Closed-form Solutions of the Time-fractional Standard Black-Scholes Model for Option Pricing using He-separation of Variable Approach

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Abstract: - The Black-Scholes option pricing model in classical form remains a benchmark model in Financial Engineering and Mathematics concerning option valuation. Though, it has received a series of modifications as regards its initial constancy assumptions. Most of the resulting modifications are nonlinear or time-fractional, whose exact or analytical solutions are difficult to obtain. This paper, therefore, presents exact (closed-form) solutions to the time-fractional classical Black-Scholes option pricing model by means of the He-Separation of Variable Transformation Method (HSVTM). The HSVTM combines the features of the He's polynomials, the Homo-separation variable, the modified DTM, which increases the efficiency and effectiveness of the proposed method. The proposed method is direct and straight forward. Hence, it is recommended for obtaining solutions to financial models resulting from either Ito or Stratonovich Stochastic Differential Equations (SDEs).

Key-Words: - Option pricing, Black-Scholes model, exact solutions, fractional calculus
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1 Introduction

Almost all aspects of applied mathematics have witnessed the emergence of Fractional Calculus (FC) as an important generalization of the classical calculus [1-4]. The derivative order(s) associated with FC go beyond the set of natural numbers, the point estimate, and so on. Instead, the orders can be defined in real and complex spaces, while an interval estimate is considered [5-8]. This creates rooms for memory settings of the systems. The applications of FC are widely seen in [9-11]. In financial mathematics, Jumaris [12,13] introduced FC to option pricing with its base in the Black-Scholes pricing model (Financial derivatives). The Black-Scholes Model (BSM) for European option pricing and valuation plays a notable role in risk and portfolio management [14-17]. Though, some of the BSM underlying assumptions when relaxed leads to more complex or nonlinear versions. Hence, the need for effective and efficient numerical, semi-approximate methods of solution. In literature, a lot of solution methods have been considered by a good number of researchers. These include: Adomian decomposition method (ADM), variational iteration method (VIM), Modified ADM (MADM), homotopy perturbation method (HPM), Differential transformation method (DTM), projected DTM (PDTM) [18-25]. He's polynomials method was initiated in [26, 27] by Ghorbani *et al.*, where the nonlinear terms were expressed as series of polynomials calculated with the aid of HPM. The He's polynomials are noted to be compatible with the so-called Adomian's polynomials. Though, they are easier to be computed and are user friendly to a greater extent.

Other numerical approaches with more extensive applications to modelling situations in sciences, engineering, finance, and environmental management are considered in terms of analytical or approximate solutions [28-37].

2 The Classical Black-Scholes Model

The solution of the Black-Scholes model is used for describing the value of option mainly of European type [38]. The solution solves the model of the form:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf=0 \qquad (1)$$

with the following as defined: $f=f(S,\tau)$ represents the value of the underlying S, at a particular time, τ such that $\tau\in[0,t],\ f\in C^{2,1}[R\times[0,T]],(S,\tau)\in R^{+}\times(0,T)$ for a payoff function $p_f(S,t)$, and expiration price, E such that:

$$p_f(S,t)=\begin{cases}(E-S)^+, & \text{for European put option,}\\ (S-E)^+, & \text{for European call option.}\end{cases} \tag{2}$$

In (2), $(S_*)^+$ represents the maximum between S_* and 0 in terms of values, for the underlying asset $S=S(t)$, the volatility is σ, r is taken as the risk-free interest rate, meanwhile, the maturity time is T.

In this work, we will look at a generalization of (1) regarding fractional order in terms of real and complex order of the derivatives. This will be regarded as a non-integer (time-fractional) Black-Scholes model (TFBSM) following the form:

$$\frac{\partial^\alpha \Psi}{\partial \tau^\alpha}+m_1(S,\delta)\frac{\partial^2\Psi}{\partial S^2}+m_2(S,r)\frac{\partial \Psi}{\partial S}=r\Psi\ ,\ \alpha>0 \tag{3}$$

subject to an attributed initial or boundary conditions, $m_i(\cdot,\cdot)$, $i=0,1,2,3,\cdots$, are non-zero functions.

Recently, Ghandehari and Ranjbar [39] presented the exact solution of the option pricing model built on the Fractional Black-Scholes (FBS) equation employing a modified Homotopy Perturbation Method (HPM). In their method, they obtained the exact solutions basically with the aid of green function by combining the separation of variables method with HPM [39].

Ouafoudi and Gao [40] introduced two solution methods viz: modified HPM and Homotopy Perturbation combined with Sumudu transform for handling the same option pricing model as considered in [39]. Both views of [39] and [40] required the application of green function. The new approach in this present work aims at providing exact solutions of the time-fractional classical Black-Scholes option pricing model by means of He-Separation of Variable Transformation Method (HSVTM). The HSVTM combines the basic features of the He's polynomials, the Homo-separation variable, and the modified Differential Transform Method without the concept and application of green function. Here, the fractional derivative is defined in the sense of Caputo.

3 Remarks on the He's Polynomial Solution Method

Suppose a general form is considered as follows:

$$\Lambda(\psi)=0 \tag{4}$$

for a differential or an integral operator, Λ and $H(\psi,p)$ denotes a convex homotopy given as:

$$H(\psi,p)=p\Lambda(\psi)+(1-p)\Omega(\psi) \tag{5}$$

where $\Omega(\psi)$ is a known operator (functional) with ψ_0 as a solution. Therefore, we get:

$$\left.\begin{aligned}H(\psi,0)&=\Omega(\psi)\\ H(\psi,1)&=\Lambda(\psi)\end{aligned}\right\} \tag{6}$$

whenever $H(\psi,p)=0$ is satisfied, and the parameter $p\in(0,1]$ is embedded. According to HPM in [26, 27], the parameter, p is used in the expansion of:

$$\psi=\sum_{j=0}^{\infty}p^j\psi_j=\psi_0+p\psi_1+p^2\psi_2+\cdots. \tag{7}$$

From (7) we have the solution as $p\to 1$. Though, the convergence of (7) as $p\to 1$ has already been considered in [24].

The method considers $N(\psi)$ as the nonlinear term given as:

$$N(\psi)=\sum_{j=0}^{\infty}p^jH_j=H_o+p^1H_1+p^2H_2+\cdots \tag{8}$$

where the He's polynomials, $H_k\text{'s}$ can be obtained using:

$$H_k(\psi_0,\psi_1,\cdots,\psi_k)=\frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{k}p^j\psi_j\right)\right)_{p=0},. \tag{9}$$

3.1 The method on FDE

Consider the general form of the time-fractional differential equation (GTFDE) of the form:

$$\begin{cases}D_t^\alpha u(x,t)=Lu(x,t)+Nu(x,t)+f(x,t),\\ u(x,0)=g(x),\end{cases} \tag{10}$$

where D_t^α is the fractional differential operator of order $\alpha\in(0,1]$ in the sense of Caputo, L is a linear operator, N is a nonlinear operator, $f(x,t)$ is a source term and $u(x,t)$ is a supposed function satisfying (3.7).

Suppose $H(\psi,p)$; $\psi(x,t)$ is defined as a convex homotopy such that:

$$H(\psi,p)=p\Lambda(\psi)+(1-p)\Omega(\psi) \tag{11}$$

where,

$$\Lambda(\psi)=D_t^\alpha u(x,t)-Lu(x,t)-Nu(x,t)-f(x,t) \quad (12)$$

and $\Lambda(\psi)$ is a functional operator with ψ_0 as known solution such that:

$$\Omega(\psi)=D_t^\alpha u(x,t)-D_t^\alpha u(x,0). \quad (13)$$

We remarked that:

$$H(\psi,p)=\begin{cases}\Omega(\psi), \text{ for } p=0,\\ \Lambda(\psi), \text{ for } p=1.\end{cases} \quad (14)$$

For $H(\psi,p)\equiv 0$, and using (12) and (13), we have:

$$p\Lambda(\psi)+(1-p)\Omega(\psi)=0$$

$$\Rightarrow \begin{array}{r} p\left(D_t^\alpha u(x,t)-Lu(x,t)-Nu(x,t)-f(x,t)\right) \\ +(1-p)\left(D_t^\alpha u(x,t)-D_t^\alpha u(x,0)\right)=0 \end{array}. \quad (15)$$

Expanding and simplifying (15) give:

$$\begin{aligned} D_t^\alpha u(x,t)&=D_t^\alpha u(x,0)\\ &+p\left(Lu(x,t)+Nu(x,t)+f(x,t)-D_t^\alpha u(x,0)\right) \end{aligned} \quad (16)$$

where $u(x,0)$ is an initial approximation of (10).

In an integral form, (16) is expressed as:

$$u(x,t)=u(x,0)+pJ_t^\alpha\begin{pmatrix}Lu(x,t)+Nu(x,t)\\ +f(x,t)-D_t^\alpha u(x,0)\end{pmatrix} \quad (17)$$

where the nonlinear term is as defined above.

Applying convex homotopy method to (17) with $p\in(0,1]$ as an embedded parameter yields:

$$\begin{aligned}\sum_{n=0}^{\infty}p^n u_n(x,t)&=u(x,0)\\ &+p^{n+1}J_t^\alpha\begin{pmatrix}L\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\\ +N\left(\sum_{n=0}^{\infty}u_n(x,t)\right)+\phi(x)\end{pmatrix}\end{aligned} \quad (18)$$

where $\phi(x)=f(x,t)-D_t^\alpha u(x,0)$.

3.2 Procedures for the Exact Solution

From (18), $p^0: u_0=u(x,0)$. It is obvious that $u_0=u(x,0)$ is the initial approximation (condition) of (10). To reduce the FDE (10) to ODE of the equivalent form, the exact solution is therefore defined as:

$$u_*(x,t)=u(x,0)\lambda_1(t)+u'(x,0)\lambda_2(t). \quad (19)$$

The functions $\lambda_1(t)$ and $\lambda_2(t)$ are to be determined. Thus, $u_*(x,t)$ satisfies (3.7). Hence,

$$D_t^\alpha u_*(x,t)=Lu_*(x,t)+Nu_*(x,t)+f(x,t). \quad (20)$$

That implies that:

$$\begin{aligned}D_t^\alpha\begin{pmatrix}u(x,0)\lambda_1(t)\\ +u'(x,0)\lambda_2(t)\end{pmatrix}&=L\begin{pmatrix}u(x,0)\lambda_1(t)\\ +u'(x,0)\lambda_2(t)\end{pmatrix}\\ &+N\begin{pmatrix}u(x,0)\lambda_1(t)\\ +u'(x,0)\lambda_2(t)\end{pmatrix}+f(x,t).\end{aligned} \quad (21)$$

Since the assumed solution:

$$u(x,t)=\sum_{\substack{i=0\\ p_i\to 1}}^{\infty}u_i p_i$$

satisfies the initial condition, we have:

$$u_*(x,0)=u(x,0)\lambda_1(0)+u'(x,0)\lambda_2(0). \quad (22)$$

Thus,

$$\begin{cases}\lambda_1(0)=1,\\ \lambda_2(0)=0.\end{cases} \quad (23)$$

Though, we have from (16) that:

$$\begin{aligned}D_t^\alpha u_1(x,t)&=D_t^\alpha u_0(x,0)\\ &-\left(Lu_0(x,t)+Nu_0(x,t)+f(x,t)\right)\equiv 0\end{aligned}. \quad (24)$$

Hence, simplifying (21) by putting (22) in (24) gives:

$$\begin{aligned}\left.\begin{cases}u(x,0)D_t^\alpha(\lambda_1(t))\\ +u'(x,0)D_t^\alpha(\lambda_2(t))\end{cases}\right\}&=L\begin{pmatrix}u(x,0)\lambda_1(t)\\ +u'(x,0)\lambda_2(t)\end{pmatrix}\\ &+N\begin{pmatrix}u(x,0)\lambda_1(t)\\ +u'(x,0)\lambda_2(t)\end{pmatrix}+f(x,t).\end{aligned} \quad (25)$$

Equation (25) is thus, the fractional ODE (FODE) resulting from the fractional PDE in (10). Solving (25) may either lead to an IVP (ODE) or a system of ODEs. We will resort to Projected Differential Transform Method (PDTM) [25, 38, 41] for ease of computation as regards (25). Hence, suppose is an analytic function at in a given domain say, D, then, the projected DTM of $\psi(x,t)$ with respect to t at t_* is defined and denoted by:

$$\Xi(x,j)=\frac{1}{j!}\left[\frac{\partial^j\psi(x,t)}{\partial t^j}\right]_{t=t_0} \quad (26)$$

such that:

$$\psi(x,j)=\sum_{j=0}^{\infty}\Xi(x,j)(t-t_0)^j \quad (27)$$

where (27) is known as the projected differential inverse transform of $\Xi(x, j)$ with respect to the time parameter t .

4 Applications

In this section, the following time-fractional Black-Scholes equations are considered.

Problem 4.1: A linear Black-Scholes equation of the following form is considered:

$$\frac{\partial^\alpha w}{\partial t^\alpha}+x^2\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}x\frac{\partial w}{\partial x}-w=0 \tag{28}$$

subject to:

$$w(x,0)=\max\left(x^3,0\right)=\begin{cases}x^3, \text{ for } x>0\\ 0, \text{ for } x\le 0\end{cases}. \tag{29}$$

Procedure w.r.t Problem 4.1:

Choose $w_*(x,t)$ as an initial approximation to (28) such that:

$$w_*(x,t)=\begin{Bmatrix}\max\left(x^3,0\right)\lambda_1(t)\\ +3\max\left(x^2,0\right)\lambda_2(t)\end{Bmatrix}. \tag{30}$$

Hence, (28) becomes:

$$0=\begin{Bmatrix}\frac{\partial^\alpha}{\partial t^\alpha}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ +x^2\frac{\partial^2}{\partial x^2}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ +\frac{1}{2}x\frac{\partial}{\partial x}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ -\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{\partial^\alpha}{\partial t^\alpha}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ +x^2\frac{\partial^2}{\partial x^2}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ +\frac{1}{2}x\frac{\partial}{\partial x}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\\ -\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)\end{Bmatrix}. \tag{31}$$

$\Rightarrow$

$$\begin{cases}0=\left(x^3\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+3x^2\frac{d^\alpha}{dt^\alpha}\lambda_2(t)\right)\\ \quad+\left(6x^3\lambda_1(t)+6x^2\lambda_2(t)\right)\\ \quad+\frac{3x^3}{2}\lambda_1(t)+3x^2\lambda_2(t)-x^3\lambda_1(t)-3x^2\lambda_2(t)\end{cases}$$

$$=\left(x^3\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+3x^2\frac{d^\alpha}{dt^\alpha}\lambda_2(t)\right)+6x^2\lambda_2(t)+\frac{13x^3}{2}\lambda_1(t).$$

Thus,

$$\begin{aligned}0\equiv x^3&\left(\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+\frac{13}{2}\lambda_1(t)\right)\\ &+3x^2\left(\frac{d^\alpha}{dt^\alpha}\lambda_2(t)+2\lambda_2(t)\right)\end{aligned}. \tag{32}$$

We therefore obtain the FODE system:

$$\begin{cases}\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+\frac{13}{2}\lambda_1(t)=0\\ \lambda_1(0)=1\end{cases}, \tag{33}$$

$$\begin{cases}\left(\frac{d^\alpha}{dt^\alpha}\lambda_2(t)+2\lambda_2(t)\right)=0\\ \lambda_2(0)=0\end{cases}. \tag{34}$$

From (34), it is obvious that $\lambda_2(t)=0$. But solving (33) using the transformation properties [38] with $\Xi_1(h)$ as the differential transform of $\lambda_1(t)$ gives:

$$\Xi_1(1+h)=\frac{\Gamma(1+\alpha h)}{\Gamma\left(1+\alpha(1+h)\right)}\left(-6.5\Xi_1(h)\right). \tag{35}$$

Thus,

$$\Xi_1(p)=\frac{(-6.5)^p}{\Gamma(1+p\alpha)},\ p\ge 1. \tag{36}$$

$$\begin{aligned}\Rightarrow\ \lambda_1(t)&=\sum_{p=0}^{\infty}\Xi_1(p)t^{\alpha p}\\ &=\sum_{p=0}^{\infty}\frac{(-6.5)^p}{\Gamma(1+p\alpha)}t^{\alpha p}\\ &=\sum_{p=0}^{\infty}\frac{\left(-6.5t^\alpha\right)^p}{\Gamma(1+p\alpha)}\\ &=E_\alpha\left(-6.5t^\alpha\right).\end{aligned} \tag{37}$$

So, using (34) and (37) in (30) gives:

$$\begin{aligned} w_*(x,t) &= \max\left(x^3,0\right)\sum_{p=0}^{\infty}\frac{\left(-6.5t^{\alpha}\right)^p}{\Gamma(1+p\alpha)} \\ &= \max\left(x^3,0\right)E_{\alpha}\left(-6.5t^{\alpha}\right) \\ &= x^3E_{\alpha}\left(-6.5t^{\alpha}\right) \end{aligned} \tag{38}$$

where $E_{\alpha}(-kt^{\alpha})$ denotes a one parameter Mittag-Leffler function.

Problem 4.2:

A linear Black-Scholes equation model the following form is considered:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \frac{\partial^2 w}{\partial x^2} + (k-1)\frac{\partial w}{\partial x} - kw \tag{39}$$

subject to:

$$w(x,0) = \max\left(e^x - 1, 0\right). \tag{40}$$

Procedure w.r.t Problem 4.2:

Choose $w_*(x,t)$ as an initial approximation to (39) such that:

$$w_*(x,t) = \begin{Bmatrix} \max\left(0, e^x - 1\right)\lambda_1(t) \\ +\max\left(e^x, 0\right)\lambda_2(t) \end{Bmatrix}. \tag{41}$$

Hence, (39) becomes:

$$0 = \begin{Bmatrix} \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\max\left(0,e^x-1\right)\lambda_1(t) + \max\left(e^x,0\right)\lambda_2(t)\right) \\ -\frac{\partial^2}{\partial x^2}\left(\max\left(0,e^x-1\right)\lambda_1(t) + \max\left(e^x,0\right)\lambda_2(t)\right) \\ +(1-k)\frac{\partial w}{\partial x}\begin{pmatrix} \max\left(0,e^x-1\right)\lambda_1(t) \\ +\max\left(e^x,0\right)\lambda_2(t) \end{pmatrix} \\ +k\left(\max\left(0,e^x-1\right)\lambda_1(t) + \max\left(e^x,0\right)\lambda_2(t)\right). \end{Bmatrix} \tag{42}$$

$\Rightarrow$

$$\begin{aligned} 0 = {} & \max\left(0,e^x-1\right)\frac{d^{\alpha}}{\partial t^{\alpha}}\lambda_1(t) + \max\left(e^x,0\right)\frac{d^{\alpha}}{\partial t^{\alpha}}\lambda_2(t) \\ & -k\max\left(e^x,0\right)\lambda_1(t) + k\max\left(0,e^x-1\right)\lambda_1(t). \end{aligned} \tag{43}$$

Thus,

$$0 = \begin{Bmatrix} \max\left(e^x-1,0\right)\left(\frac{d^{\alpha}}{\partial t^{\alpha}}\lambda_1(t) + k\lambda_1(t)\right) \\ +\max\left(e^x,0\right)\left(\frac{d^{\alpha}}{\partial t^{\alpha}}\lambda_2(t) - k\lambda_1(t)\right) \end{Bmatrix}. \tag{44}$$

We therefore obtain the FODE system:

$$\begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}\lambda_1(t) + k\lambda_1(t) = 0, \\ \lambda_1(0) = 1, \end{cases} \tag{45}$$

and

$$\begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}\lambda_2(t) - k\lambda_1(t) = 0, \\ \lambda_2(0) = 0. \end{cases} \tag{46}$$

From (45), the relation:

$$\begin{cases} \Xi_1(1+h) = \frac{\Gamma(1+\alpha h)}{\Gamma\left(1+\alpha(1+h)\right)}\left(-k\Xi_1(h)\right), \\ \Xi_1(0) = 1. \end{cases} \tag{47}$$

$$\Rightarrow \quad \Xi_1(p) = \frac{(-k)^p}{\Gamma(1+p\alpha)},\ p \ge 0. \tag{48}$$

Therefore,

$$\left.\begin{aligned} \lambda_1(t) &= \sum_{p=0}^{\infty}\Xi_1(p)t^{\alpha p} \\ &= \sum_{p=0}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)} = E_{\alpha}\left(-kt^{\alpha}\right) \end{aligned}\right\} \tag{49}$$

Similarly, from (46), the relation:

$$\begin{cases} \Xi_2(1+h) = \frac{\Gamma(1+\alpha h)}{\Gamma\left(1+\alpha(1+h)\right)}\left(k\Xi_1(h)\right), \\ \Xi_1(0) = 1,\ \Xi_2(0) = 0. \end{cases} \tag{50}$$

$$\Rightarrow \quad \Xi_2(p) = \frac{-(-k)^p}{\Gamma(1+p\alpha)},\ p \ge 1. \tag{51}$$

$$\therefore \quad \left.\begin{aligned} \lambda_2(t) &= \sum_{p=0}^{\infty}\Xi_2(p)t^{\alpha p} \\ &= -\sum_{p=1}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)} \end{aligned}\right\} \tag{52}$$

But

$$\sum_{p=0}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)} = 1 + \sum_{p=1}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)}. \tag{53}$$

$$\therefore \quad 1 - \sum_{p=0}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)} = -\sum_{p=1}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)}. \tag{54}$$

So, putting (54) in (52) gives:

$$\left.\begin{aligned} \lambda_2(t) &= 1 - \sum_{p=0}^{\infty}\frac{\left(-kt^{\alpha}\right)^p}{\Gamma(1+p\alpha)} \\ &= 1 - E_{\alpha}\left(-kt^{\alpha}\right). \end{aligned}\right\} \tag{55}$$

Hence,

$$w_*(x,t) = \left\{ \begin{matrix} \max\left(0, e^x - 1\right) E_\alpha\left(-kt^\alpha\right) \\ +\max\left(e^x, 0\right)\left(1 - E_\alpha\left(-kt^\alpha\right)\right) \end{matrix} \right\}. \tag{56}$$

Here, the presentations of the graphical views of the solutions $w = w(x,\ t)$ are given (see figures 1-4). Fig.1 and Fig.2 are for *problem 4.1*, while Fig.3 and Fig.4 are for *problem 4.2*. For each considered case, the same interval is applied for x but different intervals are used for t.

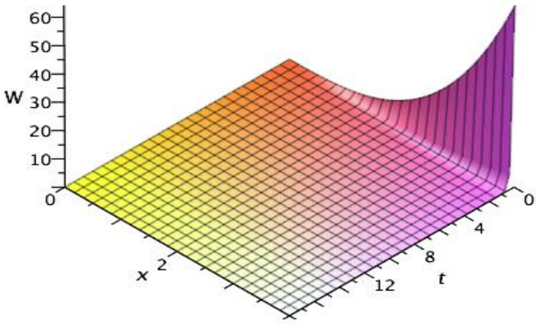

Fig. 1: w: $o \le t \le 18,\ o \le x \le 4$

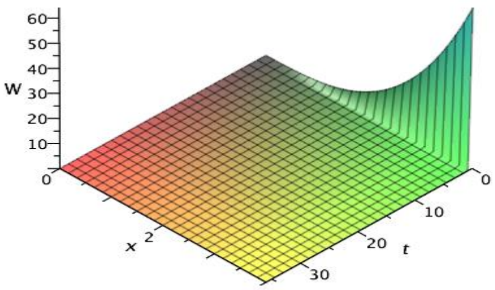

Fig. 2: w: $o \le t \le 36,\ o \le x \le 4$

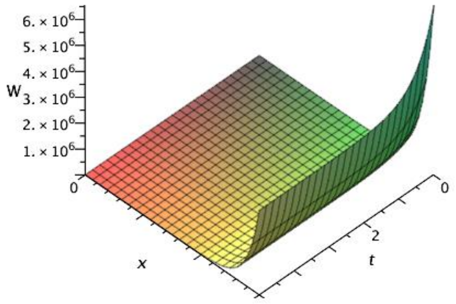

Fig. 3: w: $o \le t \le 5,\ o \le x \le 15$

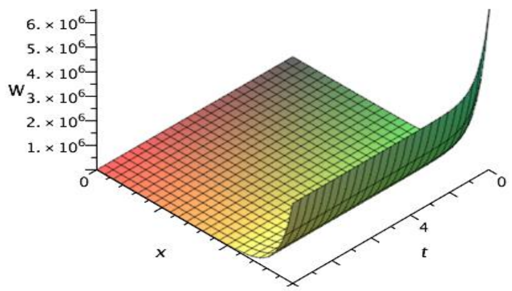

Fig. 4: w: $o \le t \le 10,\ o \le x \le 15$

5 Concluding Remarks

In this paper, a new exact solution method is proposed. The method presented the exact solutions of the time-fractional classical Black-Scholes option pricing model by means of the He-Separation of Variable Transformation Method (HSVTM). This is an extension of the conference paper-approach in [42]. The HSVTM combined the basic properties of the He's polynomials, the Homo-separation variable, and the modified DTM. The engendered fractional derivative is defined in the sense of Caputo, which increases the efficiency and effectiveness of the proposed method. The merits of the HSVTM are numerous. These include direct and straightforwardness in its application. Besides, no knowledge of green function, linearization, or Lagrange multiplier is required. Hence, it is recommended for obtaining solutions of financial models resulting from either Ito or Stratonovich Stochastic Differential Equations (SDEs).

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