Use of Fuzzy Relation Equations and of the Bloom's Taxonomy for Evaluating Student Learning Skills

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Abstract: - In the paper at hands Fuzzy Relation Equations are applied on the levels of the Bloom's Taxonomy of Educational Objectives for the assessment of student learning skills and examples are presented illustrating this application. Fuzzy Relation Equations, which are associated to the composition of fuzzy binary relations, is a dynamic tool of fuzzy mathematics that has been used by many researchers in several real life applications. The Bloom's taxonomy, which has been applied in the USA and other countries by generations of teachers and college instructors in the teaching process, refers to a classification of the different learning objectives serving as a way of distinguishing the fundamental questions within the educational system.
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Key-Words: - Fuzzy Set (FS), Membership Function (MF), Fuzzy Binary Relation (FBR), Fuzzy Relation Equations (FRE), Fuzzy Assessment Methods, Bloom's Taxonomy (BT) of Educational Objectives.

1 Introduction

The *Bloom's taxonomy (BT)*, which has been applied in the USA by generations of teachers and college instructors in the teaching process [1], refers to a classification of the different learning objectives serving as a way of distinguishing the fundamental questions within the educational system.

On the other hand, the process of learning usually involves a degree of fuzziness, created either by the instructor's uncertainty about the level of the student acquisition of a new, or by the student uncertainty concerning the good understanding of the new topic taught. This gave us several times in past the impulsion to use principles and methods of *Fuzzy Logic (FL)* for the better study and assessment of the process of learning a subject matter in the classroom (e.g. see[2], [3]: Chapters 5-7, etc.).

In the present work a new, hybrid method is applied for evaluating student learning skills, combining the use of *Fuzzy Relation Equations (FRE)* and of the BT as the main tools for its development. The rest of the paper is organized as follows: In Section 2 the headlines of the BT are presented, while in Section 3 central points of the theory of FRE are exposed being indispensable for the good understanding of this work. The hybrid model using FRE and BT is developed in Section 4 and examples are presented in Section 5 illustrating its applicability and usefulness in real situations. The paper closes with the final conclusion and some hints for future research contained in Section 6.

2 The Bloom's Taxonomy

In 1956 Benjamin Bloom with collaborators Max Englehart, Edward Furst, Walter Hill, and David Krathwohl published a framework for categorizing educational goals, the *Taxonomy of Educational Objectives* [1]. Although named after Bloom, the publication of the taxonomy followed a series of conferences from 1949 to 1953, which were designed to improve communication between educators on the design of curricula and examinations.

BT divides educational objectives into three domains: *cognitive*, *affective* and *psychomotor*, sometimes loosely described as "knowing/head", "feeling/heart" and "doing/hands" respectively. The volume published in 1956 [4] and the revision followed in 2000 [5] concern the cognitive domain, while a second volume was published in 1965 on the affective domain. A third volume was planned on the psychomotor domain, but it was never published. However, other authors published their own taxonomies on the last domain. More details can be found in [6].

The revised version of the taxonomy [5] was created by Lorin Anderson, former student of Bloom. Since the taxonomy reflects different forms of thinking and thinking is an active process, in the revised version the names of its six major levels were changed from noun to verb forms.

The six major levels of the revised taxonomy are presented in Fig. 1, taken from [6].

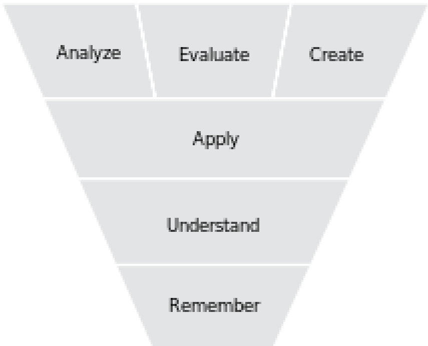

Fig 1**.** The six major levels of the Bloom's taxonomy

The above six levels in the taxonomy, moving through the lowest order processes to the highest, could be described as follows :

- *Knowing - Remembering:* Retrieving, recognizing, and recalling relevant knowledge from long-term memory. e.g. find out, learn terms, facts, methods, procedures, concepts
- *Organizing - Understanding:* Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining. Understand uses and implications of terms, facts, methods, procedures, concepts.
- *Applying:* Carrying out or using a procedure through executing, or implementing. Make use of, apply practice theory, solve problems and use information in new situations.
- *Analyzing:* Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing. Take concepts apart, break them down, analyze structure, recognize assumptions and poor logic, evaluate relevancy.
- *Generating - Evaluating:* Making judgments based on criteria and standards through checking and critiquing. Set standards, judge using standards, evidence, rubrics, accept or reject on basis of criteria.
- *Integrating - Creating:* Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing. Put things together; bring together various parts; write theme, present speech, plan experiment, put information together in a new & creative way

Most researchers and educators consider the last three levels --analyzing, evaluating and creating – as being parallel with transitions from the one to another and vice versa. It is obvious that using Bloom's higher levels helps the students become better problem solvers.

Teaching a topic, the instructor should arrange the class work in order to synchronize it with the six steps of BT. The typical questions for evaluating the student achievement at the corresponding level are the following:

Knowing questions focus on clarifying, recalling, naming, and listing: Which illustrates...? Write... in standard form.... What is the correct way to write the number of... in word form?

Organizing questions focus on arranging information, comparing similarities/ differences, classifying, and sequencing: Which shows... in order from...? What is the order...? Which is the difference between a... and a...? Which is the same as...? Express... as a...?

Applying questions focus on prior knowledge to solve a problem: What was the total...? What is the value of...? How many... would be needed for...? Solve....Add/subtract....Find....Evaluate....Estimate... .Graph....

Analyzing questions focus on examining parts, identifying attributes/ relationships /patterns, and main idea: Which tells...? If the pattern continues,.... Which could...? What rule explains/completes... this pattern? What is/are missing? What is the best estimate for...? Which shows...? What is the effect of...?

Generating questions focus on producing new information, inferring, predicting, and elaborating with details: What number does... stand for? What is the probability...? What are the chances...? What effect...?

Integrating questions focus on connecting/combining/summarizing information and restructuring existing information to incorporate new information: How many different...? What happens to... when...? What is the significance of...?

How many different combinations...? Find the number of..., ..., and ... in the figure below.

Evaluating questions focus on reasonableness and quality of ideas, criteria for making judgments and confirming accuracy of claims: Which most accurately...? Which is correct? Which statement about... is true? What are the chances...? Which would best...? Which would... the same...? Which statement is sufficient to proven...?

BT serves as the backbone of many teaching philosophies, in particular those that lean more towards skills rather than content. The emphasis on higher-order thinking inherent in such philosophies is based on the top levels of the taxonomy including analysis, evaluation, synthesis and creation. BT can be used as a teaching tool to help balance assessment and evaluative questions in class, assignments and texts to ensure all orders of thinking are exercised in student's learning.

3 Fuzzy Relation Equations

First recall that a *Fuzzy Set (FS)* A on the universe U, introduced by Zadeh in 1965 [7], is determined by a map *m*: $U \to [0, 1]$, called the *membership function (MF)* of A. The book [8] is proposed as a general reference for readers being not familiar with the basic principles of FS theory.

Definition 1: Let X, Y be two crisp sets. Then a *fuzzy binary relation (FBR)* R(X, Y) is a FS on the Cartesian product X x Y of the form:

$$R(X, Y) = \{(r, m_R(r): r = (x, y) \in X \times Y\},$$

where $m_R : X \times Y \to [0, 1]$ is the corresponding MF.-

When $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, then a FBR R(X, Y) can be represented by a n x m matrix of the form:

$$R = \begin{array}{c} \\ x_1 \\ . \\ . \\ . \\ x_n \end{array} \begin{array}{c} y_1 \ldots\ldots\ldots\ldots\ y_m \\ \begin{pmatrix} r_{11} & . & . & . & r_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ r_{m1} & . & . & . & r_{mn} \end{pmatrix} \end{array} = [r_{ij}],$$

where $r_{ij} = m_R(x_i, y_j)$, with i = 1,…, n and j =1,…, m. The matrix R is called the *membership matrix* of the FBR R(X, Y).

The basic ideas of fuzzy relations, which were introduced by Zadeh [9] and were further investigated by other researchers, are extensively covered in the book [10].

Definition 2: Consider two FBRs P(X, Y) and Q(Y, Z) with a common set Y. Then, the standard composition of these relations, which is denoted by $P(X, Y) \circ Q(Y, Z)$ produces a FBR R(X, Z) with MF m_R defined by:

$$m_R(x_i, z_j) = \underset{y \in Y}{Max} \min [m_P(x_i, y), m_Q(y, z_j)] \quad (1),$$

for all i=1,…,n and all j=1,…,m. This composition is often referred as *max-min composition.*

Compositions of FBRs are conveniently performed in terms of their membership matrices. In fact, if $P = [p_{ik}]$ and $Q=[q_{kj}]$ are the membership matrices of the relations P(X, Y) and Q(Y, Z) respectively, then by equation (1) we get that the membership matrix of $R(X, Y) = P(X, Y) \circ Q(Y, Z)$ is the matrix $R = [r_{ij}]$, with

$$r_{ij} = \max_k \min(p_{ik}, q_{kj}) \quad (2)$$

Example 1: If
$$P = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0.8 \\ 0.1 & 0.5 & 1 \\ 0.4 & 0.7 & 0.3 \end{pmatrix} \end{array}$$
and

$$Q = \begin{array}{c} \\ y_1 \\ y_2 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.2 & 0.7 & 0 & 0.4 \\ 0.8 & 0.1 & 0.5 & 0.6 \\ 1 & 0.3 & 0.2 & 0.9 \end{pmatrix} \end{array}$$
are the membership matrices of P(X, Y) and Q(Y, Z) respectively, then by equation (2) the membership matrix of R(X, Z) is the matrix

$$R = P \circ Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.3 & 0.4 & 0.8 \\ 1 & 0.3 & 0.5 & 0.9 \\ 0.7 & 0.4 & 0.5 & 0.6 \end{pmatrix} \end{array}.$$

Observe that the same elements of P and Q are used in the calculation of R as would be used in the regular multiplication of matrices, but the product and sum operations are here replaced with the min and max operations respectively.

Definition 3: Consider the FBRs P(X, Y), Q(Y, Z) and R(X, Z), defined on the sets, $X = \{x_i : i \in N_n\}$, $Y = \{y_j : j \in N_m\}$, $Z = \{z_k : k \in Ns\}$, where $N_t = \{1,2,\ldots,t\}$, for t = n, m, k, and let $P=[p_{ij}]$, $Q=[q_{jk}]$ and $R=[r_{ik}]$ be the membership matrices of P(X, Y), Q(Y, Z) and R(X, Z) respectively. Assume that the above three relations constrain each other in such a way that $P \circ Q = R$, where $\circ$ denotes the max-min composition. Therefore, for each i in N_n and each k in N_s we have that

$$\max_{j\in J} [\min(p_{ij}, q_{jk})] = r_{ik} \quad (3).$$

Therefore the matrix equation $P \circ Q = R$ encompasses n x s simultaneous equations of the form (3). When two of the components in each of the equations (3) are given and one is unknown, these equations are referred as FRE.

The notion of FRE was first proposed by Sanchez [11] and was further investigated by other researchers [12 - 14].

4 FRE on the Bloom's Taxonomy

Let us consider the crisp sets X = {M}, Y = {A, B, C, D, F} and Z = {L_1, L_2, L_2, L_4}. In those sets M denotes the **"***average student***"** of a class, A = Excellent, B = Very Good, C = Good, D = Fair and F = Failed are *linguistic labels (grades)* used for the assessment of the student performance and L_1 = Remember, L_2 = Understand, L_3 = Apply, L_4 = Analyze/Evaluate/Create, represent the levels of the BT. The three higher levels, being parallel to each other, are considered together as a unified level .

Further, let *n* be the total number of students of a certain class and let n_i be the numbers of students who obtained the grade i assessing their performance, i $\in$ Y. Then one can represent the average student of the class as a FS on Y in the form

$$M = \{(i, \frac{n_i}{n}): i \in Y\}.$$

The FS M induces a FBR P(X, Y) with membership matrix

$$P = [\frac{n_A}{n} \ \frac{n_B}{n} \ \frac{n_C}{n} \ \frac{n_D}{n} \ \frac{n_F}{n}].$$

In an analogous way the average student of the class can be represented as a FS on Z of the form

$$M = \{(j, m(j)): j \in Z\},$$

where m: Z $\to$ [0, 1] is the corresponding MF. In this case the FS M induces a FBR R(X, Z) with membership matrix

$$R = [m_R(L_1) \ m_R(L_2) \ m_R(L_3) \ m_R(L_4)].$$

We consider also the FBR Q(Y, Z) with membership matrix the 5X4 matrix Q = [q_{ij}], where m_Q: Q $\to$ [0, 1] is the corresponding MF and q_{ij} = m_Q(i, j) with i $\in$ Y and j $\in$ Z are the corresponding membership degrees, and the FRE encompassed by the matrix equation $P \circ Q = R$. When the matrix Q is fixed and the row-matrix P is known, then the equation $P \circ Q = R$ has always a unique solution with respect to R, which enables the representation of the average student of a class as a FS on the set Z of the levels of the BT . This is useful for the instructor for designing his/her future teaching plans. On the contrary, when the matrices Q and R are known, then the equation $P \circ Q = R$ could have no solution or could have more than one solutions with respect to P, which makes the corresponding situation more complicated.

All the above will be illustrated in the next section with suitable examples.

5 Examples

Example 2: Table 1 depicts the performance of a student class in a written test on a new mathematical topic (indefinite integrals), given to students a few days after the end of the presentation of this topic by the instructor in the classroom.

Table 1. Student Performance

Grade	No. of Students
A	30
B	10
C	7
D	5
F	8
Total	60

In this case the average student M of the class can be represented as a FS on Y = {A, B, C, D, F} by

$$M = \{(A, \frac{30}{60}), (B, \frac{10}{60}), (C, \frac{7}{60}), (D, \frac{5}{60}), (F, \frac{8}{60})\}$$

$\approx$ {(A, 0.5), (B, 0.17), (C, 0.12), (D, 0.08), (F, 0.13)}.

Therefore M induces a FBR P(X, Y), where X={M}, with membership matrix

$$P = [0.5 \ 0.17 \ 0.12 \ 0.08 \ 0.13].$$

Also, using statistical data of the last five years on the student performance on this topic, we fixed the membership matrix Q of the FBR Q(Y, Z), where Z = {L_1, L_2, L_2, L_4}, in the form:

$$Q = \begin{array}{c} \\ A \\ B \\ C \\ D \\ F \end{array} \begin{array}{c} \begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.6 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.1 \\ 0 & 0.1 & 0.1 & 0.2 \\ 0 & 0 & 0.2 & 0.6 \end{pmatrix} \end{array}$$

Next, using the max-min composition of FBRs one finds that the membership matrix of R(X, Z) = P(X, Y) o Q (Y, Z) is equal to

$$R = P \text{ o } Q = [0.5\ \ 0.5\ \ 0.2\ 0.13].$$

Therefore the average student of the class can be expressed as a FS on Z by

$$M = \{(L_1, 0.5), (L_2, 0.5), (L_3, 0.2), (L_4, 0.13)\}.$$

The conclusions obtained from the above expression of M are the following:

- Half of the students were able to retrieve, recognize, and recall relevant knowledge from memory (L_1) and to understand uses and implications of terms, facts, methods, procedures and concepts (L_2).
- On the contrary, only the 20% of the students were able to apply theory for solving problems and use information in new situations (L_3).
- Finally, only the 13% of the students were able to reach one or more of the three higher levels of the BT by analyzing and /or evaluating situations and/or creating new relevant situations.

It becomes evident that the above conclusions are very useful for the instructor for reorganizing his future teaching plans in order to achieve better results on the student skills to deal with the indefinite integrals.

Let us now consider the case where the membership matrices Q and R are known and we want to determine the matrix P representing the average student of the class as a fuzzy set on Y. This is a complicated case because we may have more than one solution or no solution at all. The following two examples illustrate this situation:

Example 3: Consider the membership matrices Q and R to be as in the previous example and set

$$P = [p_1\ p_2\ p_3\ p_4\ p_5].$$

Then the matrix equation P o Q = R encompasses the following equations:

max {min (p_1, 0.8), min (p_2, 0.2), min (p_3, 0), min (p_4, 0), min(p_5, 0)}= 0.5

max {min (p_1, 0.6), min (p_2, 0.2), min (p_3, 0.1), min (p_4, 0.1), min(p_5, 0)}= 0.5

max {min (p_1, 0.2), min (p_2, 0.2), min (p_3, 0.3), min (p_4, 0.1), min (p_5, 0.2)}= 0.2

max {min (p_1, 0), min (p_2, 0.1), min (p_3, 0.1), min (p_4, 0.2), min(p_5, 0.6)}= 0.13

The first of the above equations is true if, and only if, $p_1 = 0.5$, which satisfies the second and third equations as well. Also, the fourth equation is true if, and only if, $p_3 = 0.13$ or $p_4 = 0.13$ or $p_5 = 0.13$. Therefore, any combination of values of p_1, p_2, p_3, p_4, p_5 in [0, 1] such that $p_1 = 0.5$ and $p_3 = 0.13$ or $p_4 = 0.13$ or $p_5 = 0.13$ is a solution of P o Q = R.

Let S(Q, R) = {P: P o Q = R } be the set of all solutions of P o Q = R. Then one can define a partial ordering on S(Q, R) by

$$P \leq P' \Leftrightarrow p_i \leq p'_i,\ \forall\, i = 1, 2, 3, 4, 5.$$

It is well known that whenever S(Q, R) is a non empty set, it always contains a unique maximum solution and it may contain several minimal solutions [11]. It is further known that S(Q, R) is fully characterized by the maximum and minimal solutions in the sense that all its other elements are between the maximal and each of the minimal solutions [11]. A method of determining the maximal and minimal solutions of P o Q = R with respect to P is developed in [14].

Example 4: Let $Q = [q_{ij}]$, i = 1, 2, 3, 4, 5 and j = 1, 2, 3, 4 be as in Example 2 and let R = [1 0.5 0.2 0.13]. Then the first equation encompassed by the matrix equation P o Q = R is

max {min (p_1, 0.8), min (p_2, 0.2), min (p_3, 0), min (p_4, 0), min(p_5, 0)}= 1.

In this case it is easy to observe that the above equation has no solution with respect to p_1, p_2, p_3, p_4, p_5, therefore P o Q = R has no solution with respect to P.

In general, writing R = {$r_1\ r_2\ r_3\ r_4$}, it becomes evident that we have no solution if $\max_j q_{ij} < r_j$, for some j = 1, 2, 3, 4.

5 Conclusion

In this work we have considered a class of *n* students learning the indefinite integrals and we have applied a FRE model on the levels of BT for learning. This model has been developed with the help of three FBRs with membership matrices P, Q and R respectively satisfying the equation P o Q = R.

The matrix P = $[\frac{n_A}{n}\ \frac{n_B}{n}\ \frac{n_C}{n}\ \frac{n_D}{n}\ \frac{n_F}{n}]$, where n_i denotes the number of students whose progress has been assessed by the grade i = A, B, B, C, D, F, is fixed representing the "average student" of the class. On the other hand, Q = $[q_{ij}]$ is the 5 x 4 matrix in which q_{ij} denotes the membership degree of (i, j) in the FBR Q, where j = L_1, L_2, L_3 represent the three lower levels of the BT and L_4 one of its three higher and parallel to each other levels Also R= $[m_R(L_1)\ m_R\ (L_2)\ m_R\ (L_3)\ m_R\ (L_4)]$ is the matrix of the membership degrees of the above levels of the BT in the FBR Q.

When the matrix Q is known (from statistical data), then the equation P o Q = R has a unique solution with respect to R that gives useful information to the instructor about the student progress in learning the corresponding topic. On the contrary, when the matrix R is known, then the above equation has more than one or no solution at all with respect to Q, which makes the situation more complicated.

Through the presented examples, illustrating the applicability of our model in real situations, it becomes evident that the FRE theory is a powerful tool that could be used not only for student, but also for the assessment of other human or machine (e.g. computers [15]) activities.

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