# Fuzzy Systems, Extensions and Relative Theories

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*Abstract:* - The article at hands studies critically the development of the theory of Fuzzy Systems, its most common extensions and the main relative theories that have been introduced for managing the situations of uncertainty and vagueness or ambiguity appearing in Science, Technology and in the everyday life. The management of uncertainty in fuzzy systems in terms of the fuzzy probability and possibility is in particular discussed and examples are presented to illustrate the methods used for this purpose. The present work is important, because it offers a basic framework for those wanting to study deeper the above theories and the multi-valued logics connected to them that have found recently many and important applications to almost all sectors of human activity.

*Key-Words:* - Fuzzy set (FS) Logic (FL) and Number (FN), Membership Function (MF), Fuzzy Probability and Possibility, Uncertainty in Fuzzy Systems, Type-2 FS, Interval-valued FS (IVFS), Intuitionistic FS (IFS), Pythagorean FS (PFS), Hesitant FS (HFS), Complex FS (CFS), Neutrosophic Set {NS), Rough Set (RS), Soft Set (SS), Grey System (GS) and Number (GN).

## 1 Introduction

50-60 years ago *probability* theory used to be the unique tool in hands of the experts for dealing with situations of uncertainty appearing in problems of science, technology and of the everyday life. However, things have been changed nowadays with the development of the *Fuzzy Set (FS)* theory that gave to the experts the opportunity to model under conditions which are not precisely defined and therefore they cannot be tackled by the traditional probability methods.

From the time that Zadeh introduced the theory of FS in 1965 [1] many efforts have been made for improving its effectiveness to deal with uncertain, ambiguous and vague situations. As a result a series of extensions and generalizations of the ordinary FSs followed and relevant theories have been proposed as alternatives to the FS theory. The present work offers a basic framework to those wanting to study deeper the above FS extensions, the relative theories and the *multi-valued logics* generated by them that have found recently many and important applications to almost all sectors of human activity.

The rest of the paper is organized as follows: In Section 2 the basic principles of the FS theory and the connected to it *Fuzzy Logic {FL)* are presented. In Section 3 the ways of managing the *uncertainty* in fuzzy systems are described in terms of the *fuzzy probability* and *possibility* and examples are given to illustrate them. The most common extensions of FS theory are discussed in Section 4, while the highlights of the main relative to the FS theories are presented in Section 5. The paper closes with the final conclusion stated in Section 6

## 2 Fuzzy Sets and Logic

The *"Laws of Thought"*, of Aristotle (384-322 BC) [2], that dominated for centuries the human reasoning include:

- The principle of identity
- The law of the excluded middle
- The law of contradiction

The law of the excluded middle (TRUE or FALSE) was the basis for the traditional bi-valued Logic and the precision of the classical mathematics owes undoubtedly a large part of its success to it.

However, there were also strong objections about this law. The Buddha Sidhartha Gautama, who lived in India a century earlier, had already argued that almost every notion contains elements from its opposite one, while Plato (427-377 BC) discussed the existence of a third area beyond "True" and "False", where these two opposite notions can exist together. Modern philosophers like Hegel, Marx, Engels and others adopted and further cultivated Plato's beliefs.

Forms of multi-valued logic have been studied since the 1920s, notably by the Polish philosopher Jan Lukasiewicz [3] and by the Polish-American logician and mathematician Alfred Tarski [4]. However, *Fuzzy Logic (FL)* was introduced much later by Lofti Zadeh with the help of the concept of FS [1]. FL, based on the observation that people make frequently decisions in terms of imprecise and non-numerical information, is an infinite-valued logic in which the truth values of variables may be any real number between 0 (completely true) and 1 (completely false) [5].

It is recalled that a FS on the (crisp) set of the discourse U is defined with the help of its *membership function (MF) m:* $U \rightarrow [0, 1]$ as the set of the ordered pairs A $\{(x, m(x)): x \in U\}$, where $m(x)$ is called the *membership degree* of x in A. For reasons of simplicity many authors identify a FS with its MF. A crisp set A can be considered as a FS with its MF taking only the values 1, if $x \in A$, and 0, if $x \notin A$.

FSs are mathematical models that have the capability of recognizing, representing, manipulating, interpreting, and utilizing data and information which are vague and lack certainty. However, the creditability of a FS in representing a real situation depends on the proper definition of its MF. In fact, the MF is not uniquely determined, its definition depending on the personal criteria of the designer of the corresponding FS.

It must be emphasized that probabilities and membership degrees, despite to the fact that they act on the same real interval [0, 1], they *essentially differ* to each other. For example, the expression "The probability of John to be tall is 85%" has a completely different meaning from the FL statement "John's membership degree in the FS of the tall men is 0.85". In fact, the former expression, in terms of the law of the excluded middle, means that John, being an unknown to the observer person, is either tall or short, but the probability to be tall is 85%. On the contrary, the latter statement, since John's membership degree in the FS of the tall men is near to 1, means that John is a rather tall person. There are also other differences between the two theories mainly arising from the way of defining the corresponding notions and operations. For instance, whereas the sum of the probabilities of all the single events (singleton subsets) of *U* is always 1 (probability of the certain event), this is not necessarily true for the membership degrees. Consequently, a probability distribution could be used to define membership degrees, but the converse does not hold in general.

The process of reasoning with fuzzy rules involves:

- *Fuzzification* of the problem's data by utilizing the suitable MFs to define the required FSs.
- Application of FL operators on the defined FSs and combination of them to obtain the final result in the form of a unique FS.
- *Defuzzification* of the final FS to return to a crisp output value, in order to apply it on the real world situation for resolving the corresponding problem.

Among the more than 30 defuzzification methods in use, the most popular is probably the *Centre of Gravity (COG)* technique. According to it, a problem's fuzzy solution is represented by the coordinates of the COG of the level's section contained between the graph of the MF involved and the OX axis [6].

For general facts on FSs and the connected to them uncertainty we refer to the book [7].

Zadeh introduced also the *Fuzzy Numbers (FNs)* [8] as a special form of FSs on the set of the real numbers such that:

1. There exists x in U, with $m(x)=1$ (*normal* FS)
2. Their *x-cuts* are closed real intervals
3. Their membership function is piece-wise continuous

It is recalled that the x-cut $A^x$ of a FS A, $x \in [0, 1]$, is defined to be the crisp set

$$A^x = \{y \in U: m(y) \geq x\}.$$

Zadeh defined the basic arithmetic operations on FNs in terms of his *extension principle,* which provides the means for any function mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

An equivalent method of defining the arithmetic on FNs is with the help of their x-cuts and the *representation-decomposition theorem* of Ralesscou-Negoita for FS. According to the above theorem a FS A can be completely and uniquely expressed by the family of its x-cuts in the form $A = \sum_{x \in [0,1]} xA^x$ ([9], Theorem 2.1, p.16). In this way the arithmetic of FNs can be defined with the help of the well known arithmetic of the closed intervals introduced by Moore et al. [10].

However, the above two general methods are rarely used for performing arithmetic operations among FNs in practical applications, because both of them involve laborious calculations. Instead, the use of simpler forms of FNs is preferred in such cases, like the *Triangular* and *Trapezoidal FNs* (e.g. [11], Chapter 7), etc., for which the two general methods lead to much easier rules for performing arithmetic operations.

FNs play an important role in fuzzy mathematics analogous to the role of the ordinary numbers in the traditional mathematics. For general facts on FNs we refer to the book [12].

As it was expected, the far-reaching theory of FSs aroused some objections to the scientific community. While there have been generic complaints about the fuzziness of assigning values to linguistic terms, the most cogent criticisms come from Haack [13] in 1979, who argued that there are only two areas – the nature of Truth and Falsity and the fuzzy systems' utility – in which FL could be possibly needed. Further, she maintained that in both cases it can be shown that FL is unnecessary. Fox [14] responded against to her objections, his most powerful arguments being first that traditional and FL need not be seen as competitive, but as complementary, and second that FL, despite the objections of classical logicians, has found its way into practical applications to almost all fields of human activity and has been proved very successful there.

## 3 Managing the Uncertainty of Fuzzy Systems

A system's *uncertainty* can be defined as the shortage of precise knowledge and of complete information on data which describe the state of the system.

A fundamental principle of the classical *Information Theory* states that a system's uncertainty is connected to its ability to obtain new information. Shannon introduced in 1948 a formula for measuring the uncertainty and the information connected to it, which is based on the laws of the classical probability and it is widely known as the *Shannon's entropy* [15]. This name is due to the mathematical definition of the information connected to the evolution of a certain situation. In fact, when we have equally probable cases for the evolution of a particular situation, information is defined [15] by the formula $-\frac{\Delta(\log P)}{\log 2}$, where P is the probability of appearance of each case. The above expression appears to be analogous to the well known from Physics formula $\Delta S = \frac{\Delta Q}{T}$, where ΔS is the increase of a physical system's entropy caused from an increase of the heat by ΔQ, when the absolute temperature T remains constant.

For use in a fuzzy environment Shannon's formula has been adapted ([16], p. 20) to the form:

$$H = -\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s \qquad (1).$$

In equation (1) $m_s = m(s)$ denotes the membership degree of the element *s* of the set of the discourse U in the corresponding FS and *n* denotes the total number of the elements of U. Dividing the sum by the natural logarithm ln *n* one normalizes H, so that to take values in the interval [0, 1].

The uncertainty measured by equation (1) is usually termed as the system's *probabilistic uncertainty*. It is recalled that the *fuzzy probability* of an element *s* of U is defined by

$$P_s = \frac{m_s}{\sum_{s\in U} m_s} \qquad (2).$$

However, according to the British economist Shackle [17] and many other researchers after him, human reasoning can be formulated more adequately by the *possibility* rather, than by the probability theory. The possibility, say $r_s$, of an element *s* of U is defined by

$$r_s = \frac{m_s}{\max\{m_s\}} \qquad (3).$$

In equation (3) the term max $\{m_s\}$ denotes the maximum value of $m_s$, for all *s* in *U*. Therefore, the possibility of *s* expresses the relative membership degree of *s* with respect to max $\{m_s\}$.

In terms of possibility theory a system's uncertainty is measured by the sum of *strife (or discord)* and of the *non-specificity (or imprecision)*. The former is connected to the conflict created among the membership degrees, whereas the latter is connected to the conflict created among the *cardinalities* (sizes) of the various fuzzy subsets of U ([16], p. 28). It is recalled that the cardinality of a FS on U is defined to be the sum $\sum_{x\in U} m(x)$ of all membership degrees of the elements of U with respect to the particular FS.

The following example illustrates the above two types of uncertainty.

*Example 1:* Let U be the set of all mountains of a country and let H and L be the FSs of the high and low mountains with MF $m_H$ and $m_L$ respectively.

Assume now that a mountain x in U has a height of 1000 m. Then the corresponding strife is created by the existing conflict between the membership degrees $m_H(x)$ and $m_L(x)$. In fact, if the country has high mountains in general, then $m_H(x)$ should take values near to 0 and $m_l(x)$ near to 1. However, the opposite could happen, if the country had low mountains in general.

On the other hand, the non-specificity is connected to the question of how many elements of U should have zero membership degrees with respect to H and L.

Strife is measured ([16], p.28) by the function *ST(r)* on the *ordered possibility distribution*

$r$: $r_1=1\geq r_2\geq \ldots\ldots \geq r_n \geq r_{n+1}$ of the elements of U and it is defined by

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right] \quad (4).$$

Under the same conditions non-specificity is measured ([16], p.28) by the function

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\right] \quad (5).$$

The sum $T(r) = ST(r) + N(r)$ measures the fuzzy system's *total possibilistic uncertainty*.

*Example 2:* The performance of a student class has been evaluated by the linguistic grades A = excellent, B = very good, C = good, D = fair and F = unsatisfactory. The numbers of students receiving each of the above grades are the following: A =1, B=13, C=4, D=3, F=0. It is asked to calculate the existing in the class total possibilistic and probabilistic uncertainty.

*i) Total possibilistic uncertainty:* Defining the MF in terms of the frequencies of the student grades, i.e. by $m(x) = \frac{n_x}{n}$ where $n_x$ is the number of students who received the grade x and $n$ is the total number of the students, one can represent the student class C as FS on the set U = {A, B, C, D. F} in the form:

$$C = \{(A, \tfrac{1}{21}), (B, \tfrac{13}{21}), (C, \tfrac{4}{21}), (D, \tfrac{3}{21}), (F, 0)\}.$$

The maximum membership degree in C is equal to $\frac{13}{21}$, hence the possibilities of the elements of *U* in C are: $r(A) = \frac{1}{13}$, $r(B) = 1$, $r(C) = \frac{4}{13}$, $r(D) = \frac{3}{13}$, $r(F) = 0$. Thus, the ordered possibility distribution defined on C is

$$r:\ r_1 = 1 > r_2 = \frac{4}{13} > r_3 = \frac{3}{13} > r_4 = \frac{1}{13} > r_5 = 0 \ (6).$$

Therefore equation (4) gives that

$$ST(r) = \frac{1}{\log 2}\left[(r_2-r_3)\log\frac{2}{r_1+r_2} + (r_3-r_4)\log\frac{3}{r_1+r_2+r_3} + (r_4-r_5)\log\frac{4}{r_1+r_2+r_3+r_4}\right]$$

Replacing the values of the possibility distribution $r$ from (6) to the above equation one finds that

$$ST(r)=\frac{1}{\log 2}\left[\frac{1}{13}\log\left(\frac{26}{17}\right)+\frac{2}{13}\log\left(\frac{39}{20}\right)+\frac{1}{13}\log\left(\frac{42}{21}\right)\right.$$

$$\approx 0.27.$$

Also equation (5) gives for C that

$$N(r) = \frac{1}{\log 2}\left[\frac{1}{13}\log 2+\frac{2}{13}\log 3+\frac{1}{13}\log 4\right]\approx 0.48.$$

Therefore, the total possibilistic uncertainty for C is $T(r) \approx 0.27+ 0.48 =0.75$.

*ii) Probabilistic uncertainty:* Replacing the membership degrees of C to equation (1) one finds that the probabilistic uncertainty for C is equal to

$$H=-\frac{1}{\ln 5}\left(\frac{1}{21}\ln\frac{1}{21}+\frac{13}{21}\ln\frac{13}{21}+\frac{4}{21}\ln\frac{4}{21}+\frac{3}{21}\ln\frac{3}{21}\right)\approx 0.64.$$

The numerical values of the uncertainty found above are useful for comparing the performance of the particular class to the performance of other classes participating the same activity (e.g. common test). It becomes evident that the lower the existing uncertainty the better the corresponding class's performance.

## 4 Extensions of Fuzzy Sets

In 1975 Zadeh generalized the ordinary FS, otherwise termed as *type-1 FS*), to the *type-2 FS* [8], so that more uncertainty can be handled connected to the MF. The MF of a type-2 FS is three - dimensional, its third dimension being the value of the MF at each point of its two – dimensional domain, which is called *footprint of uncertainty (FOU)*. The FOU is completely determined by its two bounding functions, a *lower MF* and an *upper MF*, both of which are type-1 FSs. When no uncertainty exists about the MF, then a type-2 FS reduces to a type-1 FS, in a way analogous to probability reducing to determinism when unpredictability vanishes.

In order to distinguish between a type-1 and a type-2 FS, a tilde symbol is put over the FS, so that A denotes the type-1 FS and $\tilde{A}$ denotes the comparable type-2 FS. Nevertheless, Zadeh didn't stop there, but in the same paper [8] generalized the type-2 FS to the *type-n FS* n = 1, 2, 3, … However, when Zadeh proposed the type-2 FS in 1975, the time was not right for researchers to drop what they were doing with type-1 FS and focus on type-2 FS. This changed in the late 1990s as a result of Prof. Jerry Mendel and his student's works on type-2 FS [18]. Since then, more and more researchers around the world are writing articles about type-2 FS and systems.

Another application of FS inspired by Prof. Zadeh is the methodology of *Computing with Words (CWW)*, in which the objects of computation are words and propositions drawn from a natural language [19]. The idea is that computers would be activated by words, which would be converted into a mathematical representation using FSs and that these FSs would be mapped by a CWW engine into some other FS, after which the latter would be converted back into a word. As Mendel has argued [20] a type-2 FS should be used as a model for CWW, for which much research is under way.

The concept of the *interval-valued FS (IVFS)* was introduced in 1975 independently by Zadeh, Sambuc, Jahn and Grattan Guiness [21]. An IVFS is defined by a mapping F from the universe U to the set of closed intervals in [0, 1]. The idea behind IVFs is that the membership degrees of the traditional FSs can hardly be precise.

Kassimir Atanassov, Professor of mathematics at the Bulgarian Academy of Sciences, introduced in 1986, as a complement of Zadeh's membership degree $m(x)$, $x \in U$, the *degree of non-membership* $n(x)$ and proposed the notion of *intuitionistic FS (IFS)* for a more accurate quantification of the uncertainty [22]. An IFS A is formally defined as the set of the ordered triples

$$A = \{(x, m(x), n(x)): x \in U, 0 \leq m(x) + n(x) \leq 1\}.$$

One can write $m(x) + n(x) + h(x) = 1$, where $h(x)$ is called the *hesitation* or *uncertainty degree* of x. If $h(x) = 0$ for all x in U, then $m(x) = 1 - n(x)$ and A becomes an ordinary FS.

Similar to the intention of IFS is the *hesitant FS (HFS)* introduced by Torra and Narukawa in 2009 [23]. The difference in the definition of a HFS with respect to an ordinary FS is that the *hesitant degree* $h(x)$ of an element x of U is not a single value like its membership degree, but a set of some values in [0, 1]. For example, if U = {a, b, c}, then we could have $h(a) = \{0.2, 0.3\}$, $h(b) = \{0.75, 0.8, 0.82\}$ and $h(c) = \{0.9\}$.

A *Pythagorean FS (PFS)*, introduced by Yager in 2013 [24], considers the membership degree $m(x)$ and non-membership degree $n(x)$ satisfying the condition $m^2(x) + n^2(x) \leq 1$. HFS and PFS have stronger ability than IFS to manage the uncertainty in real-world decision-making problems [25].

The Romanian–American writer and mathematician Florentin Smarandache, Professor at the branch of Gallup of the New Mexico University, introduced in 1995 the *degree of indeterminancy / neutrality (i)* and defined the *Neutrosophic set (NS)* in three components (M, N, I), where
$M = \{m(x): x \in U\}$, $N = \{n(x): x \in U\}$ and
$I = \{i(x): x \in U\}$ are subsets of the interval [0, 1] [26]. In other words, if A is a NS on U, then each element x of U is expressed with respect to A in the form $(m(x), n(x), i(x))$. This structure makes NS an effective framework by empowering it to deal with indeterminate information which is not considered by FS and IFS. When the components m, *n* and *i* are independent, they are leaving room for incomplete information if their sum is <1, for paraconsistent information if their sum is >1 and for complete information if their sum is equal to 1.

Ramot et al. [27] introduced in 2002 the notion of *Complex FS (CFS)* characterized by a complex-valued MF, whose range is extended from the traditional fuzzy range of [0, 1] to the unit circle in the complex plane. More explicitly, the MF of a CFS is of the form

$$m(x) = r(x)e^{i\theta(x)} = r(x)[\cos[\theta(x)]+i\sin[\theta(x)]].$$

In the above formula $r(x)$ is the *amplitude term* and $\theta(x)$ is the *phase term* of the MF. The terms $r(x)$ and $\theta(x)$ are both real-valued and $r(x)$ is in [0, 1] for all x in the universal set U. Since $m(x)$ is a periodic function, one may only consider $\theta(x)$ in $[0, 2\pi]$. When $\theta(x) = 0$ for all x in U, then $m(x)$ reduces to the MF of an ordinary FS.

The catalogue of the extensions of Zadeh's ordinary FS does not end here. Several other generalizations have been introduced, some of them being hybrid approaches of the above mentioned concepts. For example, the notion of NS has been combined with IVFS to form a new hybrid set called *interval valued neutrosophic set*, etc.

## 5. Relative Theories

Apart of the extensions of the FS theory discussed in the previous section, various relative theories have been also proposed as alternative tools for managing the situations of uncertainty and vagueness or ambiguity in science, technology and the everyday life.

In 1982 Julong Deng, professor of the Huazhong University of Science and Technology, Wuhan, China, introduced the theory of *Grey System (GS)* [28] for handling the approximate data that are frequently appear in the study of large and complex systems, like the socio-economic, the biological ones, etc. **T**he systems which lack information, such as structure message, operation mechanism and behaviour document, are referred to as GSs. Usually, on the grounds of existing grey relations and elements one can identify where "grey" means poor, incomplete, uncertain, etc. The GS theory was mainly developed in China and it has found many applications in agriculture, economy, management, industry, ecology and in many other fields of the human activity [29].

An effective tool of the GS theory is the use of *Grey Numbers (GNs)* that are indeterminate numbers defined in terms of the closed real intervals. More explicitly, a GN, say A, is of the form $A \in [a, b]$, where $a$, $b$ are real numbers with $a \leq b$. In other words, the range in which a GN lies is known, but not its exact value. A GN may enrich its uncertainty representation with respect to the interval $[a, b]$ by a function $g$: $[a, b] \rightarrow [0, 1]$, which defines a *degree of greyness* $g(x)$ for each $x$ in $[a, b]$. The well known arithmetic of the real intervals [10] has been used to define the basic arithmetic operations among the GNs. The real number with the greatest probability to be the representative real value of the GN $A \in [a, b]$ is denoted by $W(A)$. The technique of determining the value of $w(A)$ is called *whitening* of A. When the distribution of A is unknown (i.e. no function $g$ has been defined for it) one usually takes

$$W(A) = \frac{a+b}{2}.$$

For general facts on GNs we refer to the book [30].

A *rough set*, first described by the Polish computer scientist Zdzislaw Pawlak in 1991 [31] is a formal approximation of a crisp set in terms of a pair of sets which give the *lower* and the *upper* approximation of the original set. In the standard version of rough set theory the lower and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be FSs. The theory of rough sets has found important applications to Informatics and to other scientific fields.

In 1999 Dmtri Molodstov, Professor of the Computing Centre of the Russian Academy of Sciences in Moscow, in order to overcome the existing difficulty in defining the proper MF of a FS, proposed the *soft sets* as a new mathematical tool for dealing with the uncertainties [32]. Let E be a set of parameters, then a pair (F, E) is called a soft set on the universe U, if, and only if, F is a mapping of E into the set of all subsets of U. In other words, the soft set is a paramametrized family of subsets of U. Every set $F(\varepsilon)$ of this family, $\varepsilon \in E$, may be considered as the set of the $\varepsilon$-elements of the soft set (F, E).

As an example, let U be the set of the girls of a high school and let E be the set of the characterizations {pretty, ugly, tall, short, clever} assigned to each of them. It becomes evident that for an $\varepsilon$ in E the corresponding set $F(\varepsilon)$ could be arbitrary depending on the observer's personal criteria, or empty, while some of them could have non empty intersection. A FS on U with membership function $y = m(x)$ is a soft set on U of the form (F, [0. 1]), where $F(\alpha) = \{x \in U: m(x) \geq \alpha\}$ is the corresponding $\alpha$ – cut of the FS, , for each $\alpha$ in [0. 1].

The topics presented in the previous two sections constitute the main extensions and relative to FS theories. In certain cases the corresponding notions have been combined to form new hybrid theories. For example, if in the definition of the soft set the set of all subsets of U is replaced by the set of all fuzzy subsets of U, one gets the notion of the *fuzzy soft set*, etc.

## 6. Conclusion

This article reviews critically the FS theory, its extensions and the main relative theories that have been proposed in the effort of managing, more effectively than the traditional probability does, the frequently existing uncertainty in problems of science, technology and of the every day life. In particular the management of the uncertainty in fuzzy systems was discussed in terms of the fuzzy probabilities and possibilities of the elements of the universal set in the corresponding FSs.

The present work offers the basic framework for studying deeper the above theories that have been found many and important applications during the last 60 years to almost all sectors of the human activity.

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