First Integrals for Crocco's Equation and hence for the Motion Equation

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Abstract: This work studies and clarifies some local phenomena in fluid mechanics, in the form of an intrinsic analytic study, regarding Crocco's equation and the motion one, for inviscid compressible fluid flows (both steady and unsteady), and finds new first integrals. It continues a series of works presented at some conferences and a congress during 2006 – 2012, representing a real deep insight into the still hidden theory of the isoenergetic rotational flow. Unlike the geometrical point of view (using a smart intrinsic coordinate system tied to flow's isentropic surfaces) previously approached to eliminate the rotational non-conservative term, this time a thermodynamic point of view is used, to evidence the above term first as a biscalar one, and further as a conservative one. Several new functions and surfaces were introduced: the 2-D velocity quasi-potential, the isentropic 3-D (**V**, **Ω**) surfaces, the polytropic integral ones, and the quasi-incompressible quasi-potential (Laplace) lines, for a quasi-uniform rotational pseudo-flow of an inviscid compressible fluid. The dependence of gas particle specific entropy on the 2-D velocity quasi-potential was established. The PDE of the polytropic special integral surfaces, and that of the isentropic ones (both in Cartesian system) were given. The newly found first integrals for the motion equation are related to D. Bernoulli's and D. Bernoulli–Lagrange ones. An extension of the new intrinsic model to MHD of a neutral plasma was also given.

Key-Words: conservative (irrotational, potential) and biscalar vectors; rotational flows; steady and unsteady flows; inviscid fluids; compressible fluids; isentropic surfaces; polytropic integral surfaces; quasi-Laplace lines

1 Introduction, nomenclature and the first approach to the new model of flow

We start from the general vector differential equation of motion for an inviscid fluid unsteady flow (Euler):

$$\frac{\partial \mathbf{V}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{V}^2\right)+\mathbf{\Omega}\times\mathbf{V}=\mathbf{f}-\frac{1}{\rho}\nabla p \quad , \quad \text{where}$$

$$\nabla=\sum_{i=1}^{3}\frac{\mathbf{k}_i}{h_i}\frac{\partial}{\partial x_i}=\mathbf{k}_x\frac{\partial}{\partial x}+\mathbf{k}_y\frac{\partial}{\partial y}+\mathbf{k}_z\frac{\partial}{\partial z} \quad - \quad \text{nabla}$$

(Hamilton's operator), in a triorthogonal system of curvilinear coordinates x_i ; $\mathbf{k}_i$ – a 3-D basis; h_i – Lamé's coefficients (the Helmholtz (see [1])–Gromeko–Lamb form binding the acceleration and the force density of a small fluid particle); **V** – the local instantaneous velocity of translation (of the small particle) – the intensity of the local fluid field; **Ω** = ∇ × **V** = 2**ω** – the vorticity (curl **V**), with: **ω** – the local instantaneous velocity of rotation (of the particle); **f** – the mass force density (conservative – a gradient): **f** = ∇(–gz) = –∇(gz); g – the acceleration of gravity; z – the geometrical height (height of the considered point above a reference horizontal plane xOy);
p – the fluid static pressure; ρ – the fluid density;
τ = 1/ρ – the fluid specific volume; t – the time.
For this inviscid fluid unsteady flow the momentum equation yields the Crocco–Vászonyi form (see [2], [3]):

$$\partial\mathbf{V}/\partial t+\mathbf{\Omega}\times\mathbf{V}=T\nabla S-\nabla(i_0+gz), \text{ with } i_0=i+V^2/2$$

– the total (stagnation) specific enthalpy; i = U + p/ρ – the static specific enthalpy; U – the specific internal energy; V – modulus of **V**; S – the specific entropy; T – static temperature (absolute) of the fluid particle. For a perfect (an ideal) gas: p = $\mathscr{R}$ρT, with $\mathscr{R}$ = const. The first law of thermodynamics states: dU = δq – δ$\mathscr{W}$; (δq) / (δ$\mathscr{W}$) are the elementary (heat supplied to) / (work done by) the system, both not being total differentials. For steady motions and resp. for gases, we have: ∂**V**/∂t = 0 and **f** = 0 , so remaining

$$\nabla\left(\frac{\mathbf{V}^2}{2}\right)+\mathbf{\Omega}\times\mathbf{V}=-\frac{\nabla p}{\rho}\,;\;\nabla\left(\frac{V^2}{2}\right)+T\nabla S=-\frac{\nabla p}{\rho} \quad (1)$$

using Crocco's equation for an *isoenergetic* ($\nabla i_0 = 0$) non-isentropic (rotational) flow: **Ω** × **V** = T∇S (see [2]).

2 First integrals of Crocco's equation and the motion one; the model for steady non-isentropic flow; general polytropic surfaces

Let us perform a scalar multiplication of Eq. (1) for the steady motion of a non-barotropic gas by a certain *virtual* elementary displacement vector d**R** (therefore generally not in the velocity direction), thus obtaining in the case of an isoenergetic flow the following ODE (with 3 terms):

$d(\mathbf{V}^2/2) + (\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = -dp/\rho$; $d(\mathbf{V}^2/2) + (\mathbf{\Omega} \times \mathbf{V})\cdot \mathbf{V}_v dt = -dp/\rho$, or: $d(V^2/2) + TdS = -dp/\rho$, (2)
for a non-isentropic (rotational) flow: $dS \neq 0$, $\mathbf{V}_v = d\mathbf{R}/dt$ being the *virtual* velocity vector (along the *virtual* elementary displacement vector $d\mathbf{R}$), where dS is a *virtual* differential: $dS = \nabla S\cdot d\mathbf{R} = (\partial S/\partial x)dx + (\partial S/\partial y)dy + (\partial S/\partial z)dz$, this meaning the integral S is a *virtual quantity* also, being more general than the corresponding *real* one. One can see that, besides the trivial cases 1 – 3 below:
1) $\mathbf{V} = 0$ – equilibrium (fluid statics) and the stagnation lines ($p = p_0$) of a plane or a special axisymmetric flow, these ones being straight lines and circles, respectively;
2) $\mathbf{\Omega} = 0$ – an irrotational motion (everywhere having $\mathbf{V} = \nabla\Phi$; Φ – the velocity potential) and the vortex-free regions (stagnation lines included) in a rotational flow; and
3) $\mathbf{\Omega} = c\mathbf{V}$ – a helicoidal (screw) motion ($\mathbf{\Omega}\|\mathbf{V}$ – vortex lines identical to the streamlines) – Beltrami flow, there is also an important case for which the non-conservative term $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = TdS$ (Crocco's equation) becomes zero:
4) $d\mathbf{R} = c_1\mathbf{V} + c_2\,\mathbf{\Omega}$ ($d\mathbf{R}$ coplanar with both $\mathbf{V}$ and $\mathbf{\Omega}$) – *the general* case, from the ***geometrical*** point of view, with two main *particular* subcases (so far, the only ones known in the world, to the best of the author's knowledge):
4.1) $c_2 = 0$; ($d\mathbf{R}_1 = d\mathbf{R} = c_1\mathbf{V} \Rightarrow c_1 = dt$, this being a *true* elementary displacement, $d\mathbf{R}$); 4.2) $c_1 = 0$; ($d\mathbf{R}_2 = c_2\mathbf{\Omega}$), $d\mathbf{R}_i$ ($i = 1, 2$) meaning an elementary displacement along a streamline and a vortex line, for which $dS = 0$ ($T \neq 0$, except for 4.3) the limit case of the expansion into a vacuum: $p = T = 0$; $V = W$). The case 4 assures the annulment of the term $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = TdS$ (see [2]), leading to a virtual elementary displacement $d\mathbf{R}$ in a local plane tangent to the *isentropic* surface ($p = K\rho^\gamma$) passing through the flow point considered (see [4] – [7]). *The most general* integrability case for this term, ***geometrical*** also, is:
G) $\mathbf{\Omega} \times \mathbf{V} = \nabla F$, or: $(\nabla \times \mathbf{V}) \times \mathbf{V} = \nabla F$ (F being a scalar function and ∇F a pure gradient – a conservative vector field). Taking into account the equivalence of the forms of Eq. (1), *the most general* integrability cases for Eq. (2), this time from the ***thermodynamic*** point of view ([8]), are those of a displacement $d\mathbf{R}$ in other local planes, tangent to some special surfaces passing through the same flow point, surfaces over which the term TdS (this time $\neq 0$) is evidenced as a total (an exact) differential (the 2nd term in Eq. (1) becomes conservative), also evidencing (for ideal gas laws reasons) the conservativity of the term $\nabla p/\rho$. The first law of thermodynamics states: $\delta q = dU + \delta\mathcal{W} = d(i - p\tau) + pd\tau = di - \tau dp$, hence: $\delta q = TdS = C_p dT - dp/\rho$ (the second law of thermodynamics). This expression is not generally a total differential (due to the term τdp), except for some cases (surfaces) when the fluid behaves barotropically ($p = f(\rho$ only)), but without being a barotropic one:
5) *isothermal* surfaces: $T = T_1 =$ const. and: $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = TdS = T_1dS = d(T_1S) = -\tau dp$, and: $p = \text{const.}_1\cdot\rho$ also;
6) *isobaric* surfaces: $p = p_1 =$ const.; $TdS = C_p\cdot dT = di$, where C_p is a constant (the ideal gas isobaric specific heat);
7) *isochoric* surfaces: $\rho = \rho_1 =$ const.; $TdS = C_v\cdot dT = dU$, where C_v is a constant (the ideal gas isochoric specific heat);
8) *polytropic* surfaces: $p/\rho^n = (p/\rho^n)_1 =$ const.; $TdS = (n - \gamma)/(n - 1)\cdot C_v dT$; γ and n are other constants (the adiabatic and polytropic exponents, resp.): $\gamma = C_p/C_v$; usually $n \in [1, 2]$. This is *the most general* usual special integral surface (a "quasi-barotropic" fluid), having as particular cases all the previous 4 – 7 ones, for various values of n. So, for $n = \gamma$ one obtains the case 4. For $n = 1$; 0; and $\to \infty$, one obtains the cases 5, 6 and 7, resp. All cases 5 – 8 are *equivalent* to case G, having: $F = T_1(S - S_1)$; $C_p(T - T_1)$; $C_v(T - T_1)$ and $(n - \gamma)/(n - 1)\cdot C_v(T - T_1)$, respectively. Any special virtual integral "i" surface can be a multi-sheet surface. An interesting particular case is:
E) $[(\tau - \tau_c)/a]^2 + [(p - p_c)/b]^2 = 1$; ($a, b, \tau_c, p_c > 0$; $\tau, p \geq 0$) – an *elliptic loop (thermodynamic cycle)* in the (τ, p) coordinates. Equating the local ordinate (p) and slope ($dp/d\tau$) of ellipse and of current polytrope $p\tau^n = K_1$, the unknowns n (for TdS expression) and K_1 are found as functions of τ and p . Any barotropic virtual evolution of an ideal gas may be regarded as being composed of a lot of successive polytropic elementary evolutions.

The vector equation $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = dF$ represents *the general PDE of the special integral surfaces*. In the second form of Eq. (1) the first vector is a conservative one and the remaining two are biscalar vectors. By biscalar vector one understands a vector $\mathbf{L} = \mathbf{k}_xP + \mathbf{k}_yQ + \mathbf{k}_zR$, so that the functions P, Q, R are bound by the integrability condition: $P\cdot(\partial R/\partial y - \partial Q/\partial z) + Q\cdot(\partial P/\partial z - \partial R/\partial x) + R\cdot(\partial Q/\partial x - \partial P/\partial y) = 0$, for a differential equation of the type: $P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$ (an orthogonality condition: $\mathbf{L}\cdot d\mathbf{R} = 0$) to have a total differential in the left-hand side ($\mathbf{L}\cdot d\mathbf{R} = dF(x, y, z) = 0$), so a first-integrability condition. This integrability condition takes the condensed form: $\mathbf{L}\cdot(\nabla \times \mathbf{L}) = 0$, representing another orthogonality condition (for the vectors $\mathbf{L}$ and curl $\mathbf{L}$ – see subsection 1.5 in [4]). This means $\mathbf{L}$ must be either a conservative ($\mathbf{L} = \nabla F$) or a biscalar ($\mathbf{L} = \Phi_1\nabla\Phi_2$) vector, where Φ_1 and Φ_2 are independent scalar functions ($\Phi_1 \in C^1$; $\Phi_2 \in C^2$), e.g.: just $\Phi_1 = T$; $\Phi_2 = S$, and $\Phi_1 = 1/\rho$; $\Phi_2 = p$ (these are not unique solutions), so that *the orthogonality condition is always satisfied*. If in some cases (even over special domains) a function F to satisfy the vector equation $\Phi_1\nabla\Phi_2 = \nabla F$ can be found (Φ_1 , Φ_2 – interdependent), the biscalar vector $\mathbf{L}$ becomes conservative. So, first, using Crocco's equation, the non-conservative vector $\mathbf{\Omega} \times \mathbf{V}$ becomes biscalar, $T\nabla S$, and further, over the *polytropic* surfaces $p/\rho^n = (p/\rho^n)_i = K_{1i}$, the biscalar vectors $T\nabla S$ and $\nabla p/\rho$ become conservative: $(n - \gamma)/(n - 1)\cdot C_v\nabla T = \nabla[(n - \gamma)/(n - 1)\cdot C_vT]$, and $K_{1i}\cdot n\rho^{n-2}\cdot\nabla\rho = \nabla[K_{1i}\cdot n/(n - 1)\cdot\rho^{n-1}]$, respectively.

In case 4 the specific entropy S remains constant (S_{0i}) on the surface containing the streamline and the vortex line passing through the flow point considered. That is, over the whole above ($\mathbf{V}$, $\mathbf{\Omega}$) surface (not crossing any discontinuity surface – shock wave, in the case of high-speed flows), one can write for the physical equation: $p = K_i\rho^\gamma$, with: $K_i = p_{0i}/(\rho_{0i})^\gamma\cdot\exp[(S - S_{0i})/C_v] = p_{0i}/(\rho_{0i})^\gamma > 0$ (the isentropic constant of the streamline and vortex line, and more, of the respective "i" isentropic virtual surface). The biscalar vector $\nabla p/\rho$ in Eq. (1) is evidenced as a conservative one, and the equation becomes (2 terms only)

$$d(\mathbf{V}^2/2) = -\,dp/\rho \quad , \quad \text{or} \quad d(V^2/2) = -\,\gamma K_i\rho^{\gamma-2}d\rho \quad , \qquad (3)$$

presenting itself like the differential equation of a potential motion (even if the vorticity is present $\mathbf{\Omega} \neq 0$ on the whole stream and vortex surface) and admitting on the above surface a first integral (identical to the energy equation): $V^2/2 + \gamma K_i/(\gamma - 1)\cdot\rho^{\gamma-1} = i_0 = W^2/2$,

$$\text{or:} \quad V^2/2 + \gamma\mathcal{R}/(\gamma - 1)\cdot T = i_0 = \gamma\mathcal{R}/(\gamma - 1)\cdot T_0 \quad , \qquad (4)$$

the D. Bernoulli integral for isoenergetic (adiabatic and steady, with constant total specific enthalpy i_0) non-isentropic (rotational) flows, integral very similar to that encountered in isentropic compressible steady aero-gasdynamics, where: K_i is the isentropic constant, different from one stream and vortex surface to another; crossing a shock wave, even on the same ($\mathbf{V}$, $\mathbf{\Omega}$) surface, the entropy S jumps, requiring change of the K_i constant and of the zero-S_{0i} surface; i_0, W are invariants for the whole flow (due to general flow steadiness and to the inviscid incident flow parallelism and uniformity), even if there are discontinuity surfaces (shock waves); $\mathcal{R}$ is the specific ideal gas constant ($= C_p - C_v$). Eq. (3) was derived rigorously for the case of the isoenergetic flows of a non-barotropic fluid, from the isentropicity condition along the flow streamlines and vortex lines (on some "isentropic" surfaces – with various constant values of the specific entropy $S = S_{0i}$ – analogous to D. Bernoulli's (Lamb's) ones ([9] – [27]), $B = (V^2/2 + \int dp/\rho + gz =)B_{0i}$, for the case of a barotropic fluid). These are rigid surfaces in the fluid, the study of the 3-D fluid motions being reduced to that of the 2-D fluid *potential* motions over these surfaces. Euler's equation may be written in the form: $\mathbf{\Omega} \times \mathbf{V} = -\,\nabla B$ ($B = -\,F$ in case G). Further, introducing a smart intrinsic coordinate system and a 2-D velocity "quasi-potential" function, a simpler form for the PDE of the velocity "quasi-potential" was obtained, for both steady and unsteady compressible flows ([4] – [8], [28]). Over some virtual polytropic surfaces, like those from cases (4 – 8), the new model for rotational flows of a compressible fluid offers advantages, e.g.: introducing a velocity "quasi-potential", and finding a special space curves net – the intersection lines of these surfaces – along which Steichen's equation becomes a Laplace's one. Knowing them is important for studying their properties. So, the most productive integrability cases for the motion equation from this viewpoint are the various combinations (intersections) of case 4 with any of the cases (5 – 8), so getting an identical net of space curves given by Selescu's isentropic & isotachic vector ($d\mathbf{R}_{ij} = k_1\cdot\nabla S|_i \times \nabla V|_j$ mainly). For a more detailed analysis see the conclusion section.

3 Introducing the 2-D velocity "quasi-potential" over the isentropic surfaces

The vector $\mathbf{V}$ has now two components only (like the vector $\mathbf{\Omega}$, both lying in the plane tangent to an isentropic sheet). Let be Oxyz the Cartesian system. In a triorthogonal curvilinear coordinate system $O\xi\eta\zeta$ tied to this isentropic surface (having $\xi O\eta$ as tangent plane) – therefore *a smart intrinsic coordinate system*, the vorticity component normal to the isentropic sheet ($\zeta = \zeta_0$) must be: $\Omega_\zeta = 0$. The analytic expression of the vector $\mathbf{\Omega}$ is:

$$\mathbf{\Omega} = \nabla\times\mathbf{V} = \frac{1}{h_\xi h_\eta h_\zeta}\begin{vmatrix} h_\xi\mathbf{k}_\xi & h_\eta\mathbf{k}_\eta & h_\zeta\mathbf{k}_\zeta \\ \dfrac{\partial}{\partial\xi} & \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\zeta} \\ h_\xi V_\xi & h_\eta V_\eta & h_\zeta V_\zeta \end{vmatrix} = \mathbf{k}_\xi\Omega_\xi + \mathbf{k}_\eta\Omega_\eta + \mathbf{k}_\zeta\Omega_\zeta ,$$

$$\text{with :} \quad V_\xi = h_\xi\dot{\xi} \quad ; \quad V_\eta = h_\eta\dot{\eta} \quad ; \quad V_\zeta = h_\zeta\dot{\zeta} = 0 \quad ,$$

where: $\mathbf{k}_i$ – a 3-D basis; h_i – Lamé's coefficients; the dotted variables are derivatives with respect to the time t, and so:

$$\Omega_\zeta = \frac{1}{h_\xi h_\eta}\left[\frac{\partial(h_\eta V_\eta)}{\partial\xi} - \frac{\partial(h_\xi V_\xi)}{\partial\eta}\right] = 0 \quad , \quad \text{or}$$

$$\frac{\partial(h_\eta V_\eta)}{\partial\xi} - \frac{\partial(h_\xi V_\xi)}{\partial\eta} = 0 \quad .$$

Between two isentropic surfaces the entropy can vary continuously (monotonously) or discontinuously (by jump, like for the cases of supersonic plane flow with direct and Mach reflected shock waves, and of axisymmetric confluent flows). Even if varying the index "i", $S = S_{0i}(i)$ is not always a strictly increasing function to be accepted as a ζ coordinate (like for symmetric plane flows), the monotony of S on some intervals of "i" may be considered, thus needing delimiters. Let introduce a scalar function $\Phi_i(M) = \Phi_i(\xi, \eta, \zeta_{0i})$, called by the author "quasi-potential", whose partial derivatives along the directions of the elementary orthogonal arcs $h_\xi d\xi$ and $h_\eta d\eta$ on the "i" isentropic surface ($\zeta = \zeta_{0i}$) are just the components $V_{\xi i}$ and $V_{\eta i}$ of the velocity $\mathbf{V}_i$ vector ($V_{\zeta i} = 0$). Let still define λ and μ as being two orthogonal arc lengths, so that: $d\lambda = h_\xi d\xi$ and $d\mu = h_\eta d\eta$, ($d\mathbf{R} = c_1\mathbf{V} + c_2\mathbf{\Omega} = \mathbf{k}_\xi h_\xi d\xi + \mathbf{k}_\eta h_\eta d\eta = \mathbf{k}_\xi d\lambda + \mathbf{k}_\eta d\mu$;) the elementary arc length ds (= $|d\mathbf{R}|$) on this surface being given by:

$$ds^2 = dx^2 + dy^2 + dz^2 = Ed\xi^2 + 2Fd\xi d\eta + Gd\eta^2$$
$$= h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 = d\lambda^2 + d\mu^2 \quad , \quad \text{with}$$

$$E = h_\xi^2 \quad ; \quad G = h_\eta^2 \quad ; \quad F = \frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta} + \frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta} + \frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} = 0$$

– due to the orthogonality condition, where:

$$h_i = \sqrt{\sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i}\right)^2} \;;\; (i = \overline{1, 3}) \;;\; \begin{cases} x_j = x, y, z \\ q_i = \xi, \eta, \zeta_0 \end{cases} \text{, having :}$$

$$h_\xi = \sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\xi\right)^2}\,\Bigg|_{\zeta=\zeta_{0i}} = f_1(\xi, \eta, \zeta_{0i}) \quad ;$$

$$h_\eta = \sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\eta\right)^2}\Bigg|_{\zeta=\zeta_{0i}} = f_2(\xi,\eta,\zeta_{0i}) \quad ;$$

$$h_\zeta = \sqrt{\sum_{j=1}^{3}\left(\partial x_j/\partial\zeta\right)^2}\Bigg|_{\zeta=\zeta_{0i}} = f_3(\xi,\eta,\zeta_{0i}) \quad ,$$

(with: $J = \dfrac{D(x,y,z)}{D(\xi,\eta,\zeta)} \neq 0$; J – the Jacobian determinant of this change of variables; $J = 0$ – gives the space curves representing the entropy singularities), so resulting on the "i" isentropic surface:

$$V_{\xi i} = \frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial\Phi_i}{\partial\lambda}; V_{\eta i} = \frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial\Phi_i}{\partial\mu}; V_{\zeta i} = \frac{1}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta} = 0;$$

$$\Rightarrow \; h_\xi V_{\xi i} = \frac{\partial\Phi_i}{\partial\xi} \quad ; \quad h_\eta V_{\eta i} = \frac{\partial\Phi_i}{\partial\eta} \quad ;$$

$$\frac{\partial(h_\eta V_{\eta i})}{\partial\xi} = \frac{\partial}{\partial\xi}\frac{\partial\Phi_i}{\partial\eta} = \frac{\partial^2\Phi_i}{\partial\xi\partial\eta} \; ; \; \frac{\partial(h_\xi V_{\xi i})}{\partial\eta} = \frac{\partial}{\partial\eta}\frac{\partial\Phi_i}{\partial\xi} = \frac{\partial^2\Phi_i}{\partial\eta\partial\xi} \; .$$

So the relation $\Omega_{\zeta i} = 0$ leads to: $\dfrac{\partial^2\Phi_i}{\partial\xi\partial\eta} - \dfrac{\partial^2\Phi_i}{\partial\eta\partial\xi} = 0$,

representing a true relation – Schwarz' theorem for the functions of two variables (the so-called theorem of "the equality of the mixed derivatives of the second order", they differing as to the order of differentiation). This relation proves that $\Omega_{\zeta i} = 0$ and the existence of a 2-D "quasi-potential" function Φ_i so that:

$$V_{\xi i} = V_{\lambda i} = \frac{\partial\Phi_i}{\partial\lambda} \quad ; \quad V_{\eta i} = V_{\mu i} = \frac{\partial\Phi_i}{\partial\mu} \quad ,$$

the entropy gradient vector ∇S_i being normal to the introduced isentropic surfaces $\zeta = \zeta_{0i}$.

Thus, introducing the scalar quasi-potential function $\Phi_i(M) = \Phi_i(x, y, z)$: $\mathbf{V}_i = \nabla\Phi_i$, the last vector ODE of motion, joined to the continuity and the physical ones, and taking into consideration the local speed of sound a_i definition, enables the determining of the total velocity vector $\mathbf{V}_i$ from Steichen's equation (see [29]) – usually a PDE (improperly called now the "velocity potential equation", taking into account that there is a vector $\mathbf{\Omega}_i \neq 0$; the flow being rotational, more appropriate would be the term "*velocity quasi-potential equation*").

4 The velocity "quasi-potential" equation for any steady flow of a compressible fluid

This vector equation may be written in a symbolic form:

$$\nabla\mathbf{V}_i = \frac{\mathbf{V}_i}{2a_i^2}\nabla\left(V_i^2\right), \text{with } \mathbf{V}_i = \nabla\Phi_i, \text{ but } \mathbf{\Omega}_i \neq 0; \; V_i = |\mathbf{V}_i|;$$

$$\Rightarrow \Delta\Phi_i = \frac{\nabla\Phi_i}{2a_i^2}\nabla\left[(\nabla\Phi_i)^2\right] \text{ or } \Delta\Phi_i = \frac{1}{a_i^2}\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i \; ,$$

where, in a Cartesian coordinate system Oxyz:

$$\Delta\Phi_i = \frac{\partial^2\Phi_i}{\partial x^2} + \frac{\partial^2\Phi_i}{\partial y^2} + \frac{\partial^2\Phi_i}{\partial z^2} \quad , \quad \text{and, symbolically:}$$

$$\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i = \left(\frac{\partial\Phi_i}{\partial x}\cdot\frac{\partial}{\partial x} + \frac{\partial\Phi_i}{\partial y}\cdot\frac{\partial}{\partial y} + \frac{\partial\Phi_i}{\partial z}\cdot\frac{\partial}{\partial z}\right)^{(2)}\Phi_i \; ,$$

and expanding the symbolic expression in the brackets:

$$\left(\nabla\Phi_i\cdot\nabla\right)^{(2)}\Phi_i = \left[\left(\frac{\partial\Phi_i}{\partial x}\right)^2\frac{\partial^2}{\partial x^2} + \left(\frac{\partial\Phi_i}{\partial y}\right)^2\frac{\partial^2}{\partial y^2} + \left(\frac{\partial\Phi_i}{\partial z}\right)^2\frac{\partial^2}{\partial z^2}\right.$$

$$\left. + 2\frac{\partial\Phi_i}{\partial x}\frac{\partial\Phi_i}{\partial y}\frac{\partial^2}{\partial x\partial y} + 2\frac{\partial\Phi_i}{\partial y}\frac{\partial\Phi_i}{\partial z}\frac{\partial^2}{\partial y\partial z} + 2\frac{\partial\Phi_i}{\partial z}\frac{\partial\Phi_i}{\partial x}\frac{\partial^2}{\partial z\partial x}\right]\Phi_i \; ,$$

the speed of sound a_i being given by the energy equation:

$$a_i^2 = \left(\frac{dp}{d\rho}\right)_{S=S_{0i}} = \frac{\gamma-1}{2}\left[W^2 - \left(\frac{\partial\Phi_i}{\partial x}\right)^2 - \left(\frac{\partial\Phi_i}{\partial y}\right)^2 - \left(\frac{\partial\Phi_i}{\partial z}\right)^2\right],$$

all the points at which is satisfied the previous PDE of the velocity potential Φ_i *belonging to a certain* "i" *isentropic surface*. This new equation is identical to the velocity potential equation (see [29]), written for a potential flow only. In a triorthogonal smart intrinsic coordinate system Oξηζ tied to these surfaces (or Oλμν, with λ, μ, ν – lengths of orthogonal arcs, with λ and μ contained in the local tangent plane and ν directed upon the normal) Laplace's operator Δ is given by the general expression below (the function Φ_i depending on ξ, η and ζ_{0i}, or on λ, μ and ν_{0i}):

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right.$$

$$\left. + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi_i}{\partial\zeta}\right)\right] \; ; \; \text{but} \; V_{\zeta i} = 0 \; \Rightarrow \; \frac{\partial\Phi_i}{\partial\zeta} = 0 \; ;$$

$$\Delta\Phi_i = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)\right]\Bigg|_{\zeta=\zeta_{0i}} \; , \text{ or:}$$

$$\Delta\Phi_i = \frac{1}{h_\zeta}\left[\frac{1}{h_\eta}\frac{\partial}{\partial\lambda}\left(h_\eta h_\zeta\frac{\partial\Phi_i}{\partial\lambda}\right) + \frac{1}{h_\xi}\frac{\partial}{\partial\mu}\left(h_\zeta h_\xi\frac{\partial\Phi_i}{\partial\mu}\right)\right]\Bigg|_{\nu=\nu_{0i}} \; ,$$

e.g.: the λ arcs taken along the streamlines and the μ arcs – along the equi(iso)-"quasi-potential" lines ≡ intersection lines of the isentropic surfaces with the iso-quasi-potential ones, these being normal to the local velocity **V**. Similarly, in Steichen's equation – a nonlinear PDE of the 2nd order in three variables – ξ, η, ζ (now written for any rotational flow – $\mathbf{\Omega} \neq 0$, but *on the* "i" *isentropic surfaces* $\zeta = \zeta_{0i}$) disappear all the terms containing the partial derivative about ζ of the potential function Φ_i, $(\partial\Phi_i/\partial\zeta)$, and also its derivatives with respect to ξ, η and ζ, thus being obtained a nonlinear PDE of the second order in only two variables – ξ, η – the "velocity quasi-potential equation" (see [4], [5], [7]), which was thoroughly treated in a recent paper ([30]).

5 The first integral of the steady motion equation over the polytropic surfaces

In order to derive this first integral, we start from Eq. (2) of a steady motion for an inviscid compressible fluid, taking into account Crocco's equation (see [2]) and the infinitesimal variation laws of the specific entropy S and of the static pressure p over a general polytropic special "i" surface: $d(\mathbf{V}^2/2) + (\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = -dp/\rho$; $d(V^2/2) + TdS + dp/\rho = 0$; $dS = (n-\gamma)/(n-1)\cdot C_v dT/T$; $TdS = (n-\gamma)/(n-1)\cdot C_v dT$; and $p/\rho^n = (p/\rho^n)_i = K_{1i} = \text{const.(i)}$, this resulting in: $dp = K_{1i}\cdot n\rho^{n-1}d\rho$, and: $dp/\rho = K_{1i}\cdot n\rho^{n-2}d\rho$, with γ, C_v , n and $K_{1i} = (p/\rho^n)_i$ – constants (for an ideal gas), so obtaining the first integrable differential equation: $d(V^2/2) + (n-\gamma)/(n-1)\cdot C_v dT + K_{1i}\cdot n\rho^{n-2}d\rho = 0$, admitting on a certain virtual polytropic "i" surface the first integral below (for an isoenergetic flow $i_0 = \text{const.}_1$):

$$V^2/2 + (n-\gamma)/(n-1)\cdot C_v T + K_{1i}\cdot n/(n-1)\cdot\rho^{n-1} = i_0 = W^2/2 = i_0 = W^2/2 = \text{const.}_1 , \qquad (5)$$

or, in a contracted form (obtained applying the ideal gas physical law, $p = \mathcal{R}\rho T$, and identical to energy equation):

$$V^2/2 + C_pT = V^2/2 + i = i_0 = W^2/2 = C_pT_0 = \text{const.}_1 , \qquad (5')$$

where $C_v = \mathcal{R}/(\gamma-1)$ and $C_p = \gamma C_v = C_v + \mathcal{R} = \gamma\mathcal{R}/(\gamma-1)$ are the isochoric and isobaric gas specific heats, assumed to be constant for an ideal gas; W is the maximum possible gas velocity, corresponding to its expansion into a vacuum, also a constant quantity. The quantities i_0 and W are invariants over the whole isoenergetic flow. The quantity K_{1i} is also a constant, but is different from one "i" polytropic sheet to another. The term $V^2/2$ is *the specific kinetic energy* of the gas particle; the term $(n-\gamma)/(n-1)\cdot C_vT = \int TdS = \int(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R}$ is *the specific energy due to the vorticity* $\mathbf{\Omega}$, while the term $K_{1i}\cdot n/(n-1)\cdot\rho^{n-1} = [n/(n-1)]\cdot(p/\rho) = \int dp/\rho$ is *the specific work due to the static pressure* p.

Remark: All the newly found first integrals for the motion vector equation – like (4) and (5) – both from the *geometrical* and *thermodynamic* points of view, were obtained by a "term-by-term" analytic integration, and not by evaluating the more complex integrand in this equation, written in the Helmholtz–Gromeko–Lamb form (after a scalar multiplication by $d\mathbf{R}$): $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} + dp/\rho$ (= $TdS + dp/\rho$) , as a whole. A contracted form of this integrand is C_pdT (due to relation: $TdS = C_pdT - dp/\rho$, representing the mathematical expression of the second law of thermodynamics), regardless of the dependence law of p on ρ (a generic fluid, not a barotropic one) and is encountered in the differential form of the energy equation, the integral form of which is: $V^2/2 + C_pT = [V^2/2 + \gamma\mathcal{R}/(\gamma-1)\cdot T = V^2/2 + \gamma/(\gamma-1)\cdot p/\rho = W^2/2 = \gamma\mathcal{R}/(\gamma-1)\cdot T_0 =]\ i_0 = \text{const.}$, $p = f(\rho, T) \neq f_1(\rho \text{ only})$ (for an isoenergetic steady flow, this meaning with a constant total specific enthalpy i_0) and does not depend on the rotationality (in other words if the vorticity is or is not present, leading or not to the entropy rise). Although in all cases treated in the approached intrinsic analytic study the new first integrals seam to be identical to the form above, they are phenomenologically different (though being important particular cases of first integrals for the motion equation, like D. Bernoulli and similar ones). So, over the isentropic virtual surfaces the specific entropy S is kept constant, even if the vorticity vector is present (in the plane tangent to a certain isentropic sheet). Analogously, over the other kinds of polytropic virtual surfaces, an important physical quantity (like the static: temperature T – and implicitly flow velocity V; pressure p; density ρ etc.) is conserved also and so the energy equation takes some interesting particular forms. The hidden difference consists in that in the energy equation, in order to conserve the p/ρ ratio (to have the same static temperature T at all points with the same velocity V), regardless of the losses due to flow vorticity and to shock waves, the new values of the static pressure p' and density ρ' should both be smaller, though having the same ratio, to satisfy the physical equation: $p'/\rho' = p/\rho = \mathcal{R}T$, (with: $p' < p$ and $\rho' < \rho$), while in the first integral (4) the quantities p and ρ conserved their values (in the absence of the shock waves only). More, integrating "term-by-term", several advantages in writing the velocity potential equation are evidenced, using the isentropic and polytropic virtual surfaces. The intersection lines of the surfaces in cases (4 – 8) are identical. So: $\{(4) \cap (5)\} \equiv \{(4) \cap (6)\} \equiv \{(4) \cap (7)\} \equiv \{(4) \cap (8)\}$, and more (for inviscid fluids) $\equiv \{(5) \cap (6)\} \equiv \{(5) \cap (7)\} \equiv \{(5) \cap (8)\} \equiv \{(6) \cap (7)\} \equiv \{(6) \cap (8)\} \equiv \{(7) \cap (8)\}$ – a star of sheets [a multi-pencil (-sheaf) of planes tangent to these sheets]. As one can see in [30], along these lines the velocity quasi-potential Φ_{ij} PDE becomes a 2-D Laplace's one for a rotational pseudo-flow of a quasi-incompressible fluid.

6 The PDE of the polytropic surfaces in the Cartesian coordinate system (a first integral of Crocco's vector equation)

One can derive this equation starting from Crocco's equation for steady flows, and performing a scalar multiplication by a certain virtual elementary $d\mathbf{R}$ displacement vector, obtaining: $(\delta q =)\ TdS = (\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = [(\mathbf{V}\nabla)\mathbf{V} - \nabla(\mathbf{V}^2/2)]\cdot d\mathbf{R}$, δq (the elementary heat loss) being not a total (an exact) differential for a certain case, but for some special cases (over some special virtual integral surfaces, like for the cases 5 – 8 in section 2) only. The general differential equation of the polytropic surfaces is (with $\mathbf{\Omega} \times \mathbf{V} = \nabla F$):

$$TdS = (\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = dF = (n-\gamma)/(n-1)\cdot C_v dT ,$$

meaning the elementary volume $|dF|$ of the parallelepiped built on the vectors $\mathbf{\Omega}$, $\mathbf{V}$, $d\mathbf{R}$ is $|(n-\gamma)/(n-1)\cdot C_v dT|$, so combining the ***geometrical*** and ***thermodynamic*** points of view (one of the reasons why we call this model a physical & mathematical one, elaborated in order to find new properties and using them further to simplify some mathematical formulations, introducing new concepts – physical & mathematical quantities), this resulting in (with: $d\mathbf{R} \perp \nabla(p/\rho^n)$, or $d\mathbf{R}\cdot\nabla(p/\rho^n) = 0$):

$$d\mathbf{R}\cdot(\mathbf{\Omega}\times\mathbf{V}) = \begin{vmatrix} dx & dy & dz \\ \Omega_x & \Omega_y & \Omega_z \\ V_x & V_y & V_z \end{vmatrix} = \frac{n-\gamma}{n-1}\cdot C_v dT \quad \text{, with:}$$

$$\Omega_x = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} ; \Omega_y = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} ; \Omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} .$$

This is a total (an exact) differential equation, namely that of a polytropic sheet written in the Cartesian system, assuming **V** as being a given function for any value of the variables x, y, z – the differential equation of a mobile plane (passing through the current point) tangent to such a surface. The searched for general differential equation of these surfaces is:

$$dF = \nabla F \cdot d\mathbf{R} = (n - \gamma)/(n - 1) \cdot C_v dT \quad . \qquad (6)$$

The rectangular coordinates of vector ∇F for a polytropic surface are the algebraic projections of this vector:

$$\frac{\partial F}{\partial x} = \Omega_y V_z - \Omega_z V_y ; \frac{\partial F}{\partial y} = \Omega_z V_x - \Omega_x V_z ; \frac{\partial F}{\partial z} = \Omega_x V_y - \Omega_y V_x .$$

With: $C_v = \frac{\mathcal{R}}{\gamma - 1}$ and $T = \frac{\gamma - 1}{2\gamma \mathcal{R}} (W^2 - V_x^2 - V_y^2 - V_z^2)$,

the expression $C_v dT$ in Cartesian coordinates is

$$\mathcal{R}/(\gamma - 1) \cdot d\left[(\gamma - 1) \cdot (W^2 - V_x^2 - V_y^2 - V_z^2)/2\gamma \mathcal{R}\right]$$
$$= -(V_x dV_x + V_y dV_y + V_z dV_z)/\gamma \quad .$$

The PDE (6) of the polytropic integral surfaces becomes

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz + \frac{n - \gamma}{\gamma (n - 1)} (V_x dV_x + V_y dV_y + V_z dV_z) = 0 .$$

or more:

$$\left[(\partial V_x/\partial z - \partial V_z/\partial x)V_z - (\partial V_y/\partial x - \partial V_x/\partial y)V_y\right]dx$$
$$+ \left[(\partial V_y/\partial x - \partial V_x/\partial y)V_x - (\partial V_z/\partial y - \partial V_y/\partial z)V_z\right]dy$$
$$+ \left[(\partial V_z/\partial y - \partial V_y/\partial z)V_y - (\partial V_x/\partial z - \partial V_z/\partial x)V_x\right]dz$$
$$+ (n - \gamma)/[\gamma (n - 1)] \cdot (V_x dV_x + V_y dV_y + V_z dV_z) = 0 . \quad (6')$$

In the last line of this equation we have:

$dV_x = (\partial V_x/\partial x)dx + (\partial V_x/\partial y)dy + (\partial V_x/\partial z)dz$;
$dV_y = (\partial V_y/\partial x)dx + (\partial V_y/\partial y)dy + (\partial V_y/\partial z)dz$;
$dV_z = (\partial V_z/\partial x)dx + (\partial V_z/\partial y)dy + (\partial V_z/\partial z)dz$.

The equation of the polytropic surfaces is obtained by integration. For the "integrability condition" of such a differential expression see section 8. The integral of PDE (6′) is

$F = (n - \gamma)/(n - 1) \cdot C_v T + \text{const.(i)}$, or successively

$F - (n - \gamma)/(n - 1) \cdot C_v T = -(n - \gamma)/(n - 1) \cdot C_v T_1$, (7)

$$\int [(\Omega_y V_z - \Omega_z V_y)dx + (\Omega_z V_x - \Omega_x V_z)dy + (\Omega_x V_y - \Omega_y V_x)dz]$$
$$- \frac{n - \gamma}{n - 1} C_v \left[\frac{\gamma - 1}{2\gamma \mathcal{R}} (W^2 - V_x^2 - V_y^2 - V_z^2)\right] = \text{const.(i)} ;$$

or more:

$$\int \left\{\left[(\partial V_x/\partial z - \partial V_z/\partial x)V_z - (\partial V_y/\partial x - \partial V_x/\partial y)V_y\right]dx\right.$$
$$+ \left[(\partial V_y/\partial x - \partial V_x/\partial y)V_x - (\partial V_z/\partial y - \partial V_y/\partial z)V_z\right]dy$$
$$\left.+ \left[(\partial V_z/\partial y - \partial V_y/\partial z)V_y - (\partial V_x/\partial z - \partial V_z/\partial x)V_x\right]dz\right\}$$
$$- (n - \gamma)/[2\gamma (n - 1)](W^2 - V_x^2 - V_y^2 - V_z^2) = \text{const.(i)} \quad (8)$$

(under the sign of integral in the left-hand side being a total differential dF), representing the general equation of an "i" *polytropic* surface written in the Cartesian system – envelope sheet of the planes tangent to such a surface (the general first integral of Crocco's equation).

In the following, denoting the integral $\int(\mathbf{\Omega} \times \mathbf{V}) \cdot d\mathbf{R}$ in Eq. (8) by F, the contracted form of this equation is:

$$F - (n - \gamma)/[2\gamma (n - 1)] \cdot (W^2 - V_x^2 - V_y^2 - V_z^2)$$
$$= \text{const.(i)} = -(n - \gamma)/(n - 1) \cdot C_v T_1 \quad , \qquad (8')$$

corresponding to case 8 in section 2. For various values of n, one obtains the equations for cases 4 – 7. On some intervals of T, where γ, C_p and C_v are practically constant (varying discontinuously, according to a step or a staircase function, but with steps of various lengths) the results obtained for an ideal gas are valid for the real gas too.

7 The dependence of gas particle specific entropy on the velocity "quasi-potential"

One can derive this, starting from Crocco's equation for steady flows, considering the elementary orthogonal arcs in the new system of curvilinear coordinates: $d\lambda = h_\xi d\xi$; $d\mu = h_\eta d\eta$; $d\nu = h_\zeta d\zeta$. At a flow point M(x, y, z) or M(ξ, η, ζ) or M(λ, μ, ν), where (ξ, η) are the local orthogonal curvilinear coordinates and the (λ, μ) orthogonal arcs are contained in a certain "i" isentropic surface $(\mathbf{V}, \mathbf{\Omega})_i \equiv (\zeta = \zeta_{0i})$, or $(S = S_{0i} = \text{const.})$, with $\mathbf{k}_i$ – a 3-D basis, the specific entropy's gradient is:

$$\nabla S = \mathbf{k}_\xi \frac{\partial S}{\partial \lambda} + \mathbf{k}_\eta \frac{\partial S}{\partial \mu} + \mathbf{k}_\zeta \frac{\partial S}{\partial \nu} = \mathbf{k}_\zeta \frac{\partial S}{\partial \nu} = \mathbf{k}_\zeta \frac{dS}{d\nu} \quad ;$$

$$\mathbf{V} = \mathbf{k}_\xi V_\xi + \mathbf{k}_\eta V_\eta + \mathbf{k}_\zeta V_\zeta = \mathbf{k}_\xi V_\xi + \mathbf{k}_\eta V_\eta \quad ;$$

$(V^2 = V_\xi^2 + V_\eta^2)$ – the velocity;

$\mathbf{\Omega} = \mathbf{k}_\xi \Omega_\xi + \mathbf{k}_\eta \Omega_\eta + \mathbf{k}_\zeta \Omega_\zeta = \mathbf{k}_\xi \Omega_\xi + \mathbf{k}_\eta \Omega_\eta$ – the vorticity (curl **V**);

$$\mathbf{\Omega} \times \mathbf{V} = \begin{vmatrix} \mathbf{k}_\xi & \mathbf{k}_\eta & \mathbf{k}_\zeta \\ \Omega_\xi & \Omega_\eta & 0 \\ V_\xi & V_\eta & 0 \end{vmatrix} = \mathbf{k}_\zeta (\Omega_\xi V_\eta - \Omega_\eta V_\xi) \quad ,$$

where, by definition : $\mathbf{\Omega} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{k}_\xi & \mathbf{k}_\eta & \mathbf{k}_\zeta \\ \frac{\partial}{\partial \lambda} & \frac{\partial}{\partial \mu} & \frac{\partial}{\partial \nu} \\ V_\xi & V_\eta & 0 \end{vmatrix}$

$$= -\mathbf{k}_\xi \frac{\partial V_\eta}{\partial \nu} + \mathbf{k}_\eta \frac{\partial V_\xi}{\partial \nu} + \mathbf{k}_\zeta \left(\frac{\partial V_\eta}{\partial \lambda} - \frac{\partial V_\xi}{\partial \mu}\right) = \mathbf{k}_\xi \Omega_\xi + \mathbf{k}_\eta \Omega_\eta ,$$

$$\text{with:} \Omega_\xi = -\frac{\partial V_\eta}{\partial \nu} \;;\; \Omega_\eta = \frac{\partial V_\xi}{\partial \nu} \;;\; \Omega_\zeta = \frac{\partial V_\eta}{\partial \lambda} - \frac{\partial V_\xi}{\partial \mu} = 0 \;;$$

$$\mathbf{\Omega} \times \mathbf{V} = \mathbf{k}_\zeta (\Omega_\xi V_\eta - \Omega_\eta V_\xi) = -\mathbf{k}_\zeta \left(V_\xi \frac{\partial V_\xi}{\partial \nu} + V_\eta \frac{\partial V_\eta}{\partial \nu}\right) =$$
$$= -\frac{\mathbf{k}_\zeta}{2} \left(\frac{\partial V_\xi^2}{\partial \nu} + \frac{\partial V_\eta^2}{\partial \nu}\right) = -\frac{\mathbf{k}_\zeta}{2} \frac{\partial (V_\xi^2 + V_\eta^2)}{\partial \nu} = -\frac{\mathbf{k}_\zeta}{2} \frac{\partial (V^2)}{\partial \nu} ;$$

so resulting for Crocco's equation:

$$T \frac{dS}{d\nu} = \Omega_\xi V_\eta - \Omega_\eta V_\xi \quad ;$$

$$dS = \frac{1}{T} (\Omega_\xi V_\eta - \Omega_\eta V_\xi) d\nu = -\frac{1}{T} \frac{\partial}{\partial \nu} \left(\frac{1}{2} V^2\right) d\nu \quad ;$$

and introducing the "quasi-potential" function Φ , one obtains at a certain flow point M(ξ, η, ζ) or M(λ, μ, ν):

$$V_\xi = \frac{\partial \Phi}{\partial \lambda};\ V_\eta = \frac{\partial \Phi}{\partial \mu};\ V^2 = V_\xi^2 + V_\eta^2 = \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 + \left(\frac{\partial \Phi}{\partial \mu}\right)^2,$$

the "quasi-potential" Φ *being a function of three orthogonal arcs* (λ, μ, ν), and which on a certain isentropic surface ($\zeta = \zeta_{0i}$) becomes a function of only two orthogonal arcs (λ, μ), both contained in the respective surface:

$$\left.\begin{aligned} dS &= -\frac{1}{2T}\frac{\partial}{\partial \nu}\left[\left(\frac{\partial \Phi}{\partial \lambda}\right)^2 + \left(\frac{\partial \Phi}{\partial \mu}\right)^2\right]d\nu \quad ; \\ S &= \frac{\gamma \mathcal{R}}{\gamma - 1}\int \frac{\partial}{\partial \zeta}\ln\left[W^2 - \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 - \left(\frac{\partial \Phi}{\partial \mu}\right)^2\right]d\zeta \\ &= \frac{2\gamma \mathcal{R}}{\gamma - 1}\int \frac{\partial \ln a}{\partial \zeta}d\zeta \quad , \quad \text{with} \quad a^2 = \left(\frac{dp}{d\rho}\right)_{S=S_{0i}} \end{aligned}\right\} \quad (9)$$

the expression of the gas static temperature T being given by D. Bernoulli integral for isoenergetic flows (4). If the compressible flow (and implicitly its isentropic surfaces, containing the flow streamlines and vortex lines) is crossing a shock wave, the values S_{01} and S_{02} at both ends (ζ_{01} and ζ_{02}) of the integral above are referring to the same side (upstream or downstream) of respective wave, these sides differing by a specific entropy jump.

8 The PDE of the isentropic surfaces in the Cartesian system (a particular first integral of Crocco's vector equation)

One can derive this equation also starting from Crocco's equation for steady flows (like in the previous sections 6 and 7), and performing a scalar multiplication by a virtual elementary d**R** displacement vector, so having: (δq =) TdS = d**R**·(**Ω** × **V**) ,
δq (the elementary heat loss) being not a total (an exact) differential.
The differential equation of the isentropic surfaces is: dS = (1/T)·d**R**·(**Ω** × **V**) = 0, leading to: S(x, y, z) = S_{0i} = ct., this resulting in the annulment of the mixed product in the right-hand side (coplanarity of three vectors):

$$\begin{vmatrix} dx & dy & dz \\ \Omega_x & \Omega_y & \Omega_z \\ V_x & V_y & V_z \end{vmatrix} = 0 \quad , \quad \text{or:}$$

$$\begin{vmatrix} dx & dy & dz \\ \dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z} & \dfrac{\partial V_x}{\partial z} - \dfrac{\partial V_z}{\partial x} & \dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y} \\ V_x & V_y & V_z \end{vmatrix} = 0 \quad , \quad (10)$$

but this is not a total (an exact) differential equation (because δq ≠ dq). To obtain the equation of the isentropic surfaces (by integration), it must be multiplied by the integrant factor 1/T(x, y, z) .

The so-called "integrability condition" for a differential expression of the type above (in the left-hand side):
P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 is:

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

for having a total (an exact) differential in the left-hand side (for existence of an integrant factor).
Defining a new vector **L** = $\mathbf{k}_x$P + $\mathbf{k}_y$Q + $\mathbf{k}_z$R ($\mathbf{k}_i$ – a 3-D basis), in the present case having: **L** = **Ω** × **V** (= T∇S) , each of these two relations may be written in the condensed form of annulment of a scalar product:
L·d**R** = 0 , and: **L**·(∇ × **L**) = 0 , respectively – two "orthogonality conditions"; ⇒ **L** ⊥ (d**R**, ∇ × **L**) plane.
Eq. (10) is the equation of an isentropic sheet written in the Cartesian system, assuming **V** as being a given function for any value of the variables x, y, z – the differential equation of a mobile plane (passing through the current point) tangent to a (**V**, **Ω**) surface.
The (first) integral of this differential equation is:

$$\int \frac{1}{T}\left[(\Omega_y V_z - \Omega_z V_y)dx + (\Omega_z V_x - \Omega_x V_z)dy + \right.$$
$$\left. + (\Omega_x V_y - \Omega_y V_x)dz\right] = S_{0i} = \text{ct.}, \quad \text{with:}$$
$$T = \frac{\gamma - 1}{2\gamma \mathcal{R}}\left(W^2 - V_x^2 - V_y^2 - V_z^2\right) \ \text{and} \ \mathbf{\Omega} = \nabla \times \mathbf{V} \quad ,$$

(under the integral in the left-hand side being now a total differential), representing the general equation of an "i" *isentropic* surface written in the Cartesian system – the envelope sheet of the planes tangent to a (**V**, **Ω**) surface (a particular first integral of Crocco's equation). So:
$\mathbf{L}_1 = \mathbf{k}_x P_1 + \mathbf{k}_y Q_1 + \mathbf{k}_z R_1$ = **L**/T = (**Ω** × **V**)/T = ∇S ,

$$\text{with:}\ P_1 = \frac{P}{T} = \frac{\partial S}{\partial x};\quad Q_1 = \frac{Q}{T} = \frac{\partial S}{\partial y};\quad R_1 = \frac{R}{T} = \frac{\partial S}{\partial z};$$

so that: $\mathbf{L}_1$ = ∇S and curl $\mathbf{L}_1$ = ∇ × $\mathbf{L}_1$ = 0 .
The integrability (orthogonality) condition gives us a PDE connecting V_x, V_y, V_z (all being functions of x, y, z).
Considering the elementary orthogonal arcs in the new system of curvilinear coordinates as being dλ , dμ and dν (the λ and μ arcs – contained in the isentropic surface $\zeta = \zeta_{0i}$), one can write Crocco's equation in the form:

$$\frac{1}{T}\begin{vmatrix} d\lambda & d\mu & d\nu \\ \Omega_\xi & \Omega_\eta & 0 \\ V_\xi & V_\eta & 0 \end{vmatrix} = 0 \quad \text{or} \quad \frac{1}{T}\begin{vmatrix} d\lambda & d\mu & d\nu \\ -\dfrac{\partial V_\eta}{\partial \nu} & \dfrac{\partial V_\xi}{\partial \nu} & 0 \\ V_\xi & V_\eta & 0 \end{vmatrix} = 0$$

$$\text{or more:}\ \frac{2\gamma \mathcal{R}}{(\gamma - 1)\left(W^2 - V^2\right)}\left(V_\xi \frac{\partial V_\xi}{\partial \zeta} + V_\eta \frac{\partial V_\eta}{\partial \zeta}\right)d\zeta = 0$$

(see section 3 in [4]): $\Omega_\xi = -\partial V_\eta/\partial \nu$; $\Omega_\eta = \partial V_\xi/\partial \nu$; and so: $V_\xi(\partial V_\xi/\partial \zeta) + V_\eta(\partial V_\eta/\partial \zeta) = 1/2 \cdot \partial(V^2)/\partial \zeta \neq 0$, this resulting in: dζ = 0, and $\zeta = \zeta_{0i}$ = ct. – the general equation of an "i" isentropic surface written in the new intrinsic coordinate system Oξηζ, as it was expected.

9 Noticeable cases (quantitative examples) of isentropic surfaces in the usual flows

One must consider three noticeable quantitative cases of rotational flows: (1) the *plane flow*; (2) the *axisymmetric flow*, in both cases having $\boldsymbol{\Omega} \perp \mathbf{V}$ (for an incompressible fluid meaning vortex lines identical to the equi(iso)-"quasi-potential" lines; (3) the general *3-D conical supersonic flow* (with its 2-D plane and axisymmetric subcases); the orthogonality to the streamlines for a compressible fluid will be treated in a next paper, dedicated to the continuity equation;

(1) For *a plane flow* $\left(V_z = 0; \dfrac{\partial V_x}{\partial z} = \dfrac{\partial V_y}{\partial z} = 0 \right)$, like the rotational one downstream of a detached cylindrical shock wave (with curved directrix, and thus with variable intensity) in supersonic regime, the analytic expressions of $\mathbf{V}$ and $\boldsymbol{\Omega}$ are:

$$\left.\begin{array}{l} \mathbf{V} = \mathbf{k}_x V_x + \mathbf{k}_y V_y \quad \text{and} \\ \boldsymbol{\Omega} = \nabla \times \mathbf{V} = \mathbf{k}_z \Omega_z = \mathbf{k}_z \left(\dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y} \right) \neq 0 \quad , \\ \text{with} \quad \dfrac{\partial \Omega_z}{\partial z} = 0 \quad ; \end{array}\right\} ,$$

this meaning: $\boldsymbol{\Omega} \perp \mathbf{V}$; $\boldsymbol{\Omega} \times \mathbf{V} \in$ (xOy) – the flow plane; (xOy) ≡ ($\mathbf{V}$, ∇S) – the osculating plane; (∇S, $\boldsymbol{\Omega}$) – the normal plane; ($\boldsymbol{\Omega}$, $\mathbf{V}$) – the rectifying plane; (all with respect to the streamlines). Here $\mathbf{k}_i$ is a 3-D basis.

The isentropic surfaces ($S = S_{0i}$ = const.) for *a plane flow* are the field (stream and vortex) current cylinders defined by ($\mathbf{V}$, $\boldsymbol{\Omega}$), having the "i" streamline as directrix and the generatrices (the vortex lines) parallel to the Oz axis – the envelope sheets of the rectifying planes for the streamlines belonging to the same family. This case was treated in [26].

(2) In the cylindrical coordinate system (r, ω, x), having (xOr) as meridian plane, for *an axisymmetric flow* ($V_\omega = 0$; $\left. \dfrac{\partial V_r}{\partial \omega} = \dfrac{\partial V_x}{\partial \omega} = 0 \right)$, like the rotational one downstream of an axisymmetric shock wave (with curved meridian line, and thus with variable intensity) in supersonic regime, the analytic expressions of the vectors $\mathbf{V}$ and $\boldsymbol{\Omega}$ are:

$$\left.\begin{array}{l} \mathbf{V} = \mathbf{k}_r V_r + \mathbf{k}_x V_x \quad \text{and} \\ \boldsymbol{\Omega} = \nabla \times \mathbf{V} = \dfrac{1}{h_r h_\omega h_x} \begin{vmatrix} h_r \mathbf{k}_r & h_\omega \mathbf{k}_\omega & h_x \mathbf{k}_x \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \omega} & \dfrac{\partial}{\partial x} \\ h_r V_r & h_\omega V_\omega & h_x V_x \end{vmatrix} \\ = \mathbf{k}_r \Omega_r + \mathbf{k}_\omega \Omega_\omega + \mathbf{k}_x \Omega_x \quad , \\ \text{with:} \quad h_r = 1; \; h_\omega = r; \; h_x = 1, \quad \text{thus resulting:} \\ \boldsymbol{\Omega} = \mathbf{k}_r \left(\dfrac{1}{r} \dfrac{\partial V_x}{\partial \omega} - \dfrac{\partial V_\omega}{\partial x} \right) + \mathbf{k}_\omega \left(\dfrac{\partial V_r}{\partial x} - \dfrac{\partial V_x}{\partial r} \right) \\ + \mathbf{k}_x \dfrac{1}{r} \left[\dfrac{\partial}{\partial r} (r V_\omega) - \dfrac{\partial V_r}{\partial \omega} \right] \quad . \end{array}\right\}$$

Taking into account that there is an axisymmetric flow:

$$\left.\begin{array}{l} \Omega_r = \Omega_x = 0 \quad \text{and thus} \\ \boldsymbol{\Omega} = \nabla \times \mathbf{V} = \mathbf{k}_\omega \Omega_\omega = \mathbf{k}_\omega \left(\dfrac{\partial V_r}{\partial x} - \dfrac{\partial V_x}{\partial r} \right) \neq 0 \quad , \\ \text{with} \quad \dfrac{\partial \Omega_\omega}{\partial \omega} = 0 \quad ; \end{array}\right\}$$

(circular rings of vortices in planes normal to the Ox axis), meaning: $\boldsymbol{\Omega} \perp \mathbf{V}$; $\boldsymbol{\Omega} \times \mathbf{V} \in$ (xOr) – a flow meridian plane; (xOr) ≡ ($\mathbf{V}$, ∇S) – the osculating plane; (∇S, $\boldsymbol{\Omega}$) – the normal plane; ($\boldsymbol{\Omega}$, $\mathbf{V}$) – the rectifying plane; (all with respect to the streamlines). Here $\mathbf{k}_i$ is a 3-D basis.

The isentropic surfaces for *an axisymmetric flow* are the field current revolution surfaces defined by ($\mathbf{V}$, $\boldsymbol{\Omega}$), having Ox as symmetry axis and the streamlines as meridian lines (and implicitly the vortex lines as parallel circles) – the envelope of the envelope sheets of the rectifying planes for the streamlines belonging to the same given family.

In the cases (1) – (2), on a certain ($\mathbf{V}$, $\boldsymbol{\Omega}$) sheet, along a streamline (a very thin stream tube), the phenomenon depending on a single variable (ξ, or λ), the new PDE of the velocity "quasi-potential" becomes an ODE.

(3) The choice of a special triorthogonal smart intrinsic coordinate curves net (called by the author "generalized spherical (conical)" R, φ, χ) results in always obtaining a first integral of the continuity equation for *any 3-D conical supersonic flow* (the flow rate equation – see [31]):

$\rho R^3 f(\varphi)\dot{\varphi} = Q_i ; \rho R^2 f(\varphi) V_\varphi = Q_i$, or $\rho R^2 f(\varphi) V' = Q_i$;

where: $\quad V_\varphi = R\dot{\varphi} = V' = dV/d\varphi \quad , \quad V = V_R \quad ,$

in the first integral above appearing the product of Lamé's coefficients: $h_R = 1$; $h_\varphi = R$; $h_\chi = R \cdot f(\varphi)$.

An ODE of the velocity "quasi-potential" function Φ_i (on any "i" sheet, so that: $V_{Ri} = V_i = \partial\Phi_i/\partial R$) instead of the usual (*approximate*) PDE of the velocity potential Φ function for these flows was rigorously obtained (see [28]):

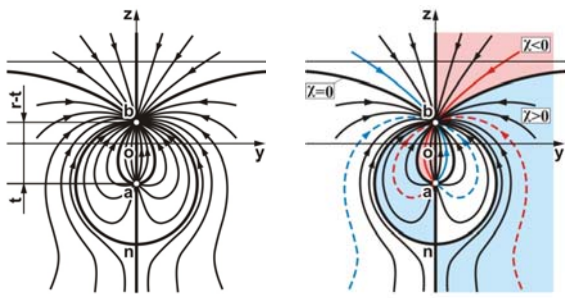
$$(\gamma - 1)\{V_i'' + [\ln|f(\varphi)|]' \cdot V_i' + 2V_i\}(W^2 - V_i^2 - V_i'^2) - 2(V_i'' + V_i)V_i'^2 = 0 \text{, with } ' = d/d\varphi \text{ and } '' = d^2/d\varphi^2 .$$

The isentropic surfaces are the conical ones having the vertex in the cone tip and the streamlines as directrices, as a result of the fact that the entropy is constant along both the streamline and any half-straight line starting from the tip.

10 A qualitative example of application; the "isentropic sheets ↔ streamlines" analogy, improved using the "mirror image" sources

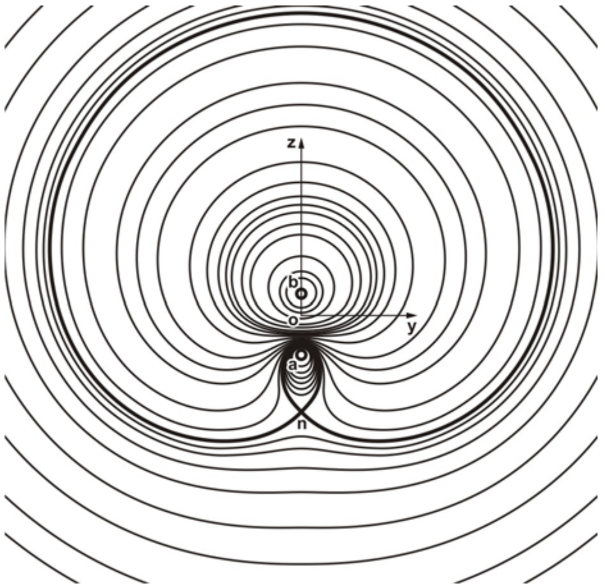
We try to apply the *isentropic surfaces* model to a special but usual case of 3-D conical supersonic flow – that around a circular cone at incidence, by an *intuitive analogy* to the *streamlines pattern* in an incompressible 2-D plane potential flow, by using an adequate smart intrinsic curvilinear coordinate system, and choosing a suitable conformal mapping in the cross section plane of a cone at incidence, so performing a flow's *qualitative mathematical modeling*.

the angle of attack $\alpha \neq 0$; r is the cone local radius;
$\bar{r} = r/x$; $\bar{t} = t/x = \tan\alpha$; $\Rightarrow$ $\alpha = \mathrm{atan}\,\bar{t}$;



$$1.a.\ \ \bar{Z} = \mathrm{Im}\,F(x_1) = \mathrm{Im}\left\{\ln\frac{[x_1 - i(\bar{r}-\bar{t})]^2}{x_1 + i\bar{t}}\right\} = \tan^{(-1)}\chi = C_2';$$

$$\chi = \frac{\bar{y}^2(\bar{z}+\bar{t}-2\bar{r}) + (\bar{z}+\bar{t}-\bar{r})^2(\bar{z}+\bar{t})}{\bar{y}[\bar{y}^2 + (\bar{z}+\bar{t})^2 - \bar{r}^2]} = \frac{1}{C_2} = \tan C_2';$$



$$1.b.\ \ \bar{Y} = \mathrm{Re}\,F(x_1) = \mathrm{Re}\left\{\ln\frac{[x_1 - i(\bar{r}-\bar{t})]^2}{x_1 + i\bar{t}}\right\} = \ln\sqrt{\varphi} = C_1' ;$$

$$\varphi = \frac{[\bar{y}^2 + (\bar{z}+\bar{t}-\bar{r})^2]^2}{\bar{y}^2 + (\bar{z}+\bar{t})^2} = C_1 = e^{2C_1'} ;$$

$x_1 = \bar{y} + i\bar{z}$ is a complex variable; $\bar{y} = y/x$; $\bar{z} = z/x$; x, y, z – Cartesian coordinates, x – abscissa of cross section current plane

Fig. 1. Intuitive example (pattern analogy) of the "generalized spherical" smart intrinsic coordinate surfaces net for the case of a circular cone at small angle of attack α (cross section) – the "incompressible" approximation (a "slender body"; no shock wave), giving the relations between the "generalized spherical" smart intrinsic coordinates (R, φ, χ) and the Cartesian ones: 1.a. the conical isentropic sheets $\chi = \chi_{0i} = c(S_{0i} - c_0) = 1/C_2$ (a smart intrinsic coordinate tied to S_{0i} – the local specific entropy value), having as remarkable directrices: the **oz** axis and a circle (the solid cone trace) centered on it (both for $C_2 = 0$), and a right strophoid ($\chi = 0$) centered on **oz** axis too (for $1/C_2 = 0$); c, S_{0i}, $c_0 > 0$; two closed curves pass through (b): a circle and a strophoid loop; 1.b. the conical sheets $\varphi = \varphi_{0j} = C_1$ (another smart intrinsic coordinate, orthogonal to the conical sheets $\chi = \chi_{0i} = c(S_{0i} - c_0)$), having as remarkable directrices: a Pascal's limaçon and the circle at infinity (both for $C_1 = 0$ and centered on the **oz** axis).

The function $F(x_1) = \ln\varphi^{1/2} + i\tan^{(-1)}\chi$, $\chi \in (-\infty, \infty)$, represents the plane x_1 on a strip in a plane X $(= \bar{Y} + i\bar{Z})$. Inside this strip, directed on the $\bar{Y}$ axis, a parallel and uniform field is obtained. This function is the complex potential of an incompressible subsonic plane flow in the cross section current plane x_1. Its real and imaginary parts are the velocity potential function $\bar{Y}(\bar{y},\bar{z})$, and the stream function $\bar{Z}(\bar{y},\bar{z})$, respectively. At any point in the yoz plane they must satisfy the orthogonality condition: $\nabla\bar{Y}(\bar{y},\bar{z}) \cdot \nabla\bar{Z}(\bar{y},\bar{z}) = (\partial\bar{Y}/\partial\bar{y})(\partial\bar{Z}/\partial\bar{y}) + (\partial\bar{Y}/\partial\bar{z})(\partial\bar{Z}/\partial\bar{z}) = 0$, expressing that in the plane flow above the streamlines and the equipotential lines are each other orthogonal. All the curves in figure 1 are the traces (directrices) of the conical surfaces of coordinates defining together with the spheres $R = (x^2 + y^2 + z^2)^{1/2} = R_{0k} = \text{const.}$ a usable system of generalized spherical coordinates. In the upper region of figure 1.a. one can see the trace of Ferri's ([32], [33]) half-straight line – a nodal singularity for the specific entropy S. Actually there are two such nodal singularities (for which the Jacobian $J = D(x, y, z)/D(R, \varphi, \chi) = 0$ and the mapping ceases to be conformal), both of the logarithmic type: one (strongest, on cone's leeward side, $\bar{z}_b = \bar{r} - \bar{t}$), for both external and internal flow, and another one (on cone's axis, $\bar{z}_a = -\bar{t}$) for the internal flow only. These flows are given by two semi-infinite line sources along cone's axis (a) and back (b) (or by three semi-infinite line sources with the sum of their intensities equal to zero, the third one being situated along the line at infinity), replacing solid cone's effect. (One can see that for $\bar{r} = 0$ this effect practically disappears, and a single semi-infinite line source remains on the (a) line, giving for the conical isentropic surfaces the well-known pattern: half-planes passing through this line.) The strength of the central source is half of the upper's one strength, and they have opposite sign (see the black curved arrowheads). Let us calculate the derivative of the complex function $F(x_1)$ with respect to x_1 (the complex velocity):

$$X = \ln\frac{[x_1 - i(\bar{r}-\bar{t})]^2}{x_1 + i\bar{t}};\ \ \frac{dX}{dx_1} = \frac{x_1 + i(\bar{r}+\bar{t})}{(x_1 + i\bar{t})[x_1 - i(\bar{r}-\bar{t})]},$$

tending to ∞ on (a) and (b) (strophoid's nodal point also): $x_{1a} = -i\bar{t}\ (\bar{y}_a = 0; \bar{z}_a = -\bar{t}); x_{1b} = i(\bar{r}-\bar{t})\ (\bar{y}_b = 0; \bar{z}_b = \bar{r}-\bar{t})$. It becomes zero on cone's windward side (limaçon's nodal point (n) also): $x_{1n} = -i(\bar{r}+\bar{t})$ $(\bar{y}_n = 0 ; \bar{z}_n = -(\bar{r}+\bar{t}))$. Along all these straight lines (a, b, n) the used mapping is not conformal (also having $J = 0$). But on the (n) line there is a false S singularity (even if $J = 0$) of a saddle point type (like a plane flow stagnation point), S_n having the same value for y and z directions of both flows. Though, actually, along **oz** axis (two half-planes), due to the different intensities of the inclined conical shock wave started from cone tip, resulting in different compressibility and vorticity effects downstream of this wave, the function S has two constant values, inside the intervals $(-\infty, a), (a, \infty)$, (a) becoming thus a discontinuity line (jump – a small conical shock wave, inclined and with variable intensity too, in the internal flow around the (a) line source) for the specific entropy S. In fact, sweeping the space around (b) in both external and internal flow, the specific entropy rises *continuously* from a minimum, along the interval (a, sw) of **oz** axis, then passing through the strophoid, finally reaching a maximum, along the circle and in interval (sw_1, a) of negative **oz** axis, so that $S = S_{0i}(i)$ is a strictly increasing function for being accepted as a coordinate χ; (sw), (sw_1) ≡ (**oz**) ∩ (shock wave). A pair of traces of isentropic sheets (red lines) and their symmetric ones with respect to **oz** axis (blue lines) was represented in figure 1.a. right. Along them the entropy S and implicitly the coordinate χ are constant. The physical significance of the constant c_0 (> 0) is a reference value for S, corresponding to, say, the isentropic sheet having as directrix the strophoid, this leading to the annulment of the coordinate χ. One can see the contradiction between the physical model and the mathematical approximate one, which can not consider the conical shock wave and its effects, so remaining a qualitative one (isentropic sheets pattern only, analogous to that of the streamlines in an incompressible potential flow). As a main inconsistency we mention the impossibility to model any entropy variation. So, in the analyzed flow, χ tends to $\pm\infty$ along the entire **oz** axis (and the circle of radius $\bar{r}$ as well), getting $\bar{Z} = \pm \pi/2$ – borders of the infinite strip in the complex X plane (except for (a) and (b), where it has undetermined values, these points being represented in the X plane by strip's left and right ends), while, physically, the quantity $c(S_{0i} - c_0)$ has various finite values (< 0, 0, or > 0). Along the strophoid $\bar{y}^2 + (\bar{z}+\bar{t}-\bar{r})^2(\bar{z}+\bar{t})/(\bar{z}+\bar{t}-2\bar{r}) = 0$ (passing through (a), and with (b) as nodal point) $\chi = 0$ and so we get $\bar{Z} = 0$ – the $\bar{Y}$ axis of the complex X plane. Above strophoid's branches and inside its loop $\chi < 0$; $\bar{Z} < 0$ ($\bar{Z} \in (-\pi/2, 0)$ – the inferior half-strip in the X plane). The parallel and uniform flow inside the infinite strip ($\pm\pi/2$) represents both external and internal flow (superposed). So, the strip's ends ($\pm\infty$) correspond to (a) and (b) in the internal flow, and to ($x_1 = -i\infty$) and (b) in the external one. The function $F(x_1)$ describes a special incompressible plane flow around any loop formed by a pair of symmetric curves in figure 1. a., taken as directrix of a solid cone at incidence. The inverse of the complex function $F(x_1)$ is multiform:

$$x_1(X) = \frac{1}{2}\left[e^X/2i\bar{r} + 1 \pm \sqrt{\left(e^X/2i\bar{r}+1\right)^2 - 1}\right] - i\bar{t}, \text{ and}$$

with : $e^X/2i\bar{r} + 1 = \cosh u; \sqrt{\left(e^X/2i\bar{r}+1\right)^2 - 1} = \sinh u$, (u – a complex variable too) one gets the compact form:

$$x_1(u) = e^{\pm u}/2 - i\bar{t}; \quad x_1(X) = e^{\pm \operatorname{arcosh}\left(e^X/2i\bar{r}+1\right)}/2 - i\bar{t}.$$

For the no-incidence motion around the circular cone the complex potential is given by an infinite number of equal positive sources uniformly distributed on the circle of radius $\bar{r}$ and a central negative source with strength of half the sum of the positive ones, describing both external and internal flow:

$$F_0(x_1) = \lim_{n\to\infty}\ln\left[\prod_{k=0}^{n-1}\left(x_1 - \bar{r}e^{ik\frac{2\pi}{n}}\right)\Big/\frac{nx_1}{2}\right] = \ln\lim_{n\to\infty}\frac{2\left(x_1^n - \bar{r}^n\right)}{nx_1}$$

having as streamlines the radii passing through the centre (a), but traversing in both senses the circle of radius $\bar{r}$, and this circle as remarkable streamline, so an infinity of saddle points. This circle is an equipotential line also (like the other concentric circles); the radii are equipotential lines too (only their small regions in the close vicinity of the circle streamline). The complex velocity has the analytic expression below:

$$\frac{dF_0(x_1)}{dx_1} = \lim_{n\to\infty}\left(\frac{nx_1^{n-1}}{x_1^n - \bar{r}^n} - \frac{1}{x_1}\right) = \lim_{n\to\infty}\frac{(n-1)x_1^n + \bar{r}^n}{x_1\left(x_1^n - \bar{r}^n\right)},$$

tending to ∞ on the (a) line (this time $x_{1a} = 0; \bar{y}_a = \bar{z}_a = 0$) and along all uniformly distributed line sources $x_1^n = \bar{r}^n$ (an infinity of singularities on the circular cone $|x_1| = \bar{r}$, for which the Jacobian $J = D(x, y, z)/D(R, \varphi, \chi) = 0$). It becomes zero for $x_1^n = -\bar{r}^n/(n-1)$; ($n \to \infty$) representing the stagnation lines (the "saddle point" type line singularities), also uniformly distributed on the circular cone surface. Like in the previous case (for the motion with incidence), at all these points the used mapping ceases to be conformal, this domain including therefore the entire circle of radius $\bar{r}$, the velocity having alternate values (either infinite or zero). Since $\bar{r} = r/x < 1$, we always and throughout have $\lim_{n\to\infty}\bar{r}^n = 1$.

So $F_0(x_1) = \ln\lim_{n\to\infty}\frac{2\left(x_1^n - 1\right)}{nx_1}$ and $\frac{dF_0(x_1)}{dx_1} = \lim_{n\to\infty}\frac{(n-1)x_1^n + 1}{x_1\left(x_1^n - 1\right)}$.

In plane polar coordinates the equipotential lines equation is $(2/n)\cdot\{|x_1|^{n-1}\cos[(n-1)\arg x_1] - (1/|x_1|)\cdot\cos(\arg x_1)\} = C_1$; ($n \to \infty$) and the one of the streamlines, respectively: $(2/n)\cdot\{|x_1|^{n-1}\sin[(n-1)\arg x_1] + (1/|x_1|)\cdot\sin(\arg x_1)\} = C_2$. Taking into account that $\lim_{n\to\infty} 2/(n|x_1|) = 0$ (except for $x_1 = 0$) it remains: $(2/n)\cdot|x_1|^{n-1}\cos[(n-1)\arg x_1] = C_1$, and: $(2/n)\cdot|x_1|^{n-1}\sin[(n-1)\arg x_1] = C_2$ (for $n \to \infty$), both lines' equations being of the type: $f_1(|x_1|)\cdot f_2(\arg x_1) = C_i$. Generally ($1/n \neq 0$) both lines are special multifolia (roses). But due to the ultra-high angular density of singularities along the circle of radius $\bar{r}$, all lines above (both streamlines and equipotential lines) have a stellar pattern (petals like some high density "Dirac δ impulses", uniformly distributed on the various concentric circles of radius $|x_1|$ and centered in (a), in both senses, centrifugal – for $|x_1| > 1$, and centripetal – for $|x_1| < 1$, as well). The proper half-straight lines were obtained for the undetermined values of the products $(|x_1|^{n-1}/n)\cdot\cos[(n-1)\arg x_1]$ and $(|x_1|^{n-1}/n)\cdot\sin[(n-1)\arg x_1]$, of the type ($\infty\cdot 0$), given by $f_1 = (|x_1|^{n-1}/n) \to \infty$, due to n, and $f_2 = 0$: $\arg x_1 = k\pi/[2(n-1)]$, for $k = \overline{0, (n-1)}$, due to x_1. Since along a regular line $\arg x_1 = $ const. the factor $f_2(\arg x_1)$ (either cosine or sine) in both products $f_1(|x_1|)\cdot f_2(\arg x_1) = C_i$ (hyperbolic dependence) has various constant values const._2, the other factor must be $f_1(|x_1|) = \text{const.}_1 = C_i/\text{const.}_2$, leading to $|x_1| = \text{const.}$ (concentric circles) for both lines' equations. Searching for the general expression of the complex potential function F_g, one can write it as a symbolic sum of self-excluding source contributions (either F_0 or F):

$$F_g \equiv (F_0)\cup(F) \equiv \left[\ln\lim_{n\to\infty}\frac{2\left(x_1^n-1\right)}{nx_1}\right]\cup\left\{\ln\frac{\left[x_1 - i(\bar{r}-\bar{t})\right]^2}{x_1 + i\bar{t}}\right\}.$$

We try further to improve this intuitive qualitative model, in order to get *a more realistic streamlines pattern*, delimiting the domain affected by the rotational supersonic flow around the solid prolate circular cone at small incidence, by evidencing a conical attached shock wave, extending the analogy to the supersonic regime. Inside this wave, in its proximity, the conical isentropic sheets (the smart intrinsic coordinate surfaces) must be normal to the wave surface (see figs. 2 in [33] and 48 in [34], which we reproduce here as fig. 2), the wave trace playing the role of an equipotential line (another analogy aspect). At any point in the new yoz plane the same orthogonality condition must be satisfied: $\nabla\bar{Y}_1(\bar{y},\bar{z})\cdot\nabla\bar{Z}_1(\bar{y},\bar{z}) = (\partial\bar{Y}_1/\partial\bar{y})(\partial\bar{Z}_1/\partial\bar{y}) + (\partial\bar{Y}_1/\partial\bar{z})(\partial\bar{Z}_1/\partial\bar{z})$ $= 0$, with $F_1(x_1) = X_1 = \bar{Y}_1(\bar{y},\bar{z}) + i\bar{Z}_1(\bar{y},\bar{z})$ – a new complex potential function, this time for the supersonic plane flow.

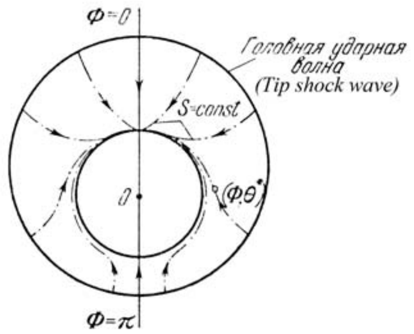


Рис. 48.

Fig. 2. Reproduction of fig. 48 from chapter 2 § 8 in [34] (in the brackets is given the English translation of the Russian text)

In this reason we introduce a third line source (c) – *mirror image* of the (b) one with respect to the conical shock wave surface. We assume (like in [33], [34]) the wave sheet due to the prolate circular cone at small incidence in a moderate supersonic stream to be a circular cone with two unknowns: the axis position and the base radius $\bar{R}$ (in the cross section current plane). The lesser the cone tip half-angle, the incidence, and the emergent supersonic stream Mach number M > 1 (however keeping an attached conical shock wave) are, the better the above approximation is. The (c) line source has the half-strength of (and opposite sign to) the (b) one strength (and sign). So, unlike in the subsonic motion, in the supersonic one the sum of sources strengths (mass flow rates) is quite zero (two pairs of equal sources in equilibrium, two of them coinciding on the (b) line, affecting the entire space and giving in the yoz plane two circles: a streamline – the solid cone trace, and an equipotential line – the wave trace). We will develop this model in a subsequent paper, dealing with determining the position of the new (c) line source and the wave quasi-circular trace base radius $\bar{R}$. A similar method of "mirror image" line sources coupled with the "isentropic sheets ↔ streamlines" analogy may be applied to give a realistic streamlines pattern for the conical rotational supersonic flow around a solid prolate elliptic cone at small incidence, assuming to produce a circular conical shock wave too (see fig. 42 in [34], which we reproduce here as fig. 3, the interrupted lines representing the Mach cone).

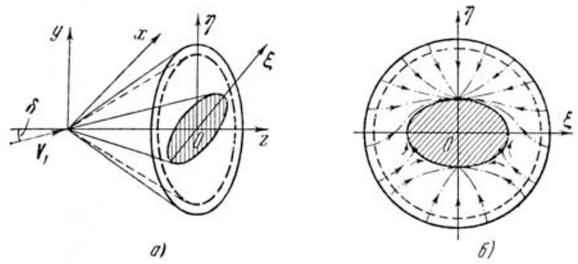


Рис. 42.

Fig. 3. Reproduction of fig. 42 from chapter 2 § 4 in [34]

In the cross section plane ξοη of this cone at small incidence two nodal (Ferri's) singularities for the specific entropy S appear in the external flow (and two saddle points also), meaning an external flow due to a pair of unequal sources of the same sign on cone's windward and leeward sides. Even if the analyzed analogous flows are potential or quasi-potential, the nature of the governing (quasi-)potential is quite different: a Laplace one for the special 2-D plane flow, and a Steichen quasi-potential for the analogous 3-D conical flow. Even for a given incidence, the streamlines pattern for a 2-D plane subsonic flow (in the cross section) of a compressible fluid depends *strongly* on the incident M_z Mach number. This problem was studied for flows without circulation (e. g.: [35], [36]), and for flows with circulation (e.g.: [37]) as well. But the qualitative geometrical analogy was applied in this example only for modeling as good as possible the pattern of the isentropic surfaces, in order to find a smart intrinsic triorthogonal curvilinear coordinate system, for using it as a starting point in a subsequent quantitative study applying the new 2-D velocity quasi-potential theory on the above surfaces, for obtaining the searched for physical quantities distribution, characterizing this rotational conical flow.

11 The model for unsteady non-isentropic flows; the new D. Bernoulli–Lagrange first integral; the PDE of the isentropic surfaces in the Cartesian coordinate system

For an unsteady flow, the term $\partial\mathbf{V}/\partial t$ in the left-hand side of the motion equation in section 1 must be considered. Taking into account that on a certain "i" isentropic $(\mathbf{V}, \mathbf{\Omega})$ surface, the velocity vector can be written as $\mathbf{V} = \nabla\Phi_i$ (where, now, the velocity "quasi-potential" Φ_i is a scalar function depending not only on ξ, η, or λ, μ, but on t also).

After a scalar multiplication of this equation by a virtual elementary d**R** vector in the (**V**, **Ω**)$_i$ tangent plane, integrating the term $\partial(\nabla\Phi_i)/\partial t\cdot d\mathbf{R}$, one gets the term $\partial\Phi_i/\partial t$ (due to Schwarz' theorem on mixed derivatives of the 2nd order):

$$\frac{\partial(\nabla\Phi_i)}{\partial t}d\mathbf{R}=\frac{\partial}{\partial t}\left(\mathbf{k}_\xi\frac{1}{h_\xi}\frac{\partial\Phi_i}{\partial\xi}+\mathbf{k}_\eta\frac{1}{h_\eta}\frac{\partial\Phi_i}{\partial\eta}\right)(\mathbf{k}_\xi h_\xi d\xi+\mathbf{k}_\eta h_\eta d\eta)$$
$$=\left(\mathbf{k}_\xi\frac{1}{h_\xi}\frac{\partial^2\Phi}{\partial t\partial\xi}+\mathbf{k}_\eta\frac{1}{h_\eta}\frac{\partial^2\Phi}{\partial t\partial\eta}\right)\cdot(\mathbf{k}_\xi h_\xi d\xi+\mathbf{k}_\eta h_\eta d\eta)$$
$$=\frac{\partial^2\Phi}{\partial\xi\partial t}d\xi+\frac{\partial^2\Phi}{\partial\eta\partial t}d\eta=\frac{\partial}{\partial\xi}\left(\frac{\partial\Phi}{\partial t}\right)\cdot d\xi+\frac{\partial}{\partial\eta}\left(\frac{\partial\Phi}{\partial t}\right)\cdot d\eta$$
$$=\left(\frac{\partial}{\partial\xi}d\xi+\frac{\partial}{\partial\eta}d\eta\right)\frac{\partial\Phi}{\partial t}=d\left(\frac{\partial\Phi}{\partial t}\right)\quad ,$$

so the first integral for an *unsteady* ($\partial\mathbf{V}/\partial t\neq 0$) and *non-isentropic* ($\mathbf{\Omega}\neq 0$) motion is slightly different with respect to that for a steady rotational motion, having the form:
$\partial\Phi/\partial t + V^2/2 + \gamma K_i/(\gamma-1)\rho^{\gamma-1} = C_i(t)$,
(K_i is the isentropic constant, $C_i(t)$ is an arbitrary function depending on the time t only; both are different from one "i" isentropic surface to another) similar to D. Bernoulli–Lagrange integral in compressible unsteady isentropic (irrotational) aero-gasdynamics. In this case, the isentropic surfaces cease to be rigid, now becoming time dependent (deformable in time – see [4], [5], [7]). Their PDE in the Cartesian system can be derived starting from Crocco–Vászonyi equation (with $\nabla i_0(t) = 0$ – an isoenergetic flow):
$T\nabla S = \partial\mathbf{V}/\partial t + \mathbf{\Omega}\times\mathbf{V}$
(see subsections 1.1 and 1.6 in [7]) and equating its right-hand side to zero. Performing a scalar multiplication by a virtual elementary displacement vector d**R** describing the new isentropic (**V**, **Ω**) deformable surface, one gets:
(TdS =) $\partial\mathbf{V}/\partial t\cdot d\mathbf{R} + (\mathbf{\Omega}\times\mathbf{V})\cdot d\mathbf{R} = 0$; $(\mathbf{\Omega}\times\mathbf{V})\cdot d\mathbf{R}\neq 0$, or in expanded form, taking into account that TdS is not a total (an exact) differential (1/T being an integrant factor):

$$\frac{1}{T}\left(\frac{\partial V_x}{\partial t}dx+\frac{\partial V_y}{\partial t}dy+\frac{\partial V_z}{\partial t}dz\right.$$
$$\left.+\begin{vmatrix} dx & dy & dz\\ \frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z} & \frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x} & \frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\\ V_x & V_y & V_z\end{vmatrix}\right)=0, \quad (11)$$

where the expression of the local instantaneous static temperature T of the fluid particle is given by D. Bernoulli –Lagrange integral for the isoenergetic unsteady flows:

$$T=\frac{\gamma-1}{2\gamma\mathcal{R}}\left[W^2(t)-V^2-2\frac{\partial\Phi}{\partial t}\right]$$
$$=\frac{\gamma-1}{2\gamma\mathcal{R}}\left[W^2(t)-V_x^2-V_y^2-V_z^2-2\frac{\partial\Phi}{\partial t}\right]\quad .$$

For a general flow ($\nabla i_0\neq 0$), Eq. (11) is valid along the "ij" space lines of intersection: $(\mathbf{V},\mathbf{\Omega})_i\cap(i_0=\text{const.})_j$. The new form of the searched for PDE of the isentropic surfaces for an unsteady rotational flow becomes thus:

$$\frac{2\gamma\mathcal{R}}{\gamma-1}\frac{1}{2C_i(t)-V_x^2-V_y^2-V_z^2-2\partial\Phi_i/\partial t}\times$$
$$\times\left\{\left[\frac{\partial V_x}{\partial t}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)V_z-\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)V_y\right]dx\right.$$
$$+\left[\frac{\partial V_y}{\partial t}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)V_z-\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)V_y\right]dy$$
$$\left.+\left[\frac{\partial V_z}{\partial t}+\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)V_y-\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)V_x\right]dz\right\}=0\,.$$

It leads to a particular first integral of Crocco–Vászonyi equation, here having $V_x=\partial\Phi_i/\partial x$; $V_y=\partial\Phi_i/\partial y$; $V_z=\partial\Phi_i/\partial z$.

12 Extension of the new model to MHD

The new intrinsic model of flow can be extended to the ideal (inviscid, both fluid and magnetic) and viscous MHD (see [38], [39], resp.) of a neutral plasma or a conducting liquid, for both steady and unsteady flow. Analogously to the before presented cases (see sections 1 and 11), in the magneto-plasma dynamics (by plasma we understand a mixture of neutral and excited atoms, ions, electrons and photons), the general form of the vector differential equation of motion (Euler) for (not as usual) *an adiabatic* but *non-isentropic flow of a barotropic inviscid electro-conducting fluid* in an external electromagnetic field, *considering the flow vorticity*, is (see [40] – [46], for the right-hand side only):

$$\frac{\partial\mathbf{V}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{V}^2\right)+\mathbf{\Omega}\times\mathbf{V}=\mathbf{f}-\frac{\nabla p}{\rho}+\mathbf{f}_{eL};\ \ \mathbf{f}=-\nabla(gz);$$
$$\frac{\partial\mathbf{V}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{V}^2\right)+\mathbf{\Omega}\times\mathbf{V}=\mathbf{f}-\frac{\nabla p}{\rho}+\frac{\rho_e}{\rho}\mathbf{E}+\frac{1}{c\rho}\mathbf{j}\times\mathbf{H};$$

where $\mathbf{f}_{eL}$ is the density of Lorentz electromagnetic force; for gases **f** = 0 (we neglect, even if it is conservative). For the left-hand side one usually writes $\partial\mathbf{V}/\partial t + (\mathbf{V}\nabla)\mathbf{V}$, (except for [45] and [46] – unexploited), with:
$\mathbf{V} = (\rho_a\mathbf{V}_a + \rho_+\mathbf{V}_+ + \rho_-\mathbf{V}_-)/\rho$, where: $\mathbf{V}_a$, $\mathbf{V}_+$, $\mathbf{V}_-$ are the velocities of the components; ρ_a, ρ_+, ρ_- are the densities of the components: $\rho_a = n_a m_a$; $\rho_+ = n_+ m_+$; $\rho_- = n_- m_-$;
$\rho = n_a m_a + n_+ m_+ + n_- m_- = \rho_a + \rho_+ + \rho_-$ – plasma density (analogously to the case of a mixture of components);
m_a is the mass of a neutral atom ($m_a = m_+ + m_-$);
m_+ is the mass of the positive ion (a single species);
m_- is the mass of the negative ion or of electron;
n_a is plasma concentration in neutral atoms, and, respectively (for a three-component neutral plasma):
n_+, n_- are the concentrations in positive and negative particles (a single species of cations and anions; in the case of a quasi-neutral plasma: $n_+\approx n_-$), all according to a simplified model proposed by the author (no collision).
Therefore the flow (mean) vorticity is given by:
$\mathbf{\Omega} = \nabla\times\mathbf{V} = \nabla\times[(\rho_a\mathbf{V}_a + \rho_+\mathbf{V}_+ + \rho_-\mathbf{V}_-)/(\rho_a + \rho_+ + \rho_-)]$;
H is the strength of the local magnetic field, using the same convention of equivalence (see for reference [40] − [43]) to

the magnetic induction **B** (as a rule variable in time), with $\nabla\mathbf{H} = 0$ ($\mathbf{H} = \nabla \times \mathbf{W}$ − a solenoidal field);
j is the density of the conduction electric current (see Maxwell's 2nd equation in [47]):
$\mathbf{j}/k = \nabla\times\mathbf{H} - 1/c\cdot\partial\mathbf{E}/\partial t = \nabla\times\mathbf{H} + 1/c^2\,\partial/\partial t(\mathbf{V}\times\mathbf{H})\,;\ k = c/4\pi$;
E is the intensity of the local induced electric field (also variable in time) in an inertial coordinate system, given by Maxwell's equations: $\nabla\mathbf{E} = 4\pi\rho_e$ (Ampère); $\nabla\times\mathbf{E} = -1/c\cdot\partial\mathbf{H}/\partial t$ (Faraday) – a Poincaré–Steklov problem (for given ρ_e and **H** functions); $\rho_e = e(n_+ - n_-)$ is the density of electric charges in the fluid medium considered; e is the magnitude of the electron charge; c is the light speed in a vacuum. The low-frequency Ampère's law neglects the displacement electric current, the density of the conduction electric current becoming thus: $\mathbf{j} = k\nabla\times\mathbf{H}$.
Searching for *a steady motion solution* we have: $\partial\mathbf{V}/\partial t = 0$, **H** being considered an oscillating field, but reaching a damped state (constant in time: $\partial\mathbf{H}/\partial t = 0$), so (according to $\nabla\times\mathbf{E} = 0$) **E** becomes an irrotational field.
In the non-relativistic theory ($V \ll c$), or *for a neutral* ($\rho_e = 0$) *gas* ($\nabla\mathbf{E} = 0$, **E** being a solenoidal field too, and also being an irrotational one, **E** is therefore a harmonic field), the term $(\rho_e/\rho)\cdot\mathbf{E}$ can be neglected.
For *a steady* but *isentropic motion* (irrotational – $\mathbf{\Omega} = 0$ in an ideal fluid – $\sigma\to\infty$, σ being the electric conductivity of the fluid medium) see [48], pp. 94 – 96; this gives a D. Bernoulli first integral for some very particular cases ($\mathbf{H}\perp\mathbf{V}$; $\mathbf{H}\parallel\mathbf{V}$ – along the **H** vector lines and $\mathbf{H}\cdot\mathbf{V}$ = const.); also see [45], [49] – [57]. The MHD problem's exact (partial) solution for *an incompressible fluid* ($\rho = \rho_0$ = const.) is given, e.g., in [57], p.177: $\mathbf{V} = \pm\,\mathbf{H}/(4\pi\rho)^{1/2}$ (meaning $\mathbf{H}\parallel\mathbf{V}$) – the specific MHD version of the one-parameter class of the solutions of the "freezing-in" (of the lines of force of the magnetic field **H** into a fluid) equation that exists for all hydrodynamic models: $(\mathbf{V}\nabla)\,\mathbf{H} = (\mathbf{H}\nabla)\,\mathbf{V}$ – for the stationary case.
Performing a scalar multiplication of the motion equation by a certain d**R**, there are some first integrability cases:
1. $\mathbf{\Omega}\times\mathbf{V} = \mathbf{j}\times\mathbf{H} = 0$, meaning either $\mathbf{\Omega} = \mathbf{j} = 0$ – an irrotational (potential) fluid field **V**, as well magnetic field **H**; or $\mathbf{\Omega}\parallel\mathbf{V}$ and $\mathbf{j}\parallel\mathbf{H}$ – ($\mathbf{\Omega} = c_1\mathbf{V}$ and $\mathbf{j} = c_2\mathbf{H}$ – helicoidal (screw) fields) – a mixed Beltrami flow;
2. $\mathbf{\Omega}\times\mathbf{V} = (\mathbf{j}\times\mathbf{H})/c\rho$, meaning that vectors **V**, **Ω**, **H** and **j** are coplanar, also being satisfied the sense and modulus conditions for the vector products – a very particular case;
3. d**R** coplanar with both the pairs (**V** and **Ω**) and (**H** and **j**); for all the considered special (but usual) cases, there are some lines (space curves) along which an elementary vector d**R** is coplanar with both the vectors **V** and **Ω** – contained in the plane tangent to the 0-work sheet of the ($\mathbf{\Omega}\times\mathbf{V}$) elementary force ($d\mathbf{R}\in(\mathbf{V},\mathbf{\Omega})$) and the vectors **H** and **j** – contained in the plane tangent to the 0-work sheet of the Lorentz electromagnetic force ($d\mathbf{R}\in(\mathbf{j},\mathbf{H})$), it being directed upon the local intersection straight line of the two tangent planes above, with the particular subcase:
3.1. $\mathbf{\Omega} = 0$ (an irrotational flow), when d**R** must be coplanar with **H** and **j**. The general case 3 leads to Selescu's *magnetohydrodynamic* vector lines, defined by: $d\mathbf{R}\parallel\$$, where $\$\equiv(\mathbf{V},\mathbf{\Omega})\cap(\mathbf{H},\mathbf{j})\equiv(\mathbf{\Omega}\times\mathbf{V})\times(\mathbf{j}\times\mathbf{H})$. The dimension of its modulus is $[|\$|] = (\mathrm{m/s^2})\cdot(\mathrm{A^2/m}) = (\mathrm{A/s})^2$. Along these space lines both **V** and **H** vector fields become potential (gradients): $\mathbf{V} = \nabla\Phi$ and $\mathbf{H} = \nabla\Xi$, for $\mathbf{j} = k\nabla\times\mathbf{H}$.
But *the most general case* for which the product
$[(\mathbf{\Omega}\times\mathbf{V}) + 1/(c\rho)\cdot(\mathbf{H}\times\mathbf{j})]\cdot d\mathbf{R} = \mathbf{N}'\cdot d\mathbf{R}$
(a virtual elementary work) becomes zero is:
4. d**R** contained in a local plane normal to the vector $\mathbf{\Omega}\times\mathbf{V} + 1/(c\rho)\cdot(\mathbf{H}\times\mathbf{j}) = \mathbf{N}_1 - 1/(c\rho)\cdot\mathbf{N}_2 = \mathbf{N}'$, the envelope sheet of the local normal planes above being the most general 0-work sheet for the sum **N′** between the ($\mathbf{\Omega}\times\mathbf{V}$) elementary force density ($\mathbf{N}_1$) and the Lorentz one ($-\mathbf{N}_2/c\rho$), work made with a virtual d**R** elementary displacement, therefore satisfying the condition:
$\mathbf{N}'\cdot d\mathbf{R} = 0$ (defining Selescu's surfaces), where:

$$\mathbf{N}' = \mathbf{k}_x N_{1x} + \mathbf{k}_y N_{1y} + \mathbf{k}_z N_{1z} - \frac{1}{c\rho}(\mathbf{k}_x N_{2x} + \mathbf{k}_y N_{2y} + \mathbf{k}_z N_{2z})$$
$$= \mathbf{k}_x N'_x + \mathbf{k}_y N'_y + \mathbf{k}_z N'_z \quad , \quad \text{with:}$$
$$\begin{cases} N'_x = N_{1x} - N_{2x}/(c\rho) = A_1 - A_2/(c\rho) \ ; \\ N'_y = N_{1y} - N_{2y}/(c\rho) = B_1 - B_2/(c\rho) \ ; \\ N'_z = N_{1z} - N_{2z}/(c\rho) = C_1 - C_2/(c\rho) \ , \quad \text{with:} \end{cases}$$
$$A_1 = \Omega_y V_z - \Omega_z V_y = N_{1x}\,;\ B_1 = \Omega_z V_x - \Omega_x V_z = N_{1y}\,;$$
$$C_1 = \Omega_x V_y - \Omega_y V_x = N_{1z}\,;$$
$$A_2 = j_y H_z - j_z H_y = N_{2x}\,;\ B_2 = j_z H_x - j_x H_z = N_{2y}\,;$$
$$C_2 = j_x H_y - j_y H_x = N_{2z}\,,\quad \text{generally} \neq 0).$$

The ODEs of the intersection curves of the 0-work sheets of ($\mathbf{\Omega}\times\mathbf{V}$) force density with the 0-work sheets of Lorentz force density, lines on which lies the searched for $d\mathbf{R}\parallel\$$ (Selescu's MHD vector) are therefore in the Cartesian and resp. in the intrinsic coordinate system:

$$\frac{dx}{\$_x} = \frac{dy}{\$_y} = \frac{dz}{\$_z} \ , \ \text{with:}\ \$_x = A_3\,;\ \$_y = B_3\,;\ \$_z = C_3\,,$$
$$\text{and}\ \frac{h_\xi d\xi}{\$_\xi} = \frac{h_\eta d\eta}{\$_\eta}\ \text{or}\ \frac{d\lambda}{\$_\xi} = \frac{d\mu}{\$_\eta}\ (\text{on a}\ \zeta = \zeta_{0i}\ \text{sheet});$$
$$A_3 = B_1 C_2 - B_2 C_1 = \$_x\,;\quad B_3 = C_1 A_2 - C_2 A_1 = \$_y\,;$$
$$C_3 = A_1 B_2 - A_2 B_1 = \$_z \quad (\text{all} \neq 0) \quad ,$$

("Selescu's curves", along which the motion equation for an adiabatic but non-isentropic *steady* flow of a barotropic inviscid electroconducting fluid in an external magnetic field admits a first integral), these rigid lines being very similar (regarding their properties) to D. Bernoulli's – (**V**, **Ω**) rigid surfaces and lying on these ones. In the Cartesian system (with A_1, B_1, C_1, A_2, B_2, C_2 and ρ – given functions of x, y, z), one obtains the ODE of Selescu's surfaces in the form: $[A_1 - A_2/(c\rho)]dx + [B_1 - B_2/(c\rho)]dy + [C_1 - C_2/(c\rho)]dz = 0$.
Over these surfaces the *unsteady* motion equation becomes: $\partial\mathbf{V}/\partial t\cdot d\mathbf{R} + d(\mathbf{V}^2/2) = -\,d(gz) - dp/\rho$. Over the (**V**, **Ω**) surfaces: $\mathbf{V} = \nabla\Phi$. Along their intersection lines one gets a D. Bernoulli–Lagrange integral (like that in section 11).

13 Conclusions and remark

The original contribution of this work to the state of the art consists in finding new first integrability cases for:

1. *Crocco's equation*, obtaining the first integrals: $S = S_{0i}$ (isentropic – section 8); $S = (n-\gamma)/(n-1)\cdot C_v \ln T|_i$ (polytropic – section 6), and $S = \int[f(T)/T]\cdot dT = f_1(T)$ (more general); for the most general one see the remark.

2. *the motion equation,* for a rotational steady flow of an inviscid compressible fluid (with $\mathbf{f} = 0$), introducing a large family of *special* integral surfaces, and not *always* eliminating the rotational non-conservative term: $(\mathbf{\Omega} \times \mathbf{V})\cdot d\mathbf{R} = TdS$. $T\nabla S$ being a biscalar vector field, the term $TdS = T\nabla S\cdot d\mathbf{R}$ over these surfaces *is always conservative* – a total (an exact) differential dF , but finding its first integrals from the geometrical point of view is more difficult than it seems at the first sight. Unlike the already known "term-by-term" integrability case of the motion equation, obtained by introducing the *isentropic* surfaces – a very particular case ($n = \gamma$) of the *polytropic* ones (having $TdS = (n-\gamma)/(n-1)\cdot C_v dT$ – the most general case of *special* integral surfaces), the integrability cases for this term and for the motion equation (on the *special* integral surfaces) show a "quasi-barotropic" fluid behavior. The integration is performed by two leaps: first, using Crocco's equation, the non-conservative vector $\mathbf{\Omega} \times \mathbf{V}$ becomes biscalar, $T\nabla S$, and further, over the *polytropic* surfaces $p/\rho^n = (p/\rho^n)_i = K_{1i}$, it becomes conservative. Thus *the conservativity is not an immovable quality* of a vector. Any *special* integral surface can be a multi-sheet surface, cutting the *isentropic* ones. The same integrability problem occurs with the non-conservative term dp/ρ (originating from the biscalar vector $\nabla p/\rho$) – *always conservative* also, but evidenced as a conservative one over the same *special* integral surfaces only. Finally, we can classify these surfaces, from the viewpoint of TdS term structure (increasing its generality degree) as follows:

a. $TdS = 0$, meaning $dS = 0$, with two interesting interpretations: a1. physical (fluido-thermodynamic); a2. mathematical;

a1. $S = S_{0i}$ = const.(i) – *isentropic* surfaces ($d\mathbf{R} \perp \nabla S$ or $\nabla S\cdot d\mathbf{R} = 0$ – the most productive ones regarding the new model of flow, covering the cases of unsteady flow and of a viscous Newtonian compressible fluid flow also). For the inviscid fluid (Euler) they contain the streamline and the vortex line passing through the flow point considered; finding the isentropic surfaces is a very difficult procedure;

a2. $dS/d\zeta = 0$ – *extrema* ($S = S_{max}$; $S = S_{min}$) and *horizontal inflexion* surfaces, meaning those *isentropic* surfaces over which the continuous function S reaches its extreme values (both relative and absolute *maxima* and *minima* with respect to the strictly increasing variable ζ) on the whole flow field. These values serve as delimiters for the monotony intervals of S (not always an increasing function with respect to ζ), in order to introduce the smart intrinsic triorthogonal curvilinear coordinate system tied to the *isentropic* surfaces, this assuming we must know the function $S = S(\zeta)$;

b. $T = T_{1i}$ = const.(i), and thus: TdS = const.$\cdot dS$ – *isothermal & isotachic* surfaces ($d\mathbf{R} \perp \nabla T$ or $\nabla T\cdot d\mathbf{R} = dT = 0$) ≡ ($d\mathbf{R} \perp \nabla V$ or $\nabla V\cdot d\mathbf{R} = dV = 0$, therefore the term *isotachic* refers to the velocity modulus V only);

c. $TdS = (n-\gamma)/(n-1)\cdot C_v dT$ and so: TdS = const.$\cdot dT$ – general *polytropic* surfaces ($d\mathbf{R} \perp \nabla(p/\rho^n)$ or $\nabla(p/\rho^n)\cdot d\mathbf{R} = 0$), including the *isentropic*, *isothermal & isotachic*, *isobaric* and *isochoric* ones, for various values of n $\in [0, \infty)$. So, for: $n = \gamma$; 1; 0; and $\to \infty$, one obtains the: *isentropic*; *isothermal & isotachic*; *isobaric* ($d\mathbf{R} \perp \nabla p$ or $\nabla p\cdot d\mathbf{R} = 0$); and *isochoric* ($d\mathbf{R} \perp \nabla\rho$ or $\nabla\rho\cdot d\mathbf{R} = 0$) virtual integral surfaces, respectively, as particular cases of the general *polytropic* ones (a "quasi-barotropic" fluid). Any barotropic virtual evolution of an ideal gas may be regarded as being composed of a lot of successive elementary polytropic evolutions. Along the intersection lines of the *isentropic* surfaces with the *isochoric* ones, the fluid has an *isobaric* and *isothermal & isotachic* behavior also (a "quasi-uniform potential" virtual flow of a "quasi-incompressible" fluid, these Laplace lines cutting the streamlines). *A rich nomenclature* was introduced in fluid mechanics and in MHD – the special virtual integral surfaces for the motion equation of an inviscid compressible fluid in steady rotational flow: particular cases of the general polytropic one, and Selescu's *magnetohydrodynamic* vector **$** and its vector lines and Selescu's MHD surfaces.

Remark (a short discussion – mathematically only)**:** $\mathbf{L} = T\nabla S$ (= $\mathbf{\Omega} \times \mathbf{V}$) being a biscalar vector, has the property: $\mathbf{L}\cdot(\nabla \times \mathbf{L}) = 0$, or: $T\nabla S\cdot[\nabla \times (T\nabla S)] = 0$; ($T \neq 0$), or: $(\mathbf{\Omega} \times \mathbf{V})\cdot[\nabla \times (\mathbf{\Omega} \times \mathbf{V})] = 0$, so that $\mathbf{V}$ *must be a solution of vector equation below* (with $\mathbf{\Omega} = \nabla \times \mathbf{V}$): $[(\nabla \times \mathbf{V}) \times \mathbf{V}]\cdot\{\nabla \times [(\nabla \times \mathbf{V}) \times \mathbf{V}]\} = 0$ for both Crocco's and motion equations (the general integrability condition, covering the cases 1 – 3 and G ($\mathbf{\Omega} \times \mathbf{V} = \nabla F$ and thus $\nabla \times (\mathbf{\Omega} \times \mathbf{V}) = 0$) from section 2, the potential field being a particular form of a biscalar one – for Φ_1 = const. in the definition from section 2 (*polytropic* sheets); performing a scalar multiplication of the ($\mathbf{\Omega} \times \mathbf{V} = 0$) condition by a virtual elementary $d\mathbf{R} \in (\mathbf{V}, \mathbf{\Omega})$ plane ($d\mathbf{R} = c_1\mathbf{V} + c_2\mathbf{\Omega}$), one also gets the case 4 (*isentropic* sheets) from section 2 (all these cases from a geometrical point of view). More, one finds the case: $[\nabla \times (\mathbf{\Omega} \times \mathbf{V})] \perp (\mathbf{\Omega} \times \mathbf{V})$. The same analysis may be performed from a thermodynamic point of view: $T\nabla S\cdot[\nabla \times (T\nabla S)] = 0$, noticing that: $\nabla \times (T\nabla S) = \nabla T \times \nabla S$, and getting similar results. Additionally, the biscalar vector $\mathbf{L}_1 = \nabla p/\rho$ in the motion Eq. (1) has an analogous property: $\nabla p/\rho\cdot[\nabla \times (\nabla p/\rho)] = 0$, noticing that: $\nabla \times (\nabla p/\rho) = \nabla(1/\rho) \times \nabla p$, representing the case 6 (*isobaric* surfaces $\nabla p = 0$) from section 2; $\nabla p/\rho = \nabla F_1$ – a potential field (F_1 being a scalar function and ∇F_1 a gradient – a conservative vector field), and thus: $\nabla \times (\nabla p/\rho) = 0$ (a *barotropic* fluid, or flow's *polytropic* sheets of a non-barotropic fluid), representing the case 8 in section 2; and $\nabla \times (\nabla p/\rho) \perp \nabla p$ (a new case).

In order to integrate the motion Eq. (1) both conditions $\mathbf{L}\cdot(\nabla \times \mathbf{L}) = 0$ above must be concurrently satisfied. This can be reached along some intersection space lines (representing a trivial solution), or over the *polytropic* sheets of a non-barotropic fluid. One notices that, for an isoenergetic flow, Eq. (1) can be written as: $T\nabla S + \nabla p/\rho = -\nabla(\mathbf{V}^2/2) = \nabla F_2$ ($F_2 = i$ – the static specific enthalpy) – a potential field. The same result can be obtained applying the curl in Eq. (1) (thus eliminating $\mathbf{V}$): $\nabla \times (T\nabla S + \nabla p/\rho) = 0$, leading to: $T\nabla S + \nabla p/\rho = \nabla F_2 = \nabla i$ (with $i = C_pT$). One gets further: $\nabla \times (T\nabla S) + \nabla \times (\nabla p/\rho) = 0$, or more: $\nabla T \times \nabla S + \nabla(1/\rho) \times \nabla p = 0$, or: $\nabla T \times \nabla S = \nabla p \times \nabla\tau$, meaning the vectors ∇T, ∇S, ∇p and $\nabla\tau$ are coplanar, also being satisfied the sense and modulus conditions for the vector products (both directed along the tangent to the space line of intersection of the (∇T, ∇S) and (∇p, $\nabla\tau$) surfaces, having the same modulus and sense), or: the normals to the *isothermal*, *isentropic*, *isobaric* and *isochoric* sheets passing through any flow's point are coplanar (*a remarkable geometrical property*), except for the cases of *isentropic*, *isothermal* and *general polytropic* sheets, with: $\nabla S = 0$, $\nabla T = 0$ and $T\nabla S = \nabla F$, all leading to: $\nabla T \times \nabla S = 0$; another exception occurs over the *isobaric*, *isochoric*, and *general polytropic* sheets, with: $\nabla p = 0$, $\nabla\tau = 0$ and $\tau\nabla p = \nabla F_1$, all leading to: $\nabla p \times \nabla\tau = 0$.
Unlike the previous biscalarity relations for the vectors $\mathbf{L} = T\nabla S$ ($= \mathbf{\Omega} \times \mathbf{V}$) and $\mathbf{L}_1 = \nabla p/\rho = \tau\nabla p$, (implying the separate irrotationality of these fields: $\nabla \times \mathbf{L} = 0$ and $\nabla \times \mathbf{L}_1 = 0$, as particular cases of: $\mathbf{L}\cdot(\nabla \times \mathbf{L}) = 0$ and $\mathbf{L}_1\cdot(\nabla \times \mathbf{L}_1) = 0$: $\mathbf{L} = \nabla F$ and $\mathbf{L}_1 = \nabla F_1$), *the last relation* (found applying the curl to Eq. (1)) *gives a compulsory irrotationality of the vector sum* (not a particular case): $\nabla \times (\mathbf{L} + \mathbf{L}_1) = 0$, and thus: $\mathbf{L} + \mathbf{L}_1 = \nabla F_2$, *no longer satisfying the integrability condition for Crocco's equation*, so that *both terms of this vector sum are not conservative*, this being the main difference with respect to the "term-by-term" integration for the motion equation given in section 2 (valid over flow's *polytropic* surfaces, while the integration as a whole is valid in the entire fluid mass). But if $\mathbf{L} = \nabla F$ (a potential field, requested by Crocco's equation, as a particular case of integration), one gets: $\mathbf{L}_1 = \nabla F_2 - \mathbf{L} = \nabla(F_2 - F) = \nabla F_1$, also a potential field. Therefore both $\mathbf{L}$ and $\mathbf{L}_1$ are potential vector fields (over the *polytropic* sheets of a non-barotropic fluid – section 2).

The purpose of this paper is not do deal with the most general case of first integrability for the motion equation, but with the cases derived from the integral of Crocco's equation only (as it was specified in the title), taking into account the advantages offered by this special treatment.

Note
This paper (the first in a series dedicated to the intrinsic analytic study of the basic equations in compressible fluid mechanics) is fully original, however having as starting point other ones with almost the same title ([58], [4] – [8]).

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