Phi_i \leftrightarrow \Psi_{ci}$:

$$V_{\xi i} = \frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi} = \frac{1}{\rho h_\eta}\frac{\partial\Psi_{ci}}{\partial\eta};\ V_{\eta i} = \frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta} = -\frac{1}{\rho h_\xi}\frac{\partial\Psi_{ci}}{\partial\xi},$$

$$\text{or:} \quad V_{\xi i} = \frac{\partial\Phi_i}{\partial\lambda} = \frac{1}{\rho}\frac{\partial\Psi_{ci}}{\partial\mu} \quad ; \quad V_{\eta i} = \frac{\partial\Phi_i}{\partial\mu} = -\frac{1}{\rho}\frac{\partial\Psi_{ci}}{\partial\lambda} \quad .$$

Let us check the orthogonality of the gradient lines above: $\nabla\Phi_i\cdot\nabla\Psi_{ci} = (\partial\Phi_i/\partial\lambda)(\partial\Psi_{ci}/\partial\lambda) + (\partial\Phi_i/\partial\mu)(\partial\Psi_{ci}/\partial\mu) = 0$; therefore these vector lines are orthogonal each other.

6 A simple example: the 3-D conical flow

Let us consider further as a simple application the case of a general 3-D conical supersonic flow, with its two well-known particular cases of 2-D conical flow: axisymmetric (Busemann–Taylor–Maccoll) and plane (Prandtl –Meyer). The isentropic surfaces are the conical ones with

the vertex in the cone tip and the streamlines as directrices, as a result of the fact that the entropy is constant along both the streamline and any half-straight line starting from the tip. Let be: (x, y, z) – the Cartesian coordinates; (R, θ, ω) – the classical spherical coordinates; (R, φ, χ) – the generalized spherical (conical) coordinates – *intrinsic*; $x = Rf_1(\theta, \omega) = R\cos\theta$; $y = Rf_2(\theta, \omega) = R\sin\theta\cos\omega$; $z = Rf_3(\theta, \omega) = R\sin\theta\sin\omega$, and analogously (since x, y, z *may be expressed as products of separated functions for any kind of spherical coordinates*):
$x = Rf_{1G}(\varphi, \chi) = Rf_1(\varphi)g_1(\chi)$; $y = Rf_{2G}(\varphi, \chi) = Rf_2(\varphi)g_2(\chi)$; $z = Rf_{3G}(\varphi, \chi) = Rf_3(\varphi)g_3(\chi)$; (also see [8], [9]), satisfying the following condition on the Jacobian J (see subsection 1.2 in [1], [3] and section 2 in [2]):

$$J = \frac{D(x,y,z)}{D(R,\varphi,\chi)} = R^2 \begin{vmatrix} f_1 g_1 & f_2 g_2 & f_3 g_3 \\ f'_{1\varphi} g_1 & f'_{2\varphi} g_2 & f'_{3\varphi} g_3 \\ f_1 g'_{1\chi} & f_2 g'_{2\chi} & f_3 g'_{3\chi} \end{vmatrix} =$$

$$R^2 f_1 g_1 f_2 g_2 f_3 g_3 \begin{vmatrix} 1 & 1 & 1 \\ (\ln|f_1|)'_\varphi & (\ln|f_2|)'_\varphi & (\ln|f_3|)'_\varphi \\ (\ln|g_1|)'_\chi & (\ln|g_2|)'_\chi & (\ln|g_3|)'_\chi \end{vmatrix} \neq 0;$$

$J = 0$ – gives entropy's S singularities (see subsection 1.3 in [1], [3] and section 5 in [2]) for a cone at incidence: a pair of half-straight lines – two nodal logarithmic singularities, and a half-straight line – a false saddle point type singularity;

$$\text{sphere}(S): x^2 + y^2 + z^2 = C_0^2;$$

$$(R = C_0;) \quad \cos^2\theta + \sin^2\theta\cos^2\omega + \sin^2\theta\sin^2\omega = 1;$$

similarly, using the generalized spherical coordinates:

$$R = C_0; \quad f_1^2(\varphi)g_1^2(\chi) + f_2^2(\varphi)g_2^2(\chi) + f_3^2(\varphi)g_3^2(\chi) = 1;$$

$$\text{cone } (C_1): \quad F_1(y/x, z/x) = 0;$$

$$F_1\left[\frac{f_2(\varphi)g_2(\chi)}{f_1(\varphi)g_1(\chi)}, \frac{f_3(\varphi)g_3(\chi)}{f_1(\varphi)g_1(\chi)}\right] = 0; \text{ or}$$

$$\begin{cases} F_1[f_4(\varphi)g_4(\chi), f_5(\varphi)g_5(\chi)] = F_1(\varphi) = 0, \\ \text{with the solution: } \quad \varphi = C_1; \end{cases}$$

$$\text{cone } (C_2): \quad F_2(y/x, z/x) = 0;$$

$$F_2\left[\frac{f_2(\varphi)g_2(\chi)}{f_1(\varphi)g_1(\chi)}, \frac{f_3(\varphi)g_3(\chi)}{f_1(\varphi)g_1(\chi)}\right] = 0; \text{ or}$$

$$\begin{cases} F_2[f_4(\varphi)g_4(\chi), f_5(\varphi)g_5(\chi)] = F_2(\chi) = 0, \\ \text{with the solution: } \quad \chi = 1/C_2; \end{cases}$$

h_R, h_φ, h_χ are Lamé's coefficients:

$$h_i = \sqrt{\sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i}\right)^2} \quad ; \quad (i=\overline{1,3}) \quad ; \quad \begin{cases} x_j = x, y, z \\ q_i = R, \varphi, \chi. \end{cases}$$

$$h_R = \sqrt{\left(\frac{\partial x}{\partial R}\right)^2 + \left(\frac{\partial y}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2} = \sqrt{f_1^2 g_1^2 + f_2^2 g_2^2 + f_3^2 g_3^2} = 1;$$

$$h_\varphi = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = R\sqrt{f_{1\varphi}'^2 g_1^2 + f_{2\varphi}'^2 g_2^2 + f_{3\varphi}'^2 g_3^2} = R;$$

$$h_\chi = \sqrt{\left(\frac{\partial x}{\partial \chi}\right)^2 + \left(\frac{\partial y}{\partial \chi}\right)^2 + \left(\frac{\partial z}{\partial \chi}\right)^2} = R\sqrt{f_1^2 g_{1\chi}'^2 + f_2^2 g_{2\chi}'^2 + f_3^2 g_{3\chi}'^2} = Rf(\varphi).$$

We will start further from the continuity equation (1) for a certain 3-D steady motion, in a triorthogonal curvilinear q_i coordinate system, which becomes in the generalized spherical *intrinsic* coordinates (R, φ, χ):

$$\frac{\partial(\rho h_\varphi h_\chi V_R)}{\partial R} + \frac{\partial(\rho h_R h_\chi V_\varphi)}{\partial \varphi} + \frac{\partial(\rho h_R h_\varphi V_\chi)}{\partial \chi} = 0 \quad ,$$

$$\text{with:} \quad h_R = 1 \quad ; \quad h_\varphi = R \quad ; \quad h_\chi = R\,f(\varphi) \quad ,$$

$$\frac{\partial}{\partial R}\left[\rho R^2 f(\varphi) V_R\right] + \frac{\partial}{\partial \varphi}\left[\rho R f(\varphi) V_\varphi\right] + \frac{\partial}{\partial \chi}\left(\rho R V_\chi\right) = 0 \quad ;$$

$$\Rightarrow f(\varphi)\frac{\partial\left(\rho R^2 V_R\right)}{\partial R} + R\left\{\frac{\partial[\rho f(\varphi)V_\varphi]}{\partial \varphi} + \frac{\partial\left(\rho V_\chi\right)}{\partial \chi}\right\} = 0 \quad ,$$

equation that we will divide by $R\cdot f(\varphi)$ ($\neq 0$), and replacing the velocity **V** components along the system axes:

$$V_R = V = \dot{R}; \quad V_\varphi = R\dot{\varphi}; \quad V_\chi = R\,f(\varphi)\,\dot{\chi}, \text{ one gets:}$$

$$\frac{1}{R}\frac{\partial\left(\rho R^2 \dot{R}\right)}{\partial R} + \frac{1}{f(\varphi)}\left\{\frac{\partial\left[\rho R f(\varphi)\dot{\varphi}\right]}{\partial \varphi} + \frac{\partial\left[\rho R f(\varphi)\dot{\chi}\right]}{\partial \chi}\right\} = 0 \, .$$

Taking into account that we are studying a conical flow, and R, φ and χ are independent variables, chosen so that along any streamline: $\chi = \chi_0$; $\dot{\chi} = 0; \Rightarrow V_\chi = 0$, we have:

$$\frac{1}{R}\frac{\partial\left(\rho R^2 \dot{R}\right)}{\partial R} + \frac{1}{f(\varphi)}\frac{\partial\left[\rho R f(\varphi)\dot{\varphi}\right]}{\partial \varphi} = 0, \text{with:} \frac{\partial \rho}{\partial R} = \frac{\partial \dot{R}}{\partial R} = 0$$

$$(\text{conical flow}); \ \frac{\partial R}{\partial \varphi} = 0 \ (R, \varphi - \text{independent variables});$$

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial \varphi}\dot{\varphi} + \frac{\partial \rho}{\partial \chi}\dot{\chi} = \frac{\partial \rho}{\partial \varphi}\dot{\varphi} \quad ; \quad \Rightarrow \quad \frac{\partial \rho}{\partial \varphi} = \frac{\dot{\rho}}{\dot{\varphi}} \quad ;$$

$$\frac{\partial f(\varphi)}{\partial \varphi} = \frac{df(\varphi)}{d\varphi} = \frac{df(\varphi)}{dt}\frac{dt}{d\varphi} = \frac{1}{\dot{\varphi}}\frac{df(\varphi)}{dt} = \frac{\dot{[f(\varphi)]}}{\dot{\varphi}} \quad .$$

Grouping conveniently the terms and dividing by ρR ($\neq 0$), we obtain the form of the continuity equation, specific to the conical flows (a first order PDE in one variable – φ):

$$\frac{\dot{\rho}}{\rho} + 2\frac{\dot{R}}{R} + \frac{\dot{[f(\varphi)]}}{f(\varphi)} + \frac{\partial \dot{\varphi}}{\partial \varphi} = 0 \quad , \tag{5}$$

similar to Eq. (3) previously established for a certain 3-D steady flow, where: $\xi = R$; $\eta = \varphi$ and $\zeta = \chi$. We will try to express in other form the partial derivative $\partial\dot{\varphi}/\partial\varphi$:

$$\ddot{\varphi} = \frac{d\dot{\varphi}}{dt} = \frac{\partial \dot{\varphi}}{\partial R}\dot{R} + \frac{\partial \dot{\varphi}}{\partial \varphi}\dot{\varphi} + \frac{\partial \dot{\varphi}}{\partial \chi}\dot{\chi} \quad , \quad \left(\frac{\partial \dot{\varphi}}{\partial R} \neq 0\right),$$

$\dot{\varphi}$ varying with respect to R, because $\partial\left(R\dot{\varphi}\right)/\partial R = 0$, whence it results in:

$$\frac{\partial \dot{\varphi}}{\partial \varphi} = \frac{\ddot{\varphi}}{\dot{\varphi}} - \frac{\partial \dot{\varphi}}{\partial R}\frac{\dot{R}}{\dot{\varphi}} - \frac{\partial \dot{\varphi}}{\partial \chi}\frac{\dot{\chi}}{\dot{\varphi}} \quad .$$

Taking into account that we study a conical flow, in a special chosen coordinate system, there are the relations:

$$\frac{\partial V_\varphi}{\partial R} = \frac{\partial\left(R\dot{\varphi}\right)}{\partial R} = 0 \text{ and } \dot{\chi} = 0 \, .$$

Developing the first one, we get: $\dot{\varphi} + R\dfrac{\partial \dot{\varphi}}{\partial R} = 0$.

Therefore, in the case of a certain conical flow, we have: $\partial\dot{\varphi}/\partial R = -\dot{\varphi}/R$.

Replacing this in the expression of $\partial\dot{\varphi}/\partial\varphi$, one obtains:

$$\frac{\partial\dot{\varphi}}{\partial\varphi}=\frac{\ddot{\varphi}}{\dot{\varphi}}+\frac{\dot{\varphi}}{\mathrm{R}}\frac{\dot{\mathrm{R}}}{\dot{\varphi}}=\frac{\dot{\mathrm{R}}}{\mathrm{R}}+\frac{\ddot{\varphi}}{\dot{\varphi}}\ ,$$

the previous continuity Eq. (5) for conical flows getting the first integrable general form below (a second order ODE):

$$\frac{\dot{\rho}}{\rho}+3\frac{\dot{\mathrm{R}}}{\mathrm{R}}+\frac{\dot{[\mathrm{f}(\varphi)]}}{\mathrm{f}(\varphi)}+\frac{\ddot{\varphi}}{\dot{\varphi}}=0,\text{or}\ \frac{\mathrm{d}}{\mathrm{dt}}\left[\ln\left|\rho\mathrm{R}^3\mathrm{f}(\varphi)\dot{\varphi}\right|\right]=0,\quad(6)$$

valid for any conical flow (either potential or rotational), representing the new form of its continuity equation, admitting a remarkable first integral (the flow rate equation):

$$\rho\mathrm{R}^3\mathrm{f}(\varphi)\dot{\varphi}=\mathrm{Q_i}\ ,\ \text{or}\ \ \rho\mathrm{R}^2\mathrm{f}(\varphi)\mathrm{V}_\varphi=\mathrm{Q_i}\ ,\ \text{or more}:$$

$$\rho\mathrm{R}^2\mathrm{f}(\varphi)V'=\mathrm{Q_i}\quad(\text{with}\ {}'=\mathrm{d}/\mathrm{d}\varphi)\qquad(7)$$

(see example 3 at the end of subsection 1.2 in [1], [3]), the constants $\mathrm{Q_i}$ being kept over the field surfaces $\psi(\mathrm{R},\varphi)=\mathrm{Q_i}$. The quantity $\mathrm{R}^2\mathrm{f}(\varphi)$ is proportional to the infinitesimal side area of the current cone (C_1): $\varphi=\varphi_{0j}=\mathrm{C}_1$ (fig. 1.b in [1] – [3], [8]), between two closely near (isentropic) cones (C_2): $\chi=\chi_{0i}=\mathrm{c}(\mathrm{S}_{0i}-\mathrm{c}_0)=1/\mathrm{C}_2$ (fig. 1.a in [1] – [3], [8]); the tip half-angle φ of this quasi-circular cone (compression simple wave) is an intrinsic coordinate. One can apply a qualitative analogy of conical isentropic sheets traces to the streamlines in an incompressible 2-D plane potential flow. In fig. 1 the flow due to a two source system is represented. A more realistic flow pattern can be obtained adding a third source. In the particular case of the axisymmetric conical flow (Busemann–Taylor–Maccoll: see [10], [11]), we have:

$$\varphi=\theta\quad;\quad\dot{\varphi}=\dot{\theta}\quad;\quad\ddot{\varphi}=\ddot{\theta}\quad;\quad\mathrm{f}(\varphi)=\mathrm{f}(\theta)=\sin\theta\quad;$$

$$\dot{[\mathrm{f}(\varphi)]}=\dot{[\mathrm{f}(\theta)]}=\dot{(\sin\theta)}=\cos\theta\cdot\dot{\theta}\quad,$$

so being obtained for the ODE (6) the first integrable form:

$$\frac{\dot{\rho}}{\rho}+3\frac{\dot{\mathrm{R}}}{\mathrm{R}}+\frac{\dot{(\sin\theta)}}{\sin\theta}+\frac{\ddot{\theta}}{\dot{\theta}}=0,\text{or}:\ \frac{\dot{\rho}}{\rho}+3\frac{\dot{\mathrm{R}}}{\mathrm{R}}+\cot\theta\cdot\dot{\theta}+\frac{\ddot{\theta}}{\dot{\theta}}=0$$

and respectively, for the first integral (7) (also see [8], [12]):

$$\rho\mathrm{R}^3\sin\theta\dot{\theta}=\rho\mathrm{R}^2\sin\theta\mathrm{V}_\theta=\rho\mathrm{R}^2\sin\theta\cdot V'=\mathrm{Q}_3=m/\pi\quad.$$

The quantity $\pi\mathrm{R}^2\sin\theta$ represents just the side area of the current circular cone of vertex half-angle θ, in the constant Q_3 expression intervening the mass flow rate m. In case of the plane 2-D conical flow, it can be easily proved that:

$$\varphi=\omega\quad;\quad\dot{\varphi}=\dot{\omega}\quad;\quad\ddot{\varphi}=\ddot{\omega}\quad;\quad\mathrm{f}(\varphi)=\mathrm{f}(\omega)=\frac{\mathrm{C}}{\mathrm{R}(\omega)}\quad;$$

$$\dot{[\mathrm{f}(\omega)]}=-\frac{\mathrm{C}\dot{\mathrm{R}}(\omega)}{\mathrm{R}^2(\omega)}\quad;\quad\frac{\dot{[\mathrm{f}(\omega)]}}{\mathrm{f}(\omega)}=-\frac{\dot{\mathrm{R}}}{\mathrm{R}}\quad,$$

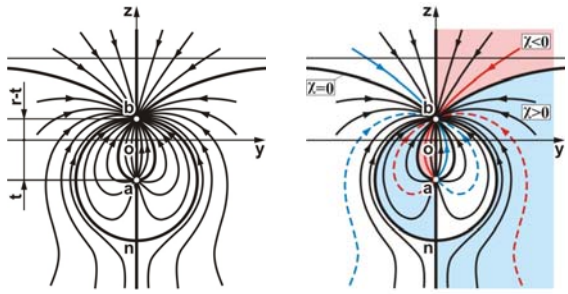
representing the plane conical flow (with Prandtl–Meyer expansion or isentropic compression; see [13] – [17]):

$$\varphi=\omega;\ {}'=\frac{\mathrm{d}}{\mathrm{d}\varphi}=\frac{\mathrm{d}}{\mathrm{d}\omega};\ \mathrm{f}(\varphi)=\mathrm{f}(\omega)=\cos^{\frac{\gamma+1}{\gamma-1}}\left[\sqrt{\frac{\gamma-1}{\gamma+1}}(\omega+\omega_0)\right];$$

$$\left[\ln\left|\mathrm{f}(\varphi)\right|\right]'=-\sqrt{\frac{\gamma+1}{\gamma-1}}\tan\left[\sqrt{\frac{\gamma-1}{\gamma+1}}(\omega+\omega_0)\right]\quad;$$

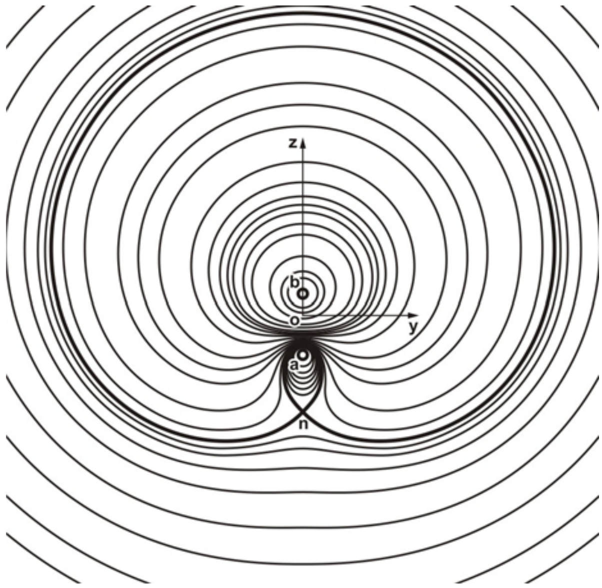
the angle of attack $\alpha\neq 0$; r is the cone local radius;

$$\bar{\mathrm{r}}=\mathrm{r}/\mathrm{x}\quad;\quad\bar{\mathrm{t}}=\mathrm{t}/\mathrm{x}=\tan\alpha\quad;\quad\Rightarrow\quad\alpha=\operatorname{atan}\bar{\mathrm{t}}\quad;$$



$$\text{1.a.}\ \ \bar{\mathrm{Z}}=\operatorname{Im}\mathrm{F}(\mathrm{x}_1)=\operatorname{Im}\left\{\ln\frac{\left[\mathrm{x}_1-i(\bar{\mathrm{r}}-\bar{\mathrm{t}})\right]^2}{\mathrm{x}_1+i\bar{\mathrm{t}}}\right\}=\tan^{(-1)}\chi=\mathrm{C}_2';$$

$$\chi=\frac{\bar{\mathrm{y}}^2(\bar{\mathrm{z}}+\bar{\mathrm{t}}-2\bar{\mathrm{r}})+(\bar{\mathrm{z}}+\bar{\mathrm{t}}-\bar{\mathrm{r}})^2(\bar{\mathrm{z}}+\bar{\mathrm{t}})}{\bar{\mathrm{y}}\left[\bar{\mathrm{y}}^2+(\bar{\mathrm{z}}+\bar{\mathrm{t}})^2-\bar{\mathrm{r}}^2\right]}=\frac{1}{\mathrm{C}_2}=\tan\mathrm{C}_2';$$



$$\text{1.b.}\ \ \bar{\mathrm{Y}}=\operatorname{Re}\mathrm{F}(\mathrm{x}_1)=\operatorname{Re}\left\{\ln\frac{\left[\mathrm{x}_1-i(\bar{\mathrm{r}}-\bar{\mathrm{t}})\right]^2}{\mathrm{x}_1+i\bar{\mathrm{t}}}\right\}=\ln\sqrt{\varphi}=\mathrm{C}_1'\ ;$$

$$\varphi=\frac{\left[\bar{\mathrm{y}}^2+(\bar{\mathrm{z}}+\bar{\mathrm{t}}-\bar{\mathrm{r}})^2\right]^2}{\bar{\mathrm{y}}^2+(\bar{\mathrm{z}}+\bar{\mathrm{t}})^2}=\mathrm{C}_1=\mathrm{e}^{2\mathrm{C}_1'}\ ;$$

$\mathrm{x}_1=\bar{\mathrm{y}}+i\bar{\mathrm{z}}$ is a complex variable; $\bar{\mathrm{y}}=\mathrm{y}/\mathrm{x}$; $\bar{\mathrm{z}}=\mathrm{z}/\mathrm{x}$; x, y, z – Cartesian coordinates, x – abscissa of cross section current plane

Fig. 1. Simple qualitative example of "generalized spherical" smart intrinsic coordinate surfaces for the case of a circular cone at a small angle of attack (cross section) – the incompressible approximation (slender body; no shock wave), giving the relations between the new spherical coordinates (R, φ, χ) and the Cartesian ones; the flow is due to two semi-infinite line sources along cone's axis (a) and back (b), replacing it; (a), (b), (n) – nodal and saddle lines; 1.a. the conical isentropic sheets $\chi=\chi_{0i}=\mathrm{c}(\mathrm{S}_{0i}-\mathrm{c}_0)=1/\mathrm{C}_2$ (a smart intrinsic coordinate tied to S_{0i} – the local specific entropy value), having as remarkable directrices: the **oz** axis and a circle (the solid cone trace) centered on it (both for $\mathrm{C}_2=0$), and a right strophoid ($\chi=0$) centered on **oz** axis too (for $1/\mathrm{C}_2=0$); c, S_{0i}, $\mathrm{c}_0>0$; 1.b. the conical sheets $\varphi=\varphi_{0j}=\mathrm{C}_1$ (smart intrinsic coordinate), orthogonal to the conical isentropic ones: $\nabla\bar{\mathrm{Y}}(\bar{\mathrm{y}},\bar{\mathrm{z}})\cdot\nabla\bar{\mathrm{Z}}(\bar{\mathrm{y}},\bar{\mathrm{z}})=0$, having as remarkable directrices: a Pascal's limaçon and the circle at infinity (both for $\mathrm{C}_1=0$ and centered on the **oz** axis).

(the isentropic surfaces are the planes parallel to that represented by the degenerate cone $\theta = \pi/2$, that means: $\chi = x = x_0$; the $\varphi = \varphi_0 = C_1$ intrinsic coordinate surfaces are the planes $\varphi = \omega = \omega_0$), being thus proved that:

$$f(\omega) = \frac{R_0}{R(\omega)} \quad \text{and} \quad \frac{f'(\omega)}{f(\omega)} = -\frac{R'(\omega)}{R(\omega)} \quad , \quad \text{or} \quad \frac{\dot{f}}{f} = -\frac{\dot{R}}{R} \quad ,$$

so being obtained the first integrable form below:

$$\frac{\dot{\rho}}{\rho} + 2\frac{\dot{R}}{R} + \frac{\ddot{\omega}}{\dot{\omega}} = 0 \quad , \quad \text{the first integral (7) becoming:}$$

$\rho R^2\dot{\omega} = \rho R V_\omega = \rho R V' = Q_2 = m/l$ (also see [8], [12]), where l is the thickness (depth) of the plane flow layer.

7 The flow rate equation; special and general cases of first integrability for the continuity equation in a thick stream tube

The first integrability of the general (even unsteady) continuity equation (like Eqs. (2), (3)) will be treated in section 12. The *fluid flow* across a stationary surface S is the mass m of the fluid passing across this surface in unit time (the *mass flow rate*). Thus:

$$m = \int_S \rho V_n \, dS = \int_S \mathbf{jn} \, dS \ ,$$

where: $\mathbf{n}$ – versor (unit vector) of the outward normal to the element of surface dS; $V_n = \mathbf{Vn}$ – projection of the fluid velocity on the vector $\mathbf{n}$; $\mathbf{j} = \rho\mathbf{V}$ – vector of the density of fluid flow (the mass flux density); $\mathbf{jn} = \rho\mathbf{Vn}$ – the normal mass flux density. It follows from the continuity equation that for a steady flow of a fluid in a pipe:

$$m = \int_S \rho V_n \, dS = \text{const.}$$

With a steady motion, the flow of fluid through a cross section of a stream filament does not depend on the location of the cross section. For two arbitrary cross sections dS_1 and dS_2 of an elemental filament the following condition holds:

$\rho_1|V_{n1}|[dS_1] = \rho_2|V_{n2}|[dS_2]$,

also called "the continuity equation for the steady flow in thin stream tubes". In some very special cases, if the quantities ρ_j and V_{nj} are rigorously constant on each "j" section (sections of uniform flow of a thick stream tube – also isochoric surfaces), then the ODE above admits in these sections a first integral – the (mass) flow rate equation:

$\rho_1|V_{n1}|[S_{u1}] = \rho_2|V_{n2}|[S_{u2}] = m = \text{const.}$, or more:

$\rho_j|\mathbf{V}_j\cdot\mathbf{n}_j|[S_{uj}] = \rho_j|V_{nj}|[S_{uj}] = m = \text{const.}$. (8)

Here $[S_{uj}]$ is the stream tube cross section area in the "j" section of uniform flow S_{uj} , crossing the "i" surfaces ($\zeta = \zeta_{0i}$), and V_{nj} is the component of $\mathbf{V}_j$ taken on the local normal to the respective "j" section of uniform flow. This is a special first integral (obtained for particular cases) of the continuity equation for a steady flow of a compressible fluid. Simple examples of such flows are:

- the flow due to a source of constant mass flow rate (plane and spherical – in a conical tubular domain);
- the plane 2-D conical supersonic flow (Prandtl–Meyer flow): with expansion, or with isentropic compression in a streamlined channel (a reverse of the expansion flow);
- the axisymmetric 2-D conical supersonic flow (Busemann –Taylor–Maccoll flow) with isentropic compression:
  1. downstream of an axisymmetric attached conical shock wave – around an infinite circular solid cone without incidence (between the shock wave and the solid cone), and
  2. upstream of an axisymmetric conical shock wave with the same tip as the conical simple compression waves fan – inside Busemann's nozzle connecting two parallel and uniform co-axial supersonic flows.

The first integral (8) is valid over the volume inside any stream tube surface (a tubular surface composed of streamlines as generatrices and a certain closed curve contained in a "j" section of uniform flow as directrix). The most general case for which the continuity equation in its global form:

$$\oint_{\mathscr{S}} \rho\mathbf{Vn}\, d\mathscr{S} = \oint_{\mathscr{S}} \mathbf{jn}\, d\mathscr{S} = \int_{S_{in}} \mathbf{jn}\, dS + \int_{S_{out}} \mathbf{jn}\, dS = 0 \quad ,$$

admits as first integral a similar one to Eq. (8) (the same for the compressible flow in thin as well thick stream tubes) is:

$|\mathbf{jn}| = \rho|\mathbf{V}\cdot\mathbf{n}| = \rho|V_n| = \text{const.(j)}$,

on some sections S_j crossing a closed surface $\mathscr{S}$ of a thick stream tube (the lids of this thick stream tube, delimiting a control volume). That introduces the surfaces S_j of *iso*-normal mass flux *density* $(\mathbf{jn})_j$ (a scalar quantity):

$(|\mathbf{jn}|)_j = (\rho|\mathbf{V}\cdot\mathbf{n}|)_j = (\rho|V_n|)_j = m/[S_j] = \text{const.(j)}$, (9)

whose envelopes are just the "j" sections of uniform flow – similarly to Huygens' principle (see [18]) explaining the propagation of a wavefront due to the spherical wavelets emanating from every point on the wavefront at the prior instant.

8 A method for determining the integral surfaces (lines) of the continuity equation in a thick stream tube for the 2-D flows

Assuming the local values of density ρ and velocity $\mathbf{V}$ as being given (known) all over the respective domain of plane flow, the problem solution is given by a system of two ODEs, expressing the constancy of $\mathbf{jn}$ over (along) the integral "j" surfaces (lines) – see Eq. (9), and the length of an elementary arc ds (s being the intrinsic curvilinear coordinate) in the Cartesian coordinates (x, y), respectively, where $\rho(x, y)$, $V_x(x, y)$ and $V_y(x, y)$ are given functions of x and y:

$$\begin{cases} \rho V_x(dy/ds) - \rho V_y(dx/ds) = \text{const.(j)} = k_j \quad ; \\ (dx/ds)^2 + (dy/ds)^2 = 1 \quad , \end{cases}$$

the normal form (the solution) of which is given by:

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{1}{\rho\left(V_x^2 + V_y^2\right)}\left[-k_j V_y \pm V_x\sqrt{\rho^2\left(V_x^2 + V_y^2\right)^2 - k_j^2}\right] \ ; \\ \dfrac{dy}{ds} = \dfrac{1}{\rho\left(V_x^2 + V_y^2\right)}\left[k_j V_x \pm V_y\sqrt{\rho^2\left(V_x^2 + V_y^2\right)^2 - k_j^2}\right] \ . \end{cases}$$

For the case of an axisymmetric flow, replacing y by r in a certain meridian plane in cylindrical coordinates (x, r), the above solution remains valid, so obtaining the meridian line of the axisymmetric integral "j" surfaces. The sign (±) before the square root in the solution (dx/ds, dy/ds) means that through any flow's point are passing two "j" surfaces (lines) of *iso*-normal mass flux *density*. The quantity under this square root must be positive or zero, the physical significance of the root being the tangential mass flux density. If this quantity is just zero, a single solution remains, representing the "j" surface (line) of normal uniform flow passing through the same flow's point. Along this line we have $|\mathbf{V}_j| = k_j/\rho_j$, the solution not depending on ρ:

$$\begin{cases} \left(\dfrac{dx}{ds}\right)_j = -\dfrac{k_j V_{jy}}{\rho_j\left(V_{jx}^2+V_{jy}^2\right)} = -\dfrac{k_j V_{jy}}{\rho_j\left|\mathbf{V}_j\right|^2} = -\dfrac{V_{jy}}{\left|\mathbf{V}_j\right|} \quad ; \\ \left(\dfrac{dy}{ds}\right)_j = \dfrac{k_j V_{jx}}{\rho_j\left(V_{jx}^2+V_{jy}^2\right)} = \dfrac{k_j V_{jx}}{\rho_j\left|\mathbf{V}_j\right|^2} = \dfrac{V_{jx}}{\left|\mathbf{V}_j\right|} \quad . \end{cases}$$

Therefore through any 2-D flow's point three integral "j" lines (two S_j and one normal S_{uj}) are passing. If the root has a constant value k_{1j}, one gets a pair of solutions (±), representing other two "j" lines of uniform flow S_{uj}, this time not normal, like the half-straight lines for the plane (Prandtl–Meyer) and axisymmetric (Busemann–Taylor–Maccoll) conical flows. In these cases more conveniently is to use the polar plane (R, ω), and polar spherical (R, θ) coordinates (see section 6) instead of the Cartesian ones. Furthermore, this method can be extended for the 3-D flows, in the new smart intrinsic orthogonal coordinates (ξ, η, ζ), or (λ, μ, ν), used in sections 1 – 5 of this work, in order to find one (that in the plane (ξ, η), or (λ, μ) tangent to the (**V**, **Ω**) surface) of the two orthogonal lines which determine the local integral "j" surface (**jn** = const.) for the continuity equation.

9 A simple example: the plane flow due to a source of constant mass flow rate

Let us consider as simple example the case of a plane flow of a compressible fluid due to a source of constant mass flow rate (between two radii – half-straight lines). The flow is not a conical one, but is closely related to this (circular cylindrical), the main physical quantities (velocity, static pressure, static temperature, density, entropy) being kept constant on some concentric circles (co-axial cylinders). The surfaces S_j of *iso*-normal mass flux *density* $(\mathbf{jn})_j$ are the families of circles (circular cylinders) with the same diameter d_j and passing through the respective source trace (origin, or the Oz axis – a singularity for **jn**: $(\mathbf{jn})_O = 0$), given by the Cartesian equation below: $(x - d_j/2\cdot\cos\theta_k)^2 + (y - d_j/2\cdot\sin\theta_k)^2 = (d_j/2)^2$, or $x^2 + y^2 - d_j(x\cos\theta_k + y\sin\theta_k) = 0$, with $\theta_k \in (0, 2\pi)$, or by the polar equation $R = d_j\cos(\theta - \theta_k)$, with $\theta, \theta_k \in (0, 2\pi)$. Their envelopes (the surfaces S_{uj} of uniform flow) are the concentric circles (co-axial circular cylinders) of radius $R = d_j$, having the source trace as center (the Oz axis as symmetry axis), given by the Cartesian equation: $x^2 + y^2 - d_j^2 = 0$.

But this flow (either subsonic or supersonic) takes place outside the circle (circular cylinder) $R = R_c$ only (R_c is the critical radius, on which $V = a = c$ – the critical speed at the considered flow point), so that one must have: $d_j \geq R_c$; ($R < R_c$ – a no-motion region), by physical reasons (see fig. 2). Therefore the surfaces S_j of *iso*-normal mass flux *density* $(\mathbf{jn})_j$ are the families of circles (circular cylinders) above, except for their regions inside the circle (circular cylinder) of equation $R = R_c$, so including the singularity in origin for the *iso*-normal mass flux *density* $(\mathbf{jn})_O$. The flow rate Eqs. (8), (9) take in this case the form: $2\pi d_j(\mathbf{jn})_j = m = \text{const.}$, for both kinds of integral surfaces (S_j and S_{uj}).

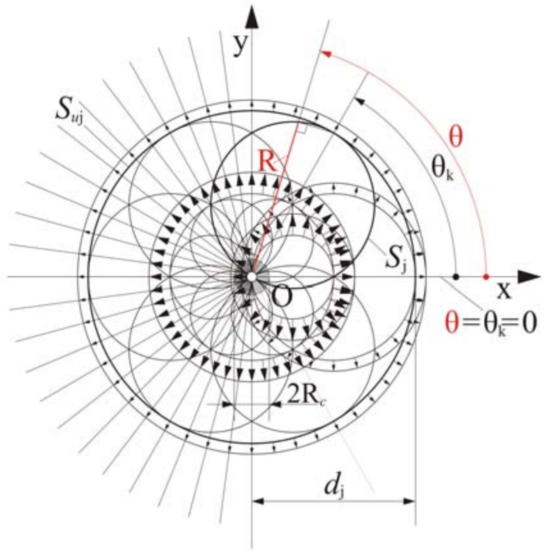


Fig. 2. The pattern of the *iso*-normal mass flux *density* surfaces and the sections of uniform flow for a simple plane flow of a compressible fluid due to a source of constant mass flow rate *m*

O – the source trace in the xOy plane (orthogonal projection of the source filament directed on the Oz-axis); S_j – the lines of *iso*-normal mass flux *density* $|\mathbf{jn}| = \rho|\mathbf{Vn}| = \rho|V_n| = \text{const(j)}$ – circles passing through O (traces in the xOy plane of the right circular cylinders "j" with their axes parallel to the Oz-axis and of diameter d_j), given by the polar equation: $R = d_j\cos(\theta - \theta_k)$, with $\theta_k \in (0, 2\pi)$; along a certain circle θ_k takes a constant value; θ is the current polar angle; S_{uj} – the lines of uniform flow (on which $\rho = \rho_j = \text{const}_1(j)$ and $V_n = V_{nj} = \text{const}_2(j)$) – circles centered in O (traces in the xOy plane of the right circular cylinders "j" with their axes directed on the Oz-axis and of diameter $2d_j$ – the envelope sheets of the cylinders "j" of *iso*-normal mass flux *density*), given by the polar eq. $R = d_j$. O is a singular point for $|\mathbf{jn}|$. But this flow (either subsonic or supersonic) takes place outside the "critical" cylinder (circle) only: $R \geq R_c = m/(2\pi\rho_c c)$ – the critical radius, with $\rho_c c = \{\gamma p_0\rho_0[2/(\gamma+1)]^{(\gamma+1)/(\gamma-1)}\}^{1/2}$ – the critical *iso*-normal mass flux *density* $(|\mathbf{jn}|)_c$, so that one must have: $d_j \geq R_c$, by physical reasons; $R < R_c$ – a no-motion region (a solid nucleus). The lengths of radial arrowheads are proportional to density $|\mathbf{jn}|$. Through any "j" flow's point three integral surfaces (lines) are passing: one of normal uniform flow (S_{uj}) and other two of *iso*-normal mass flux *density* (S_j), all for the same value of $|\mathbf{jn}|$. These results can be generalized for a spherical source.

10 New global forms for the flow rate equation and continuity equation, using the 3-D “stream function” vector; analogy with Helmholtz’ 2nd theorem on vortices

Considering $\mathbf{\Psi}_c$ definition and applying a Stokes’ formula, we get for the normal flux of curl $\mathbf{\Psi}_c$ through the surface S_j:

$$m = \int_{S_j} \mathbf{jn}\mathrm{d}S = \int_{S_j} (\nabla\times\mathbf{\Psi}_c)\cdot\mathbf{n}\mathrm{d}S = \oint_{l_j} \mathbf{\Psi}_c \cdot \mathrm{d}\mathbf{R} = (\Gamma_{\mathbf{\Psi}_c})_{l_j} = \text{const.}$$

– the circulation of the vector $\mathbf{\Psi}_c$ along the closed curve l_j bordering the surface S_j, where, this time S_j is an arbitrary cross section of the thick stream tube above – *a new global form of the flow rate equation,* leading to:

$$\oint_{\mathscr{S}} \mathbf{jn}\mathrm{d}\mathscr{S} = \int_{S_{in}} \mathbf{jn}\mathrm{d}S + \int_{S_{side}} \mathbf{jn}\mathrm{d}S + \int_{S_{out}} \mathbf{jn}\mathrm{d}S = 0 \quad \text{, or :}$$

$$\int_{S_{in}} (\nabla\times\mathbf{\Psi}_c)\cdot\mathbf{n}\mathrm{d}S + 0\,(\text{stream tube}) + \int_{S_{out}} (\nabla\times\mathbf{\Psi}_c)\cdot\mathbf{n}\mathrm{d}S$$

$$= \oint_{l_{in}} \mathbf{\Psi}_c\cdot\mathrm{d}\mathbf{R} + \oint_{l_{out}} \mathbf{\Psi}_c\cdot\mathrm{d}\mathbf{R} = (\Gamma_{\mathbf{\Psi}_c})_{in} + (\Gamma_{\mathbf{\Psi}_c})_{out} = 0 \quad ,$$

($\mathrm{d}\mathbf{R}$ is an elementary displacement vector; $\mathscr{S}$ is defined in section 7, being composed of S_{in}, S_{side} and S_{out}), so getting:

$$(\Gamma_{\mathbf{\Psi}_c})_{out} = -\,(\Gamma_{\mathbf{\Psi}_c})_{in} = \text{const.} \tag{10}$$

– *a new global form of the continuity equation*, like Helmholtz’ 2nd theorem (see [19] – [24]) about the velocity circulation $\Gamma_\mathbf{V}$ along any closed curve surrounding a vortex tube in an ideal (inviscid) fluid (see fig. 3, helping to give an alternative demonstration of the theorem above).

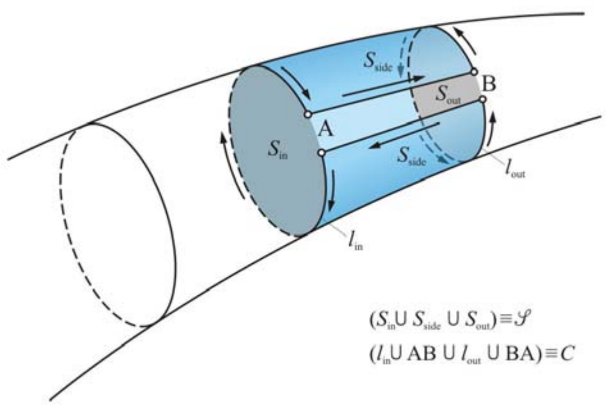


Fig. 3. The circulation $\Gamma_{\mathbf{\Psi}_c}$ of the 3-D “stream function” vector $\mathbf{\Psi}_c$ along any complex closed curve C bordering a cut surface S_{side} of a thick stream tube of a compressible perfect fluid flow, $\mathbf{\Psi}_c$ defined by: $\nabla\times\mathbf{\Psi}_c = \rho\mathbf{V}$ ($\mathbf{\Psi}_c \neq \nabla G$) – the analogy with the circulation $\Gamma_\mathbf{V}$ of the velocity $\mathbf{V}$ along any closed curve surrounding a vortex tube, according to Helmholtz’ 2nd theorem about vortices in a perfect fluid, which states that $\Gamma_\mathbf{V}$ is constant along this vortex tube and constant in time also; $\mathscr{S}$ – a closed *surface* of control volume of the tube (*bordered* by the lids S_{in} and S_{out}), so that *it* includes both lids (S_{in} and S_{out}): $S_{in}, S_{out} \subset \mathscr{S}$; $\mathscr{S} \equiv (S_{in} \cup S_{side} \cup S_{out})$; AB – a cut on the tube surface, between l_{in} and l_{out} (the border curves of S_{in} and S_{out}), so that the closed curve $C \equiv (l_{in} \cup \text{AB} \cup l_{out} \cup \text{BA})$ surrounds a simply-connected domain S_{side}, allowing to apply a Stokes’ formula for the normal flux of $\nabla\times\mathbf{\Psi}_c$ through the surface S_{side}.

So: $\int_{S_{side}} \mathbf{jn}\mathrm{d}S = \int_{S_{side}} (\nabla\times\mathbf{\Psi}_c)\mathbf{n}\mathrm{d}S = \int_C \mathbf{\Psi}_c\mathrm{d}\mathbf{R} = \int_{in} \mathbf{\Psi}_c\mathrm{d}\mathbf{R} + \int_{AB} \mathbf{\Psi}_c\mathrm{d}\mathbf{R} + \int_{out} \mathbf{\Psi}_c\mathrm{d}\mathbf{R} + \int_{BA} \mathbf{\Psi}_c\mathrm{d}\mathbf{R} = (\Gamma_{\mathbf{\Psi}_c})_{in} + (\Gamma_{\mathbf{\Psi}_c})_{out} = 0$, leading to Eq. (10).

11 A first integrability case for the system of two equations for the steady rotational flow of an inviscid compressible fluid

The general (unsteady, rotational) flow of a viscous Newtonian compressible fluid is described by the system

$$\begin{cases} \partial\mathbf{V}/\partial t + \nabla(\mathbf{V}^2/2) + \mathbf{\Omega}\times\mathbf{V} = \mathbf{f} - [\nabla p - \mu_1\Delta\mathbf{V} \\ \quad -(\mu_2 + \mu_1/3)\cdot\nabla(\nabla\mathbf{V})]/\rho\,, \quad \mu_1 = \text{ct.}; \quad \mu_2 = \text{ct.}; \\ \partial\rho/\partial t + \nabla(\rho\mathbf{V}) = 0 \quad , \end{cases}$$

in first equation (Navier–Stokes) using classical symbols (also see [24], p. 193 – the viscous Crocco–Vázsonyi eq.). One must search for first integrals for this system of simultaneous equations above (motion and continuity). For the case of a steady flow the system becomes:

$$\begin{cases} \nabla(\mathbf{V}^2/2) + \mathbf{\Omega}\times\mathbf{V} = \mathbf{f} - [\nabla p - \mu_1\Delta\mathbf{V} - (\mu_2 + \mu_1/3)\cdot\nabla(\nabla\mathbf{V})]/\rho\,; \\ \nabla(\rho\mathbf{V}) = 0 \quad . \end{cases}$$

Both equations admit first integrals: the first one – along the intersection lines of the zero-work surfaces (for the non-conservative force densities – see [1], [3], [25]) with the isentropic ones (see the same); the second one – over the sections of uniform flow (if they exist), and over the surfaces of *iso*-normal mass flux *density*. These integrals do not represent the system’s first integrals, due to their different definition (with no intersection domain). In the case of an inviscid fluid flow the system above becomes simpler: $\nabla(\mathbf{V}^2/2) + \mathbf{\Omega}\times\mathbf{V} = \mathbf{f} - \nabla p/\rho \quad ; \quad \nabla(\rho\mathbf{V}) = 0 \quad .$

The first integrals for these equations are given: by Eq. (11) in [1], [3], valid over any ($\mathbf{V}$, $\mathbf{\Omega}$) isentropic “i” surface (a), and by Eq. (5) in [9], valid over any polytropic surface (b) – a more general case, including the previous one, and, on the other hand, by the flow rate equation (8), valid over the “j” sections of uniform flow, on which:

$\rho = \text{const.}$ and $|V_n| = \text{const.}$ as well (c),

and over the surfaces S_j of *iso*-normal mass flux *density*:

$\rho|\mathbf{V}\cdot\mathbf{n}| = \rho|V_n| = \text{const.}$ (d),

whose envelope sheets are just the sections of uniform flow of a thick stream tube (if they exist), respectively. For an incompressible fluid flow the sheet families (c) and (d) coincide: (c) ≡ (d). On the intersection lines of these two pair of surface families: [(a) ∩ (c)], or [(a) ∩ (d)], and: [(b) ∩ (c)], or [(b) ∩ (d)], one obtains the validity domain of the first integrals for the system of equations for the steady rotational flow of an inviscid compressible fluid. But, since the various polytropic surfaces passing through a certain point are in infinite number (a star of sheets), this results in satisfying the integrability conditions for both equations over the surfaces (c) and (d) only. A similar problem of a first integrability case for a system of simultaneous vector equations occurs in MHD and was treated in subsection 2.10 in [26]. Besides the two equations above, the “magnetic induction” one must be considered.

12 The first approach to the new unsteady model (the complete continuity equation); two related rare cases of first integrability

Let us start from the equivalent local form of this equation $d\rho/dt + \rho\nabla\mathbf{V} = 0$ – the complete continuity equation (see section 1), for the unsteady flow of a compressible fluid. Dividing the equation by ρ ($\neq 0$) one obtains the form: $(1/\rho)\cdot(d\rho/dt) + \nabla\mathbf{V} = 0$, or more: $d(\ln\rho)/dt + \nabla\mathbf{V} = 0$. We apply further the same procedure as in section 1, using Lamé's coefficients and the total time-derivatives. In a triorthogonal system of curvilinear coordinates $\nabla\mathbf{V} =$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(h_\eta h_\zeta V_\xi\right)+\frac{\partial}{\partial\eta}\left(h_\xi h_\zeta V_\eta\right)+\frac{\partial}{\partial\zeta}\left(h_\xi h_\eta V_\zeta\right)\right]$$

with $V_\xi = h_\xi\dot{\xi}$; $V_\eta = h_\eta\dot{\eta}$; $V_\zeta = h_\zeta\dot{\zeta}$, so having $\nabla\mathbf{V} =$

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(h_\xi h_\eta h_\zeta\dot{\xi}\right)+\frac{\partial}{\partial\eta}\left(h_\xi h_\eta h_\zeta\dot{\eta}\right)+\frac{\partial}{\partial\zeta}\left(h_\xi h_\eta h_\zeta\dot{\zeta}\right)\right]$$

$$=\frac{1}{h_\xi h_\eta h_\zeta}\left[\left(\dot{\xi}\frac{\partial}{\partial\xi}+\dot{\eta}\frac{\partial}{\partial\eta}+\dot{\zeta}\frac{\partial}{\partial\zeta}\right)\left(h_\xi h_\eta h_\zeta\right)\right.$$
$$\left.+h_\xi h_\eta h_\zeta\left(\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\right)\right]=$$

and, taking into account the local time-derivative ($\partial/\partial t$),

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\left(\frac{d}{dt}-\frac{\partial}{\partial t}\right)\left(h_\xi h_\eta h_\zeta\right)+h_\xi h_\eta h_\zeta\left(\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\right)\right]$$

$$=\frac{1}{h_\xi h_\eta h_\zeta}\left(\frac{d}{dt}-\frac{\partial}{\partial t}\right)\left(h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}$$

$$=\frac{d}{dt}\ln\left(h_\xi h_\eta h_\zeta\right)-\frac{\partial}{\partial t}\ln\left(h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\text{, or}$$

$$\left[\ln\left(h_\xi h_\eta h_\zeta\right)\right]^{\bullet}-\left(\frac{\partial\ln h_\xi}{\partial t}+\frac{\partial\ln h_\eta}{\partial t}+\frac{\partial\ln h_\zeta}{\partial t}\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}$$

$$=\frac{\dot{h}_\xi}{h_\xi}+\frac{\dot{h}_\eta}{h_\eta}+\frac{\dot{h}_\zeta}{h_\zeta}-\left(\frac{1}{h_\xi}\frac{\partial h_\xi}{\partial t}+\frac{1}{h_\eta}\frac{\partial h_\eta}{\partial t}+\frac{1}{h_\zeta}\frac{\partial h_\zeta}{\partial t}\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}$$

the complete equation becoming thus successively:

$$\frac{d}{dt}\ln\rho+\frac{d}{dt}\ln\left(h_\xi h_\eta h_\zeta\right)-\frac{\partial}{\partial t}\ln\left(h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}$$
$$=0;\frac{d}{dt}\ln\left(\rho h_\xi h_\eta h_\zeta\right)-\frac{\partial}{\partial t}\ln\left(h_\xi h_\eta h_\zeta\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0,$$

differing from Eq. (2) by the 2nd term. In an expanded form

$$\frac{\dot{\rho}}{\rho}+\frac{\dot{h}_\xi}{h_\xi}+\frac{\dot{h}_\eta}{h_\eta}+\frac{\dot{h}_\zeta}{h_\zeta}-\left(\frac{1}{h_\xi}\frac{\partial h_\xi}{\partial t}+\frac{1}{h_\eta}\frac{\partial h_\eta}{\partial t}+\frac{1}{h_\zeta}\frac{\partial h_\zeta}{\partial t}\right)+\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}=0$$

which can accept a first integral in the cases where:

$$\frac{1}{h_\xi}\frac{\partial h_\xi}{\partial t}+\frac{1}{h_\eta}\frac{\partial h_\eta}{\partial t}+\frac{1}{h_\zeta}\frac{\partial h_\zeta}{\partial t}-\left(\frac{\partial\dot{\xi}}{\partial\xi}+\frac{\partial\dot{\eta}}{\partial\eta}+\frac{\partial\dot{\zeta}}{\partial\zeta}\right)=\dot{F},$$

F being a scalar function depending on ξ, η, ζ and t, having as trivial case: F = const. In the Cartesian coordinates: $h_\xi = h_x = 1$; $h_\eta = h_y = 1$; $h_\zeta = h_z = 1$, so that the integrability condition becomes $\partial\dot{x}/\partial x + \partial\dot{y}/\partial y + \partial\dot{z}/\partial z = \nabla\mathbf{V} = -\dot{F}$.

Further two related rare cases of integrability for the continuity equation were analyzed. In all these cases the integrating procedure acts on either first or second term in the left-hand side of complete equation in vector form. Both cases (1 & 2) have a pair of subcases.

So, for the case 1 there are the subcases:

1.a. considering the usual local form: $\partial\rho/\partial t + \nabla(\rho\mathbf{V}) = 0$ and setting: $\rho = \nabla\mathbf{P}$, and with $\rho\mathbf{V} \neq \nabla\times\mathbf{\Psi}_c$ ($\rho\mathbf{V}$ is not a solenoidal vector, in order to keep the complete equation);

1.b. using the equivalent local form as in the beginning and setting: $d(\ln\rho)/dt = \nabla\mathbf{Q}$, and with $\mathbf{V} \neq \nabla\times\mathbf{\Psi}$ ($\mathbf{V}$ is not a solenoidal vector, with the same reason as at 1.a).

Here $\mathbf{P}(\xi, \eta, \zeta, t) \neq \nabla\times\mathbf{W}$ is the Greek "Rho" vector and $\mathbf{Q}(\xi, \eta, \zeta, t) \neq \nabla\times\mathbf{W}$ also, respectively (in order to assure the existence of functions ρ and $d(\ln\rho)/dt$, respectively, $\mathbf{W}$ being a certain vector field). So both $\mathbf{P}$ and $\mathbf{Q}$ are not solenoidal vectors. Thus we get further:

1.a. $\nabla(\partial\mathbf{P}/\partial t + \nabla\mathbf{P}\cdot\mathbf{V}) = 0$, and 1.b. $\nabla(\mathbf{Q} + \mathbf{V}) = 0$.

Hence the following first integrals are obtained:

1.a. $\partial\mathbf{P}/\partial t + \nabla\mathbf{P}\cdot\mathbf{V} = \nabla\times\mathbf{Y}$, with $\mathbf{Y} \neq \nabla F_1$, and

1.b. $\mathbf{Q} + \mathbf{V} = \nabla\times\mathbf{Z}$, with $\mathbf{Z} \neq \nabla F_1$ also, (F_1 being a certain scalar field). So both $\mathbf{Y}$ and $\mathbf{Z}$ are not conservative vectors, and thus the vectors $(\partial\mathbf{P}/\partial t + \nabla\mathbf{P}\cdot\mathbf{V})$ and $(\mathbf{Q} + \mathbf{V})$ are both solenoidal. One can see that among the main working hypotheses considered in this section were:

1.a. $\rho\mathbf{V} \neq \nabla\times\mathbf{\Psi}_c$, and respectively 1.b. $\mathbf{V} \neq \nabla\times\mathbf{\Psi}$, in both cases meaning that the mass flux density $\mathbf{j} = \rho\mathbf{V}$ and the velocity $\mathbf{V}$ respectively are no longer derived from the new introduced 3-D "stream function" vector $\mathbf{\Psi}_c$ (for a compressible fluid flow) and the similar 3-D true stream function vector $\mathbf{\Psi}$ (for a compressible fluid flow also), and all the results obtained in the previous sections cease to be valid for the complete equation. The subcases 1.a and 1.b correspond to an unsteady flow of a special anisotropic fluid.

A second pair of subcases is:

2.a. considering the usual local form: $\partial\rho/\partial t + \nabla(\rho\mathbf{V}) = 0$ and setting: $\rho\mathbf{V} = \partial\mathbf{T}/\partial t$, where $\mathbf{T}(\xi, \eta, \zeta, t)$ is a special vector;

2.b. using the equivalent local form as in the beginning and setting: $\nabla\mathbf{V} = dU/dt$, where $U(\xi, \eta, \zeta, t)$ is a special scalar function. We get further the following equations:

2.a. $\partial(\rho + \nabla\mathbf{T})/\partial t = 0$, and 2.b. $d(\ln\rho + U)/dt = 0$.

In order to preserve the complete continuity equation in its both initial local forms (2.a) and (2.b), a special condition must be imposed on both functions $\mathbf{T}$ and U:

2.a. $\mathbf{T} \neq \nabla\times\mathbf{W}$ and 2.b. $dU/dt \neq 0$ also ($\mathbf{W}$ being a certain vector field). Hence the following first integrals:

2.a. $\rho + \nabla\mathbf{T} = G(\xi, \eta, \zeta) + C_0$, (G being a scalar field and C_0 a scalar constant), meaning a steady general behavior, and

2.b. $\ln\rho + U = C_1$, (C_1 being another scalar constant).

We will determine for the subcase (2.b) the dependence $\rho \leftrightarrow \mathbf{V}$, for simplicity reasons using Cartesian coordinates

$$U = \int \nabla \mathbf{V} dt = \int \frac{\partial V_x}{\partial x}\frac{dt}{dx}dx + \int \frac{\partial V_y}{\partial y}\frac{dt}{dy}dy + \int \frac{\partial V_z}{\partial z}\frac{dt}{dz}dz$$
$$= \int \frac{1}{V_x}\frac{\partial V_x}{\partial x}dx + \int \frac{1}{V_y}\frac{\partial V_y}{\partial y}dy + \int \frac{1}{V_z}\frac{\partial V_z}{\partial z}dz$$
$$= \int \frac{\partial(\ln V_x)}{\partial x}dx + \int \frac{\partial(\ln V_y)}{\partial y}dy + \int \frac{\partial(\ln V_z)}{\partial z}dz; (F = -U$$
$$-\text{solution of equation } \nabla \mathbf{V} = -\dot{F}), \text{and then } \rho/\rho_i = e^{C_1 - U}.$$

13 The vorticity equation and its first integral; introducing the 2-D vorticity quasi-potential and Selescu's roto-viscous vector; analogy to "magnetic induction" equation

This equation describes the time variation of the vorticity intensity $\mathbf{\Omega}$ in a viscous Newtonian incompressible fluid flow. It can be derived from the motion equation (Navier–Stokes – see subsection 1.7 in [1], [3]), applying the curl and taking into account that the mass force density is conservative $\mathbf{f} = -\nabla(gz)$ and the static pressure $p = f(\rho \text{ only})$:

$$\partial\mathbf{\Omega}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{\Omega}) + \nu\Delta\mathbf{\Omega} \quad ; \qquad (11)$$

$\nu = \mu_d/\rho$ is the kinematic viscosity of the fluid medium – a constant quantity. The first term in the right-hand side is a convective one, while the second term is a diffusive one. So far, first integrals were found for a single term in the right-hand side only. Taking into account that $\mathbf{\Omega}$ is a solenoidal field ($\nabla\mathbf{\Omega} = 0$), and that $\Delta\mathbf{\Omega} = \nabla(\nabla\mathbf{\Omega}) - \nabla \times (\nabla \times \mathbf{\Omega})$, one gets: $\Delta\mathbf{\Omega} = -\nabla \times (\nabla \times \mathbf{\Omega})$, and so: $\partial\mathbf{\Omega}/\partial t = \nabla \times [\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})]$ or (with $\mathbf{\Theta} = \nabla \times \mathbf{\Omega}$): $\partial\mathbf{\Omega}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{\Omega} - \nu\mathbf{\Theta})$.

On any $(\mathbf{\Omega}, \mathbf{\Theta})_i$ surface we can introduce a new smart triorthogonal intrinsic coordinate system $O\xi_1\eta_1\zeta_1$ tied to these surfaces (or $O\lambda_1\mu_1\nu_1$; with λ_1, μ_1, ν_1 – orthogonal arcs lengths; λ_1, μ_1 – contained in the local tangent plane and ν_1 directed along the normal). Let us introduce the scalar function $X_i(M) = X_i(\xi_1, \eta_1, \zeta_{10i})$ – the Greek "Chi", a 2-D "quasi-potential", whose partial derivatives along the directions of the elementary orthogonal arcs $d\lambda_1 = h_{\xi 1}d\xi_1$ and $d\mu_1 = h_{\eta 1}d\eta_1$ on the "i" $(\mathbf{\Omega}, \mathbf{\Theta})$ surface ($\zeta_1 = \zeta_{10i}$) are just the components $\Omega_{\xi 1i}$ and $\Omega_{\eta 1i}$ of the vorticity $\mathbf{\Omega}$ vector:

$$\Omega_{\xi 1i} = \frac{1}{h_{\xi 1}}\frac{\partial X_i}{\partial \xi_1} = \frac{\partial X_i}{\partial \lambda_1} \quad ; \quad \Omega_{\eta 1i} = \frac{1}{h_\eta}\frac{\partial X_i}{\partial \eta_1} = \frac{\partial X_i}{\partial \mu_1} \quad ;$$
$$\Omega_{\zeta 1i} = \frac{1}{h_{\zeta 1}}\frac{\partial X_i}{\partial \zeta_1} = \frac{\partial X_i}{\partial \nu_1} = 0 \quad , \quad \mathbf{\Omega}_i = \nabla X_i \quad .$$

1) A trivial case of first integrability is given by: $\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega}) = \nabla F$ (F being a scalar function and ∇F a pure gradient), getting for the right-hand side of the vorticity equation the form: $\nabla \times \nabla F = 0$ – a particular case ($\partial\mathbf{\Omega}/\partial t = 0$ – a steady vortex field). Performing a scalar multiplication of this equation by a virtual elementary displacement $d\mathbf{R}$ normal to the vector $\mathbf{v_i.v.} = \mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})$ – the viscous incompressible vector, one gets: $[\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})]\cdot d\mathbf{R} = 0$, therefore a "zero-work" condition for the vector $\mathbf{v_i.v.}$ ($\mathbf{v_i.v.}\cdot d\mathbf{R} = 0$), this meaning a surface – envelope sheet of the local planes normal to the $\mathbf{v_i.v.}$ vector, one obtains: $\nabla F \cdot d\mathbf{R} = dF = 0$, and the first integral for the steady case: $F = \text{const.}$

2) An important case of integrability, after performing a scalar multiplication of the equation by a virtual $d\mathbf{R}$, is that one consisting in simultaneously satisfying two conditions, to eliminate the non-conservative terms:

2a) $d\mathbf{R}$ – coplanar with both $\mathbf{\Omega}$ and $\mathbf{\Theta}$; this may be expressed as: $d\mathbf{R} = c_3\mathbf{\Omega} + c_4\mathbf{\Theta}$ (with: $d\mathbf{R} \perp (\mathbf{\Theta} \times \mathbf{\Omega})$);

2b) $d\mathbf{R}$ – normal to the vector $\mathbf{W_i} = \nabla \times \mathbf{v_i.v.} = \nabla \times [\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})]$, or in the form of a zero-scalar product: $\{\nabla \times [\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})]\}\cdot d\mathbf{R} = 0$, therefore a "zero-work" condition for $\mathbf{W_i}$ ($\mathbf{W_i}\cdot d\mathbf{R} = 0$).

The first condition leads to a $(\mathbf{\Omega}, \mathbf{\Theta})$ surface – envelope sheet of the $(\mathbf{\Omega}, \mathbf{\Theta})$ local planes, having $(\mathbf{\Theta} \times \mathbf{\Omega})\cdot d\mathbf{R} = 0$. Like in subsection 1.6 in [1], introducing a 2-D vorticity "quasi-potential" X_i, and writing $\mathbf{\Omega}_i = \nabla X_i$ (where, this time X_i is a scalar function depending not only on ξ_1, η_1 or λ_1, μ_1 but on t also), if the virtual elementary $d\mathbf{R}$ vector is contained in the $(\mathbf{\Omega}, \mathbf{\Theta})$ tangent plane, integrating the term $\partial(\nabla X_i)/\partial t\cdot d\mathbf{R}$, one gets the term $\partial X_i/\partial t$ (due to Schwarz' theorem – similarly to the case in subsection 1.6 in [1]). The second condition means a surface – envelope sheet of the local planes normal to $\mathbf{W_i}$ vector. Solving this system one gets some space curves – intersection lines of the envelope sheets above – along which the searched for $d\mathbf{R}$ must be directed, for the vorticity equation to admit a first integral. These lines are the unique solution of the system given by the conditions 2a and 2b:

$d\mathbf{R} \in (\mathbf{\Omega}, \mathbf{\Theta})$ and also $d\mathbf{R} \perp \mathbf{W}_i$, this resulting in:

$d\mathbf{R} \in ((\mathbf{\Omega}, \mathbf{\Theta}) \cap (\text{plane normal to } \mathbf{W}_i)) \parallel \mathbf{\&}$ – the $\mathbf{\&}$ (ampersand) vector lines, $d\mathbf{R}$ being therefore parallel to the vector: $\mathbf{\&} = (\mathbf{\Theta} \times \mathbf{\Omega}) \times [\nabla \times (\mathbf{V} \times \mathbf{\Omega} - \nu\mathbf{\Theta})]$, with: $\mathbf{\&}$ – Selescu's vector, representing *a new physical quantity (incompressible roto-viscous)*, and giving for the most general case of vorticity equation the first integral:

$\partial X_i/\partial t = C_{0i}(t)$,

for the entire equation (the local derivative $\partial\mathbf{\Omega}/\partial t$ and both convective $\nabla \times (\mathbf{V} \times \mathbf{\Omega})$ and diffusive $\nu\Delta\mathbf{\Omega}$ term).

3) Another particular case of first integrability, also after performing a scalar multiplication of the equation by a virtual elementary displacement $d\mathbf{R}$ vector, is that one consisting in simultaneously satisfying other two conditions, to eliminate the non-conservative terms:

3a) ≡ 2a) $d\mathbf{R}$ – coplanar with both $\mathbf{\Omega}$ and $\mathbf{\Theta}$; this may be expressed as: $d\mathbf{R} = c_3\mathbf{\Omega} + c_4\mathbf{\Theta}$ (with: $d\mathbf{R} \perp (\mathbf{\Theta} \times \mathbf{\Omega})$);

3b) $\mathbf{W_i} = \nabla F_1 + \mathbf{C}_1(t)$ or: $\nabla \times [\mathbf{V} \times \mathbf{\Omega} - \nu(\nabla \times \mathbf{\Omega})] = \nabla F_1 + \mathbf{C}_1(t)$ (F_1 being a scalar function and ∇F_1 a gradient), getting for the differential equation of the vorticity vector: $\partial\mathbf{\Omega}/\partial t = \nabla F_1 + \mathbf{C}_1(t)$, or performing a scalar multiplication by the virtual displacement $d\mathbf{R}$ given by the condition 3a): $[\partial(\nabla X_i)/\partial t]\cdot d\mathbf{R} = [\nabla F_1 + \mathbf{C}_1(t)]\cdot d\mathbf{R}$ ($\mathbf{C}_1$ – a vector), or (due to Schwarz' theorem applied to the left-hand side): $d(\partial X_i/\partial t) = dF_1 + \mathbf{C}_1(t)d\mathbf{R}$, so yielding the first integral:

$\partial X_i/\partial t = F_1 + \mathbf{C}_1(t)\mathbf{R} + C_2(t)$

(again for the entire equation); the scalar function F_1 must be determined from the condition 3b), this giving a special relation between $\mathbf{V}$ and $\boldsymbol{\Omega}$. $\mathbf{C}_1$ and C_2 are arbitrary functions of t. By analogy the "magnetic induction" equation in MHD (with the analogy laws: $\mathbf{H} \leftrightarrow \boldsymbol{\Omega}$; $\nu_m \leftrightarrow \nu$) was studied:

$$\partial\mathbf{H}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{H}) + \nu_m\Delta\mathbf{H} \quad ; \quad (12)$$

(the equation of the magnetic field intensity; see for reference Eq. (11)) which was treated in [26] (subsection 2.9), where: $\mathbf{H}$ is the intensity of the local magnetic field, making no distinction between $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ (in the Gaussian system of units), since for all electro-conducting fluids the magnetic permeability (μ) is approximately equal to 1 (see for reference [27] – [30]); $\mathbf{V}$ is the mean instantaneous velocity of translation of the ionized fluid (plasma) particles (atoms, ions etc.) contained in a local small volume element (see subsection 2.1 in [26]): $\mathbf{V} = (\rho_a\mathbf{V}_a + \rho_+\mathbf{V}_+ + \rho_-\mathbf{V}_-)/\rho$, where: $\mathbf{V}_a$, $\mathbf{V}_+$, $\mathbf{V}_-$ are the velocities of the components; ρ_a, ρ_+, ρ_- are the densities of the components: $\rho_a = n_a m_a$; $\rho_+ = n_+ m_+$; $\rho_- = n_- m_-$; $\rho = n_a m_a + n_+ m_+ + n_- m_- = \rho_a + \rho_+ + \rho_-$ – plasma density (analogously to the case of a mixture of components); m_a is the mass of a neutral atom ($m_a = m_+ + m_-$); m_+ is the mass of the positive ion (a single species); m_- is the mass of the negative ion or of electron; n_a is plasma concentration in neutral atoms, and respectively (for a three-component neutral plasma): n_+, n_- are the concentrations in positive and negative particles (a single species of cations and anions; in the case of a quasi-neutral plasma: $n_+ \approx n_-$), all according to a simplified model proposed by the author; ν_m is the magnetic viscosity of the electroconducting fluid. Analogously to $X_i(M) = X_i(\xi_1, \eta_1, \zeta_{10i})$, with $\boldsymbol{\Omega}_i = \nabla X_i$, we can introduce a 2-D magnetic "quasi-potential" $\Xi_i(M) = \Xi_i(\xi_2, \eta_2, \zeta_{20i})$, with $\mathbf{H}_i = \nabla\Xi_i$, whose partial derivatives along the directions of the elementary orthogonal arcs $d\lambda_2 = h_{\xi 2}d\xi_1$ and $d\mu_2 = h_{\eta 2}d\eta_2$ on any "i" ($\mathbf{H}$, $\nabla \times \mathbf{H}$) surface ($\zeta_2 = \zeta_{20i}$) are just the components $H_{\xi 2i}$ and $H_{\eta 2i}$ of the vector $\mathbf{H}$. *A new physical quantity* (similar to the incompressible roto-viscous vector **&**) was introduced – Selescu's *magnetic* vector: $\mathbf{\&}_m = [(\nabla \times \mathbf{H}) \times \mathbf{H}] \times \{\nabla \times [\mathbf{V} \times \mathbf{H} - \nu_m(\nabla \times \mathbf{H})]\}$. The low-frequency Ampère's law neglects displacement current, being: $\mathbf{j} = k\nabla \times \mathbf{H}$, with $k = c/4\pi$; c is the light speed in a vacuum; $\mathbf{j}$ is the density of conduction current. So we get: $\mathbf{\&}_m = (\mathbf{j}/k \times \mathbf{H}) \times [\nabla \times (\mathbf{V} \times \mathbf{H} - \nu_m\mathbf{j}/k)]$.

So far, to the best of the author's knowledge, *only two particular cases of first integrability* are known for Eq. (12):

1. *ideal conducting plasma* (with zero-resistivity and thus zero-magnetic viscosity, missing the diffusion term $\nu_m\Delta\mathbf{H}$), leading to the concept of magnetic field "frozen" in plasma (given by an equation with the same form as Helmholtz' one [19] for the vorticity in the ideal fluid in hydrodynamics) *and*
2. *non-moving real plasma* (with finite conductivity and thus finite magnetic viscosity, now missing the convection term $\nabla \times (\mathbf{V} \times \mathbf{H})$), getting a standard diffusion equation.

14 Conclusions and remarks

The contribution of this work to the state of the art in fluid mechanics (especially aerothermodynamics and MHD), as well as in electromagnetic theory, consists in finding new first integrability cases and first integrals for:

1. *the continuity equation* (for the steady flow of an inviscid compressible fluid), obtaining some special general forms of the flow rate equation;
2. *the system of equations (motion and continuity)* for the steady flow of an inviscid compressible fluid;
3. *the general case* and *two particular cases of complete continuity equation* (for the unsteady flow of an inviscid compressible fluid), each of them with two subcases;
4. *the complete vorticity equation* (with both convection and diffusion terms) for a viscous incompressible fluid flow;
5. *the complete "magnetic induction" equation* in MHD (with both terms) for the flow of a magnetic viscous electroconducting fluid (plasma included), by analogy to previous one.

Three smart intrinsic curvilinear coordinate systems were used: $O\xi\eta\zeta$ (or $O\lambda\mu\nu$), for the continuity equation, $O\xi_1\eta_1\zeta_1$ (or $O\lambda_1\mu_1\nu_1$), for the vorticity equation, and $O\xi_2\eta_2\zeta_2$ (or $O\lambda_2\mu_2\nu_2$), for the "magnetic induction" one. In this work *a rich nomenclature* was introduced in fluid mechanics (for both inviscid and viscous fluids), as well as in viscous MHD and electromagnetic theory:

- the 2-D "quasi-stream" F_i function on the 3-D ($\mathbf{V}$, $\boldsymbol{\Omega}$) "i" surfaces, for the continuity equation;
- the surfaces of *iso*-normal mass flux *density* $|\mathbf{jn}| = \rho|\mathbf{Vn}| = (\rho|V_n| =)$const. (on which the continuity equation of the steady flow of a compressible fluid in a thick stream tube admits the same first integral as for the fluid flow in a thin stream tube, and whose envelope sheets are just the sections of uniform flow, on which ρ = const. and $|V_n|$ = const. as well, if they exist), for the flow rate equation;
- the 3-D stream function vector $\boldsymbol{\Psi}_c$ (Selescu, defined by: $\nabla \times \boldsymbol{\Psi}_c = \rho\mathbf{V}$, with $\boldsymbol{\Psi}_c \neq \nabla G$), allowing a new global form for the continuity equation of a compressible fluid steady flow (a stream function for *the mass flux density* $\rho\mathbf{V}$), using the circulation of $\boldsymbol{\Psi}_c$ along a closed curve l_j bounding a certain cross section S_j of a thick stream tube and applying the classical Kelvin–Stokes theorem (a reciprocal one), similarly to Helmholtz' 2nd theorem about vortices in an inviscid fluid, using the circulation of $\mathbf{V}$ along any closed curve surrounding a vortex tube: $(\Gamma_{\boldsymbol{\Psi}c})_{out} = -(\Gamma_{\boldsymbol{\Psi}c})_{in}$ = const.; $(\Gamma_{\boldsymbol{\Psi}c})_{in,\,out} = \int_{l_{in},\,l_{out}}\boldsymbol{\Psi}_c\cdot d\mathbf{R}$;
- three 2-D "quasi-potential" scalar functions: a velocity one, $\Phi_i(M)$, on the 3-D ($\mathbf{V}$, $\boldsymbol{\Omega}$) surfaces ($\mathbf{V}_i = \nabla\Phi_i$); a vorticity one, $X_i(M)$, on the 3-D ($\boldsymbol{\Omega}$, $\boldsymbol{\Theta}$) surfaces ($\boldsymbol{\Omega}_i = \nabla X_i$); and a magnetic one, $\Xi_i(M)$, on the 3-D ($\mathbf{H}$, $\mathbf{j}$) surfaces ($\mathbf{H}_i = \nabla\Xi_i$);
- the virtual "zero-work" surfaces for the vectors $\mathbf{v_i.v.} = \mathbf{V} \times \boldsymbol{\Omega} - \nu(\nabla \times \boldsymbol{\Omega})$ and $\mathbf{W_i} = \nabla \times \mathbf{v_i.v.}$ (in vorticity equation).

Two new vector physical quantities were introduced: $\mathbf{\&} = (\boldsymbol{\Theta} \times \boldsymbol{\Omega}) \times [\nabla \times (\mathbf{V} \times \boldsymbol{\Omega} - \nu\boldsymbol{\Theta})]$ – Selescu's *incompressible roto-viscous* one, for a certain 3-D flow, and by analogy, in MHD: $\mathbf{\&}_m = (\mathbf{j}/k \times \mathbf{H}) \times [\nabla \times (\mathbf{V} \times \mathbf{H} - \nu_m \mathbf{j}/k)]$ – Selescu's *magnetic* one, to find lines along which we get first integrals. *The interdependence* between the 2-D quasi-potential Φ_i and the 2-D quasi-stream function Ψ_{ci} *was also established*. *Any continuity equation* (in electromagnetic theory and MHD) *can be treated in a manner similar to this one*. In plasma-MHD the equation has the same form as in fluid mechanics, with a slightly different nomenclature (section 13). Finally we can cite a remarkable sentence from [31]: "*Not always the simplest is the best explanation. The analogies sometimes work and sometimes not. Old or conventional theories may be misleading.*"

Note
This paper (the second in a series dedicated to the intrinsic analytic study of the basic equations in compressible fluid mechanics) is fully original, however having as starting point other ones with almost the same titles (see [32], [4]).

Acknowledgements
The author is fully indebted to some world renowned scientists in the theoretical mechanics of inviscid fluid: Daniel Bernoulli, Leonhard Euler, Joseph-Louis Lagrange, Hermann von Helmholtz, Ippolit Stepanovich Gromeka, Horace Lamb, Ernst Mach, Ludwig Prandtl, Theodor Meyer, Adolf Busemann, Geoffrey Ingram Taylor, J. W. Maccoll, Antonio Ferri, and especially to: Luigi Crocco, Andrew Vázsonyi, Adolf Steichen, Henri Poincaré and Victor Vâlcovici, for their extremely valuable physical & mathematical theories, serving as starting ideas in the elaboration of this work and of the two related ones (about Crocco's, motion, and velocity potential equations). As regards the extension of the new model to: a) the viscous Newtonian fluid flow and b) electromagnetic theory & MHD, this indebtedness also addresses to: a) Isaac Newton, Augustin-Louis Cauchy, Gabriel Lamé, Claude-Louis Navier, George Gabriel Stokes, Maurice Couette, Jean Louis Marie Poiseuille, Osborne Reynolds (the last three also being talented experimentalists) and b) Hans Christian Ørsted, André-Marie Ampère, James Prescott Joule, Heinrich Lenz, Edwin Hall, Hendrik Lorentz, Michael Faraday, James Clerk Maxwell, Oliver Heaviside, Heinrich Hertz, Joseph Larmor and Hannes Alfvén.

References:
[1] R. Selescu: *An intrinsic study on a certain isoenergetic flow of a compressible fluid, with extension to some cases in magneto-plasma dynamics*, Nonlinear Analysis; Theory, Methods & Applications; Series A: Theory and Methods, doi:10.1016/j.na.2008.12.015, Vol. 71, Issue 12, December 2009, pp. e872 – e891, Elsevier Science Ltd, (Proceedings of the 5th World Congress of Nonlinear Analysts – WCNA '08, Orlando, Florida, U.S.A., July 2 – 9, 2008); also see the hyperlink: http://www.sciencedirect.com/science?_ob=ArticleURL&_udi=B6V0Y-4V47C9M-1&_user=10&_rdoc=1&_fmt=&_orig=search&_sort=d&view=c&_acct=C000050221&_version=1&_urlVersion=0&_userid=10&md5=ab73509e23e1e72cf7e3aa26f6c6e227.

[2] R. Selescu: *An Intrinsic Study on a Certain Flow of an Inviscid Compressible Fluid, with Extension to Some Cases in Magneto-Plasma Dynamics; Part One – The Isentropic Surfaces and their Applications in Aerogas-dynamics*, in "New Aspects of Fluid Mechanics and Aerodynamics" – Proceedings of the 6th IASME/ WSEAS International Conference FMA '08, Ixia, Rhodes, Greece, August 20 – 22, 2008, pp. 271 – 276, WSEAS Press, Stevens Point, Wisconsin, U.S.A.; also see the hyperlink: http://www.wseas.us/e-library/conferences/2008/rhodes/fma/fma41.pdf; BEST PAPER of the Conference; see "Prof.htm" at: http://www.wseas.us/reports/2008/best2008.htm.

[3] R. Selescu: *Intrinsic analytic study on an isoenergetic flow of a compressible fluid; Part 1: "Isentropic" and "zero-work" surfaces in aero-gas dynamics*, in Proceedings of the XXXIInd "Caius Iacob" Conference on Fluid Mechanics and its Technical Applications – INCAS, Bucharest, Romania, October 16 – 17, 2009, pp. 224 – 238, INCAS "Elie Carafoli", 2009; also see: http://www.incas.ro/images/stories/english/news/Proceedings__Caius_Iacob_Internet_15_Ian%20.pdf.

[4] R. Selescu: *New First Integrals for the Motion Equation; The Vortex Equation*, in "Advances in Fluid Mechanics & Heat & Mass Transfer", including: Proceedings of the 10th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment (HTE '12); Proceedings of the 10th WSEAS International Conference on Fluid Mechanics & Aerodynamics (FMA '12), Istanbul, Turkey, August 21 – 23, 2012, pp. 293 – 298, WSEAS Press; also see: www.wseas.us/e-library/conferences/2012/Istanbul/FLUHE/FLUHE-46.pdf.

[5] L. Crocco: *Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, Z.A.M.M. (Zeitschrift für angewandte Mathematik und Mechanik), 17, 1, S. 1 – 7, 1937; (also in Rendiconti dell'Accademia Nazionale dei Lincei, 23, pp. 115 –, 1936).

[6] A. Steichen: *Beiträge zur Theorie der zweidimensionalen Bewegungsvorgänge in einem Gase, das mit Überschallgeschwindigkeit strömt*, Doctoral dissertation, "Georg August" University, Göttingen, Germany (German Empire), 1909.

[7] R. Selescu: *The Velocity Potential PDE in a Certain Curvilinear Coordinate System*, in "Advances in Fluid Mechanics & Heat & Mass Transfer", including: Proceedings of the 10th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment (HTE '12); Proceedings of the 10th WSEAS International

Conference on Fluid Mechanics & Aerodynamics (FMA ’12), Istanbul, Turkey, August 21 – 23, 2012, pp. 305 – 310, WSEAS Press, Stevens Point, Wisconsin, U.S.A.; also see: www.wseas.us/e-library/conferences/2012/Istanbul/FLUHE/FLUHE-48.pdf.

[8] R. Selescu: *Getting the General ODE of the Velocity Potential for any Conical Flow; Part One: The Coordinate System and the First Integrable Continuity Equation*, in “Mathematical Problems in Engineering & Aerospace Sciences” – Proceedings of the 6th International Conference on Nonlinear Problems in Aviation and Aerospace Sciences (ICNPAA) – Budapest, Hungary, June 21 – 23, 2006”, pp. 729 – 739, Cambridge Scientific Publishers, Cottenham, Cambridge, U.K., 2007; see abstract at: http://atlas-conferences.com/c/a/s/p/61.htm.

[9] R. Selescu: *New first integrals for the motion equation; The vortex equation; The continuity equation; Its first integral: the flow rate equation*, in Proceedings of the XXXIIIrd “Caius Iacob” Conference on Fluid Mechanics and its Technical Applications – INCAS, Bucharest, Romania, September 29 – 30, 2011, pp. 253 – 273, INCAS “Elie Carafoli”, 2011; also see: http://www.incas.ro/images/stories/Caius_Iacob_2011/Volume_Proceedings_Caius_Iacob_2011_internet.pdf.

[10] G. I. Taylor, J. W. Maccoll: *The Air Pressure on a Cone Moving at High Speeds, I – II*, Proceedings of the Royal Society, Series A: Mathematical, Physical & Engineering Sciences, Vol. 139, No. 838, pp. 278 297 (part I) and 298 – 311 (part II), London, 1933.

[11] J. W. Maccoll: *The Conical Shock Wave formed by a Cone Moving at a High Speed*, Proceedings of the Royal Society, Series A: Mathematical, Physical & Engineering Sciences, Vol. 159, No. 899, pp. 459 – 472, London, 1937.

[12] R. Selescu: *The General Properties of the Conical Motions*, Nonlinear Analysis; Theory, Methods and Applications, Vol. 30, No. 6, pp. 3437 – 3448, December 1997, Elsevier Science Ltd, Kidlington, Oxford, U.K.; also in “Proceedings of the 2nd WCNA: part 6, 1996, Athens, Greece”, pp. 3437 – 3448, 1997, Pergamon Press, Inc., Elmsford, New York, U.S.A.; also see: doi:10.1016/S0362-546X(97)00241-1 and hyperlink http://dl.acm.org/citation.cfm?id=283320.283387&coll=GUIDE&dl=GUIDE.

[13] L. Prandtl: *Neue Untersuchungen über die strömende Bewegung der Gase und Dämpfe*, Physikalische Zeitschrift, Vol. 8(1), S. 23 – 30, 1907.

[14] Th. Meyer: *Über zweidimensionale Bewegungs-vorgänge in einem Gas, das mit Überschallgeschwin-digkeit strömt*, Doctoral dissertation, “Georg August” University, Göttingen, Germany (German Empire), 1908; together with:
E. Magin: *Optische Untersuchung über den Ausfluß von Luft durch eine Laval - Düse*, Doctoral Dissertation, “Georg August” University, Göttingen, Germany (German Empire), 1908; both in Vorschungsheft des VDI (Vereins Deutscher Ingenieure) 62 (Series: Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, insbesondere aus den Laboratorien der technishen Hochschulen), 67 S., S. 31 – 67 (Meyer’s thesis), J. Springer, Berlin, 1908.

[15] G. I. Taylor, J. W. Maccoll: *The mechanics of compressible fluids*, Section H of *Aerodynamic theory; a general review of progress*, Vol. 3, pp. 209 – 250, edited by W. F. Durand, J. Springer, Berlin, 1935; reprinted: California Institute of Technology, 1943.

[16] G. F. Carrier: *Foundations of high-speed aerody-namics*, pp. 50 – 89, Dover Publications, New York, 1951.

[17] G. S. Settles, E. Krause, H. Fütterer: *Theodor Meyer Lost pioneer of gas dynamics*, Progress in Aerospace Sciences, Vol. 45 (6-8), pp. 203 – 210, Elsevier Ltd., 2009; also see doi:10.1016/j.paerosci.2009.06.001 and hyperlink http://www.mne.psu.edu/psgdl/Pubs/2009-Settles-PAS.pdf.

[18] Chr. Huygens, *Traité de la Lumière*, completed in 1678, Leyden, The Netherlands (Dutch Republic), Pieter van der Aa, 1690.

[19] H. v. Helmholtz: *On the Integrals of Hydrodynamic Equations to which Vortex Motions conform* (in German: *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, Journal für die reine und angewandte Mathematik (Crelle’s Journal), Vol. 55, pp. 25 – 55, 1858); a rough English translation by P. G. Tait in “Philosophical Magazine and Journal of Science”, Supplement to Vol. 33, Fourth Series, 18??.

[20] M. J. Lighthill: *An Informal Introduction to Theoretical Fluid Mechanics*, Clarendon Press, Oxford, U.K., 1986.

[21] P. G. Saffman: *Vortex Dynamics*, Series “Cambridge Monographs on Mechanics and Applied Mathematics”, Cambridge University Press, 1995.

[22] G. K. Batchelor: *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967; first Cambridge Mathematical Library edition 2000, reprinted 2001, 2002.

[23] P. K. Kundu, I. M. Cohen (with contributions by P. S. Ayyaswamy and H. H. Hu): *Fluid Mechanics*, 4th edition, Academic Press, Elsevier, Inc., 2008.

[24] P. K. Kundu, I. M. Cohen, D. R. Dowling (with contributions by P. S. Ayyaswamy and H. H. Hu): *Fluid Mechanics*, 5th edition, Academic Press, Elsevier, Inc., 2012.

[25] R. Selescu: *An Intrinsic Study on a Certain Flow of a Viscous Compressible Fluid, with Extension to Some Cases in Magneto-Plasma Dynamics; Part One – The Isentropic and 0-Work Surfaces and their Applications in Aerodynamics*, in “Computers and Simulations in Modern Science – Volume II” – Selected Papers from some WSEAS Conferences in 2008 (a post-conference book), pp. 89 – 93, WSEAS Press, Stevens Point, Wisconsin, U.S.A.; also see: http://www.wseas.us/e-library/conferences/2008/tomos2/papers/vol00.pdf.

[26] R. Selescu: *Intrinsic analytic study on an isoenergetic flow of a compressible fluid; Part 2: Extension to some cases in magneto-plasma dynamics*, in Proceedings of the XXXIInd "Caius Iacob" Conference on Fluid Mechanics and its Technical Applications – INCAS, Bucharest, Romania, October 16 – 17, 2009, pp. 239 – 258, INCAS "Elie Carafoli", 2009; also see: http://www.incas.ro/images/stories/english/news/Proceedings__Caius_Iacob_Internet_15_Ian%20.pdf.
[27] L. Sédov: *Mécanique des milieux continus, tome I*, Éditions Mir, Moscou, 1975; in Russian: *Mekhanika sploshnoĭ sredy*, 3-e izd., Izdatel'stvo Nauka, Moscow, 1976.
[28] L. Landau, E. Lifshitz: *Electrodynamics of Continuous Media*, Pergamon Press, New York, 1960; [also: *Physique théorique, tome VIII (Électrodynamique des milieux continus)*, Éditions Mir, Moscou, 1969.]
[29] L. D. Landau, E. M. Lifshitz, L. P. Pitaevskiĭ: *Electrodynamics of Continuous Media*, 2nd edition revised and enlarged by E. M. Lifshitz and L. P. Pitaevskiĭ (Landau and Lifshitz Course of Theoretical Physics Volume 8), translated from the second edition of *Elektrodinamika sploshnykh sred*, Izdatel'stvo Nauka, Moscow, 1982 (translation from the Russian by J. B. Sykes, J. S. Bell and M. J. Kearsley), printed and bound in India by Replika Press Pvt. Ltd., Elsevier, 2004.
[30] B. Yavorsky, A. Detlaf: *Handbook of Physics*, 2nd edition, Mir Publishers, Moscow, 1975; (also *Aide-mémoire de physique*, 5-ième édition, Éditions Mir, Moscou, 1986.)
[31] R. F. Crespo, D. Heymann, F. Tohmé: *Abduction and Qualitative Model Building in Economics*, XLIV Reunión Annual, Anales | Asociacion Argentina de Economia Politica (AAEP), Noviembre de 2009, pp. 1 – 19; http://www.aaep.org.ar/anales/works/works2009/crespo.pdf.
[32] R. Selescu: *The Continuity Equation and its First Integral: the Flow Rate Equation*, in "Advances in Fluid Mechanics & Heat & Mass Transfer", including: Proceedings of the 10th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment (HTE '12); Proceedings of the 10th WSEAS International Conference on Fluid Mechanics & Aerodynamics (FMA '12), Istanbul, Turkey, August 21 – 23, 2012, pp. 299 – 304, WSEAS Press, Stevens Point, Wisconsin, U.S.A.; also see the hyperlink: http://www.wseas.us/e-library/conferences/2012/Istanbul/FLUHE/FLUHE-47.pdf.