Laminar boundary layer model for power-law fluids with non-linear viscosity

JACOB NAGLER
Faculty of Aerospace Engineering, Technion, Haifa 32000,
ISRAEL
syankitx@Gmail.com syanki@tx.technion.ac.il

Abstract: - In this paper, analytical solutions are obtained for the steady laminar boundary layer of non-Newtonian flow with non-linear viscosity over a flat moving plate. The power-law fluid model was adopted for the non-Newtonian fluid representation. The governing non-dimensional boundary layer equations are transformed into ordinary differential equations using similarity transformation which are then solved analytically. The analytical results are obtained for different values of the constant n representing the power-law index and flow consistency parameter K which is assumed to be function type of η and n in this study. The effects of various values on the velocity profiles are presented and discussed.

Key-Words: - Analytical solution, non-Newtonian, non-Linear viscosity, Boundary Layer, Bingham model.

1 Introduction

Boundary layer flow is discussed and investigated for many years. Its forms are widely used in fields like chemistry, aero-space and bio-medical engineering. The original studies that were dealing with boundary layer of non-Newtonian fluid are discussed in [1-2].

Development of two and three-dimensional boundary-layer equations for pseudo-plastic non-Newtonian fluids which characterized by a power-law relation is shown in Ref. [1]. In this later study, the types of potential flows which are necessary for similar solutions of the boundary-layer equations have been determined. It was found that for two-dimensional flow the results are similar to those obtained for Newtonian fluids. However, for three-dimensional flow, the possibility to find similar solutions is dependent on expression type nature which accompanied to effective viscosity of the fluid. Mostly, similar solutions are possible only for the case of flow past a flat plate where the potential velocity vector is not perpendicular to the leading edge of the plate; this is a much more restrictive condition than for Newtonian fluids obtained solutions.

Ref. [2] presents a theoretical analysis of the laminar non-Newtonian fluid past arbitrary external surfaces, which is modelled by power-law model. Acrivos *et al.* predicts the drag and the rate of heat transfer from an isothermal surface to the fluid by inspectional analysis of the modified boundary-layer equations. Also, flow past a horizontal flat plate is studied in detail numerically.

Discontinues in boundary layer flow due to power law index ($n > 2$) were investigated by Ref. [3]. It was found that for replacing the point which fulfills the correct outer boundary condition, one should replace the point where the asymptotic behavior of the boundary layer flow has to be applied.

Usually, no-slip boundary conditions are applied as appear in [3-8]. Additionally, Ref. [5] presents an asymptotic approach for a boundary-layer flow of a power-law fluid. On the contrary, some studies do not assume no-slip condition as presented by [9-10].

Ref. [9] presents flow analysis of momentum and heat transfer in laminar boundary layer flow of non-Newtonian fluids past a semi-infinite flat plate with the thermal dispersion in the presence of a uniform magnetic field for two different types of boundary conditions: static plate and moving plate. The analysis is done by solving system of coupled non-linear ordinary equations with Quasi-linearization technique together with numerically calculation based on finite difference scheme.

Ref. [10] presents analysis of steady, two-dimensional laminar flow of a power-law fluid passing through a moving flat plate under the influence of transverse magnetic field. The solution is found to be dependent on various governing parameters including magnetic field parameter *M*, power-law index *n* and velocity ratio parameter ε. A systematically study is carried out to illustrate the effects of these major parameters on the velocity profiles. It is found that dual solutions exist when the plate and the fluid move in opposite directions, near the region of separation.

In the present study a non-Newtonian fluid which characterized by a power-law constitutive relation together with non-linear viscosity distribution parameter over a flat plate is investigated. Moreover, analytically and numerically calculations of f, f' and f'' are obtained for various cases of n, α and K, respectively.

2 Boundary Layer Equations Formulation

The Ostwald-de Waele power-law model for non-Newtonian shear stress τ_{xy} is described by:

$$\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{1}$$

while (x, y) are the Cartesian coordinates of any point in the flow domain, where x-axis is along the plate and y-axis is normal to it. Flow consistency parameter is considered to be function $K(x, y)$. u represents the velocity component in the positive x direction. n is the power law index. μ is defined by $K\left|\frac{\partial u}{\partial y}\right|^{n-1}$.

The boundary layer continuity and momentum equations in case of 2D laminar flow of incompressible fluid with constant density ρ are [2]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \tag{3}$$

while pressure gradient and body forces are neglected. Where u and v represent the components of the fluid velocity in positive x, y direction and τ_{xy} denotes non-Newtonian shear stress. Also, ρ is the constant flow density and is normalized, by $\rho = 1$.

The boundary conditions for Eq. (2-3) are:

$$u(x,0) = \frac{U_w}{U_\infty}A(x) \tag{4}$$

$$v(x,0) = \frac{V_w}{U_\infty}B(x) \tag{5}$$

$$\lim_{y\to\infty} u(x,y) = C \tag{6}$$

while C is constant.
Using transformation as shown in [11] by stream function $\psi(x, y)$ leads to:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \tag{7}$$

Substitution of (7) in Eq. (2-3) gives:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(K\left|\frac{\partial^2 \psi}{\partial y^2}\right|^{n-1}\frac{\partial^2 \psi}{\partial y^2}\right) \tag{8}$$

The transformed boundary conditions relations are:

$$\frac{\partial \psi}{\partial y}(x,0) = \frac{U_w}{U_\infty}A(x) \tag{9}$$

$$\frac{\partial \psi}{\partial x}(x,0) = -\frac{V_w}{U_\infty}B(x) \tag{10}$$

$$\lim_{y\to\infty} u(x,y) = C \tag{11}$$

Similarity solution will be defined by the following parameters [11]:

$$\psi(x,y) = bx^{-\alpha}f(\eta), \quad \eta = ayx^{-\beta} \tag{12}$$

where ψ is the stream function and η is the similarity variable. While a and b are constants. Substituting relations (12) in Eq. (8) yields:

$$\begin{aligned} &x^{-1-\alpha-2\beta}\left[a^2b^2\alpha ff'' - a^2b^2(\alpha+\beta)f'^2\right] + \\ &x^{\alpha-\beta}\left(a^2bx^{-\alpha-2\beta}\right)^n\left|f''\right|^n\left[K_y + anK\frac{f^{(3)}}{f''}\right] = 0 \end{aligned} \tag{13}$$

while $\eta = yax^{-\beta}$.

$$\begin{aligned} &a^{2-2n}b^{2-n}x^{-1-2\alpha-\beta+n\alpha+2n\beta}\left[\alpha ff'' - (\alpha+\beta)f'^2\right] \\ &+\left|f''\right|^n\left[K_y + anK\frac{f^{(3)}}{f''}\right] = 0 \end{aligned} \tag{14}$$

In order to have simple ODE equations, two relations should be satisfied:

$$\alpha(n-2) + \beta(2n-1) = 1 \text{ and } a^{2-2n}b^{2-n} = 1 \tag{15}$$

Fulfilling (15) converts Eq.(14) to the following form:

$$\alpha ff^{"} - (\alpha+\beta) f^{'2} + \left|f^{"}\right|^{n}\left[K_y + anK\frac{f^{(3)}}{f^{"}}\right] = 0 \quad (16)$$

Now we will note the following parameters as:

$$P = \alpha+\beta, \alpha = \frac{P(2n-1)-1}{n+1} \quad (17)$$

After using substitution of (17) in (16) we have:

$$\alpha ff^{"} - Pf^{'2} + \left|f^{"}\right|^{n}\left[K_y + anK\frac{f^{(3)}}{f^{"}}\right] = 0 \quad (18)$$

For further analytic development and fulfilling $y \to \infty$ condition, the next parameters will be chosen:

$$y = \eta,\ K = K(\eta),\ a = 1,\ P = 0 \text{ while } n \neq 2 \quad (19)$$

$$\alpha ff^{"} - \alpha f^{'2} + n\frac{\left|f^{"}\right|^{n}}{f^{"}}\left(Kf^{"}\right)^{'} = 0 \quad (20)$$

In order to fulfill similarity rules, the next parameters will be chosen:

$$A(x) = 1,\ B(x) = 0,\ C = 1 \quad (21)$$

with following B.C.:

$$f^{'}(0) = \frac{U_w}{U_\infty}, f(0) = 0, f^{'}(\infty) = 1 \quad (22)$$

3 Method of Solution

Next step is to find analytic solution by coefficients investigation of main relevant cases.

Eq.(20) will be written as following:

$$n\frac{\left|f^{"}\right|^{n}}{f^{"}}\left(Kf^{"}\right)^{'} = 0 \quad (23)$$

There are two possibilities of solution.
Option 1:

$$n\left(Kf^{"}\right)^{'} = 0 \to f = C_1\int_0^\eta\int_0^\eta \frac{1}{K} d\eta d\eta + C_2\eta + C_3 \quad (24)$$

Applying boundary conditions (22) on (24) leads to the form:

$$f(0) = 0: \quad C_3 = 0 \quad (25)$$

$$f^{'}(0) = \frac{U_w}{U_\infty}: \quad C_2 = \frac{U_w}{U_\infty} \quad (26)$$

$$f^{'}(\infty) = 1: \quad C_1\int_0^\infty \frac{1}{K} d\eta + \frac{U_w}{U_\infty} = 1 \quad (27)$$

Which gives the following relations:

$$f = C_1\frac{\int_0^\eta\int_0^\eta \frac{1}{K} d\eta d\eta}{\int_0^\infty \frac{1}{K} d\eta} + \frac{U_w}{U_\infty}\eta, \qquad C_1 = \frac{U_\infty - U_w}{U_\infty\int_0^\infty \frac{1}{K} d\eta} \quad (28)$$

where $U_w \neq U_\infty$.

Option 2: for $n > 0$

$$\frac{\left|f^{"}\right|^{n}}{f^{"}} = 0 \quad (29)$$

Integration together with applying B.C. on (29) leads to:

$$f = \frac{U_w}{U_\infty}\eta = \eta \quad (30)$$

where $U_w = U_\infty$ (in order to fulfill $f^{'}(\infty) = 1$). It seems that Eq.(30) behaves like Bingham Plastic-Liquid model. Also, it has no dependence on n and K.

4 Results

The following Figures in this section are for the following parameters: $U_\infty = 1, U_w = 0.5$ while $K = e^{\eta/n}$ or $K = \frac{(\eta+n)^2}{n}$ for various α values.

Substituting K in Eq.(28) yields:

$$f\big|_{K=e^{-\eta/n}} = \left(\frac{U_\infty - U_w}{U_\infty}\right)\left(\eta + ne^{-\eta/n} - n\right) + \frac{U_w}{U_\infty}\eta \quad (31)$$

Also,

$$f\Big|_{K=-\frac{(\eta+n)^2}{n}} = \left(\frac{U_\infty - U_w}{U_\infty}\right)\left(\eta - nln\left(n+\eta\right) + nln(n)\right) + \frac{U_w}{U_\infty}\eta \quad (32)$$

In this section f, f' represent normalized u and v velocities, respectively. f'' represents normalized viscosity μ.

Analyzing Eq.(31-32) or Fig.(1-3) shows that f profile for $K = exp(\eta/n)$ or for $K = \frac{(\eta+n)^2}{n}$ decreases with n. Eq.(30) has no dependency in K and n, so no change in value is exist as appear in Fig. (1-3).

Additionally, f' is converged to 1 with increasing η. The difference between the solutions for f, f' and f'' decreases for different K functions as shown in Fig.(1-9). It seems that this fluid can be categorized by Bingham plastic model behaviour according to Fig.(1-9) compared to Mitsoulis study [15]. Other numerical results can be compared with BOGNÁR [11], Schetz [16], Naikoti and Borra [17]. It was found that f' is perfectly matched with these references.

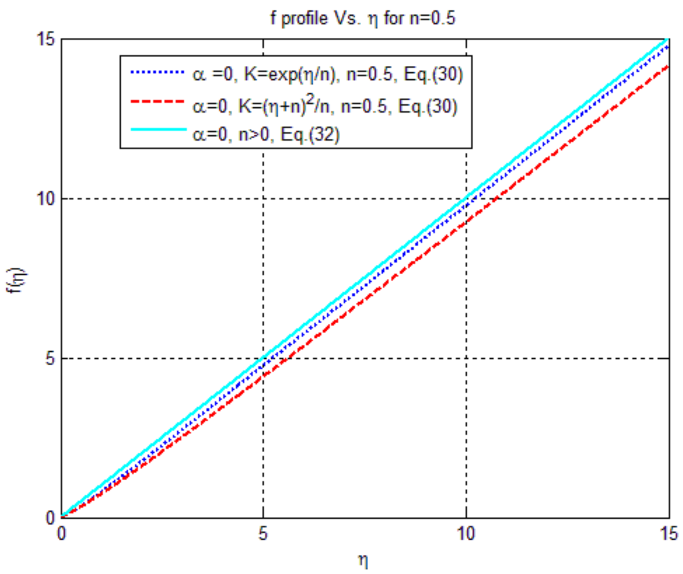

Fig. 1: f profile velocity Vs. η for $n = 0.5$

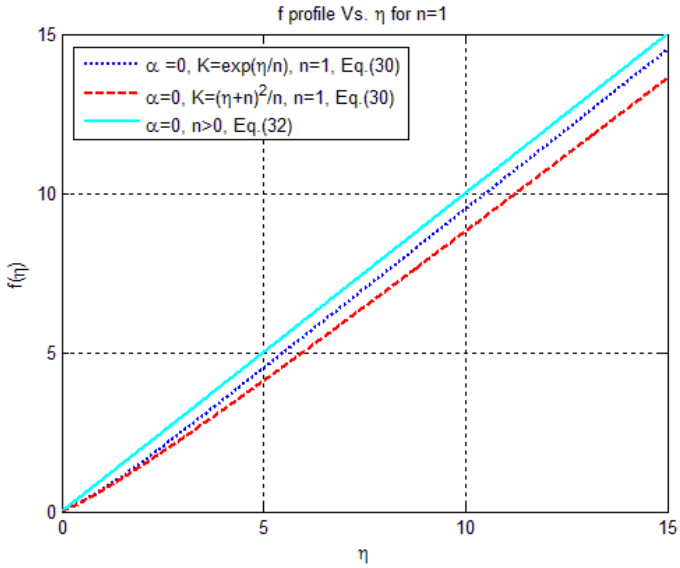

Fig. 2: f profile velocity Vs. η for $n = 1$

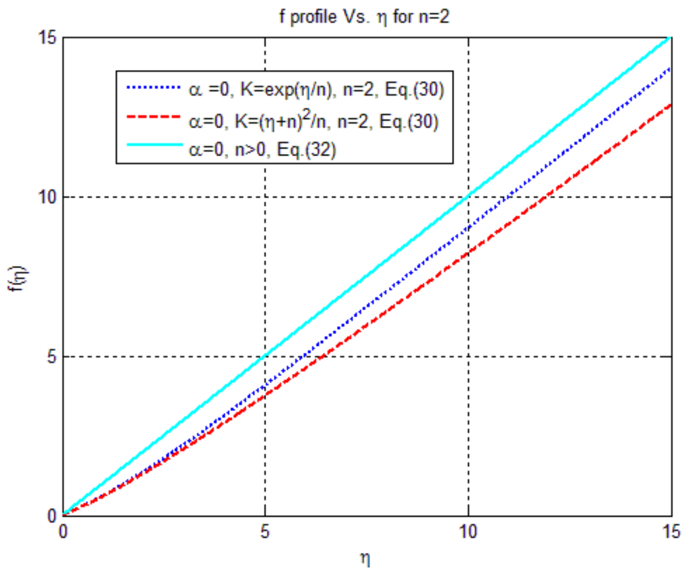

Fig. 3: f profile velocity Vs. η for $n = 2$

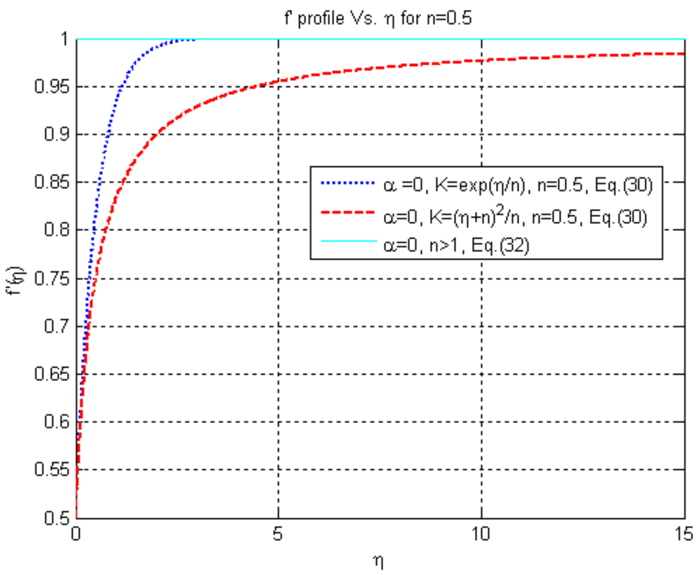

Fig. 4: f' profile velocity Vs. η for $n = 0.5$

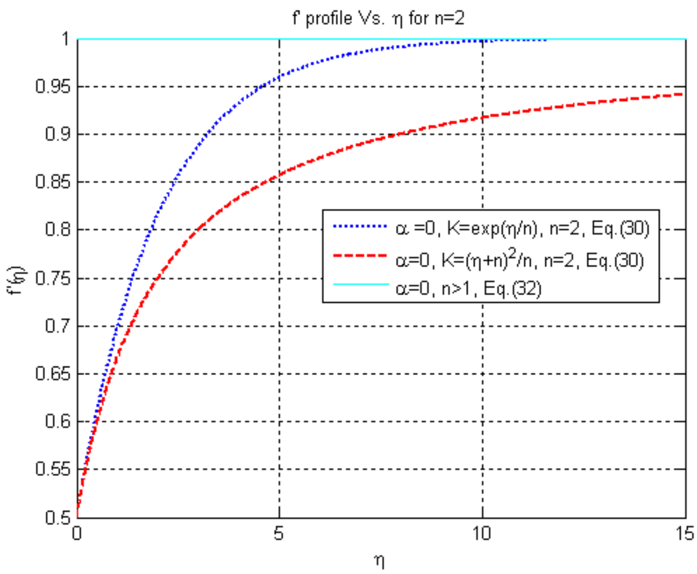

Fig. 6: f' profile velocity Vs. η for $n = 2$

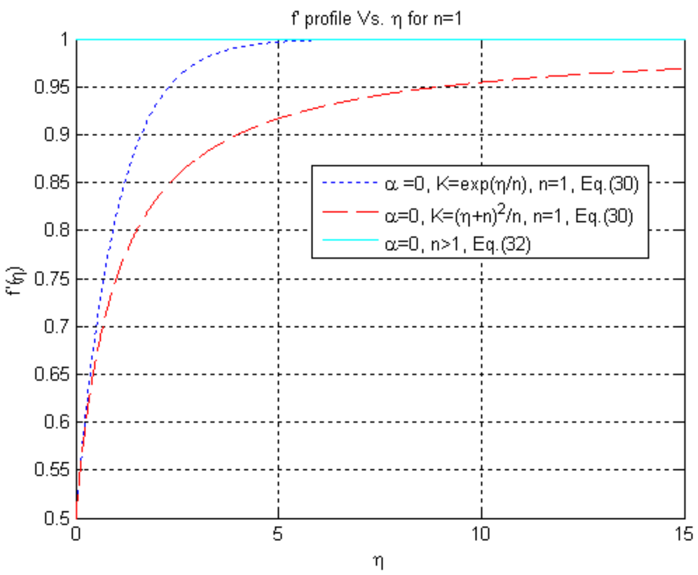

Fig. 5: f' profile velocity Vs. η for $n = 1$

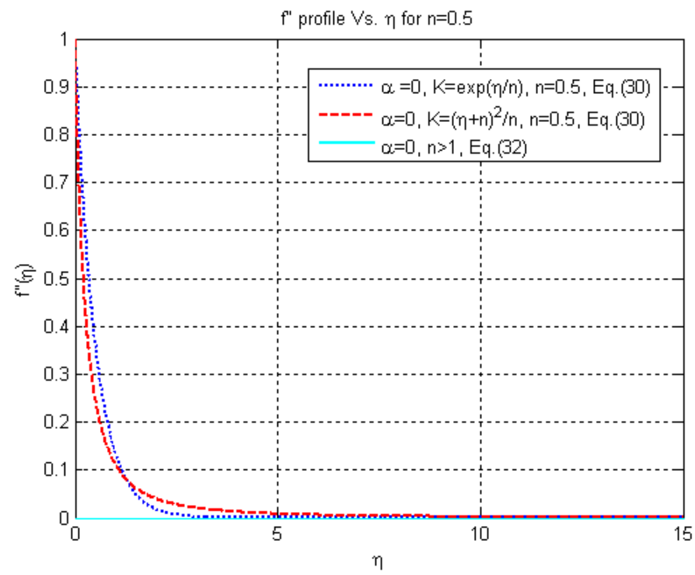

Fig. 7: f'' profile velocity Vs. η for $n = 0.5$

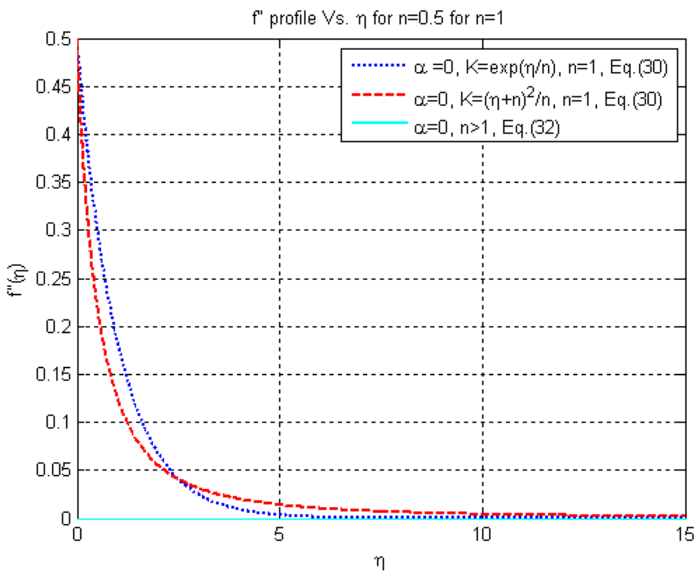

Fig. 8: f'' profile velocity Vs. η for $n=1$

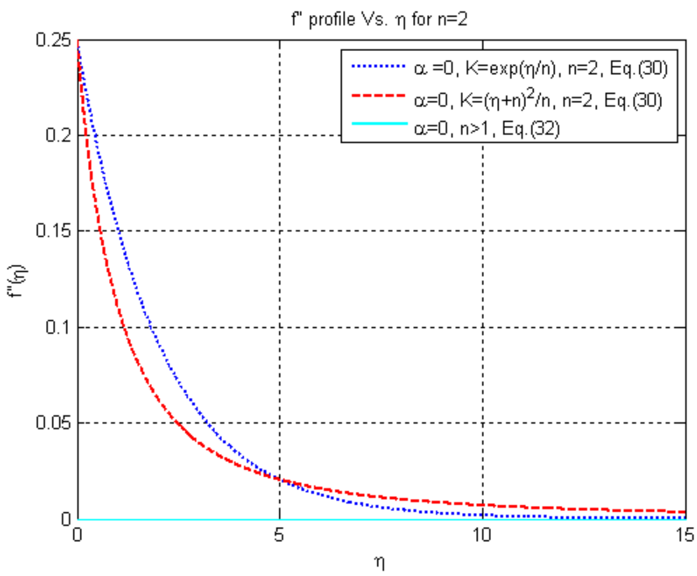

Fig. 9: f'' profile velocity Vs. η for $n=2$

4 Conclusions

This study deals with steady laminar boundary layer of Newtonian and non-Newtonian fluids with non-linear viscosity over a flat plate. The power-law fluid model was adopted for the non-Newtonian fluid representation and analytical solutions were found for $\alpha = 0$.

B.C. represents moving plate state since $f'(0) \neq 0$. Additionally, B.C. were supplied by showing that f' is converged to 1 with increasing η. The difference between solutions of f, f' and f'' decreases with η for different K functions ($K = exp(\eta / n), K = \frac{(\eta+n)^2}{n}$).

It was found that results can be categorized by Bingham plastic model behaviour compared to Ref. [15]. Moreover, numerical results can be compared with Ref. [11, 16-17]. It was found that f' is perfectly matched with these references.

Finally, this study presents general observation on boundary layer over moving flat plate with general viscosity function which isn't necessarily constant. Further research should be done in context of B.C. influence - like changing U_W, U_∞ values and/or using other surface geometry (general curve) which may contribute to the comprehension of boundary layer behaviour and enhance the understanding of viscosity role.

References:

[1] W. R. Schowalter, *AIChE J.*, Vol. 6, 1960.

[2] A. Acrivos, M. J. Shah, and E. E. Petersen, Momentum and Heat Transfer in Laminar Boundary-Layer Flows of Non-Newtonian Fluids Past External Surface, *AIChE J.*, Vol. 6, No. 2, 1960, pp. 312–317.

[3] C.C. Hsu, A simple solution for boundary layer flow of power law fluids past a semi-infinite flat plate, *AIChE J.*, Vol. 15, 1969, pp. 367–370.

[4] T.G. Myers, An approximate solution method for boundary layer flow of a power law fluid over a flat plate, *International Journal of Heat and Mass Transfer*, 2011.

[5] J P. Denier, and R E. Hewitt, Asymptotic matching constraints for a boundary-layer flow of a power-law fluid, *J. Fluid Mech.*, Vol. 518, 2004, pp. 261–279.

[6] J. P. Denier and P. P. Dabrowski, On the boundary-layer equations for power-law fluids, *Proc. R. Soc. Lond. A 460*, 2004, pp. 3143-3158.

[7] K. B. Pavlov and A. P. Shakhorin, On the Spatial Localization of the Laminar Boundary Layer in a Non-Newtonian Dilatant Fluid, *Pleiades Publishing, Ltd.*, ISSN 0015-4628, Fluid Dynamics, Vol. 42, No. 1, 2007, pp. 44–49.

[8] M. Pakdemirli, Similarity analysis of boundary layer equations of a class of non-Newtonian fluids, *Inl. J. Non-Linear Mechanics*. Vol. 29, No. 2, 1994, pp. 187-196.

[9] K. Naikoti, S. R. Borra, Quasi-linearization Approach to MHD Effects on Boundary Layer

Flow of Power-Law Fluids Past A Semi Infinite Flat Plate with Thermal Dispersion, ISSN 1749-3889 (print), *International Journal of Nonlinear Science*, Vol.11, No.3, 2011, pp.301-31.

[10] B. Shashidar Reddy, N. Kishan and M. N. Rajasekhar, MHD boundary layer flow of a non-newtonian power-law fluid on a moving flat plate, Pelagia Research Library Advances in Applied Science Research, 3 (3):1472-1481 (2012).

[11] G. BOGNÁR ,"Analytic Solutions to a Boundary Layer Problem for Non-Newtonian Fluid Flow Driven by Power Law Velocity Profile", WSEAS TRANSACTIONS on FLUID MECHANICS, Issue 1, Vol. 6, 2011, pp. 22-31.

[12] C. L. Huang, Application of Quasi-Linearization Technique to the Vertical Channel Flow and Heat Convection, *International Journal of Non-Linear Mechanics*, Vol. 13, No. 2, 1978, pp. 55-60.

[13] W. Ibrahim and B. Shanker, Unsteady Boundary Layer Flow and Heat Transfer Due to a Stretching Sheet by Quasilinearization Technique, World Journal of Mechanics, Vol. 1, No. 6, 2011, pp. 288-29.

[14] E. S. Lee, Quasi-linearization, non-linear boundary value problems and optimization, *Chemical Engineering Science* - *Pergamon Press Ltd., Oxford*, 1966, Vol. 21, pp. 183-194.

[15] E. Mitsoulis, Flows of Viscoplastic Materials: Models and Computations, Rheology Reviews, London, UK, *Brit. Soc. Rheol*., 2007, pp. 135-178.

[16] J. A. Schetz, *Boundary Layer Analysis*, Prentice-Hall, New-Jersey, 1993, pp. 82-93.

[17] K. Naikoti, S. R. Borra, Quasi-linearization Approach to MHD Effects on Boundary Layer Flow of Power-Law Fluids Past A Semi Infinite Flat Plate with Thermal Dispersion, *International Journal of Nonlinear Science*, Vol.11, No.3, 2011, pp. 301-311.