A Research-Based Pedagogical Approach to Introduction to Differential Equations for Undergraduate Students at an American Two-Year College

JOHN A. GORDON
Department of Mathematics and Computer Science
City University of New York
Queensborough Community College
Bayside, New York, 11364
USA
jgordon@qcc.cuny.edu

MAJID HAGHVERDI
Department of Mathematics
Islamic Azad University
Arak Branch, IRAN
majid_haghverdi@yahoo.com

JOHATHAN MARTINEZ[1], YANG HE[2], YANYAN CHEN[3]
City University of New York
Queenborough Community College
Bayside, New York, 11364
USA
[1]jmartinez91@tigermail.qcc.cuny.edu,[2]yhe42@tigermail.qcc.cuny.edu,[3]ychen81@tigermail.qcc.cuny.edu

Abstract: Undergraduate students pursuing degrees in STEM (Science, Technology, Engineering, and Mathematics) at two-year colleges in the United States are usually required to take an introductory course in ordinary differential equation. The pedagogical strategy in most cases has been based primarily on presenting various standard techniques used in solving differential equations. This approach involves very little student research.
The Mathematics Department at Queensborough Community College (QCC) has adopted a new student research-based approach to its introductory course in ordinary differential equations. In this approach students learn very early in the course to apply technology and research techniques to solve real-world problems. This paper presents some of the student research and our preliminary findings.

Keywords: Auxiliary equations, integrating factor, first-order, eigenvalues, homogeneous equation, non-homogeneous equation, linear system, research-based.

1 Community College in the United States

In the United States, community colleges, sometimes called junior colleges, technical colleges, two-year colleges, or city colleges, are primarily two-year public institutions providing lower-level tertiary education also known as continuing education, granting certificates, diplomas, and associate degrees. After graduating from a community college, some students transfer to a four-year liberal arts college or university for two to three years to complete a bachelor's degree. [1]

1.1 Prerequisite for Introduction to Ordinary Differential Equations at QCC

At QCC the prerequisite for the introductory course in ordinary differential equations is a third course in elementary calculus with a grade of C or better.

1.2 Course Objectives/Expected Student Learning Outcomes

Students learn how to analyze ordinary differential equations quantitatively and qualitatively and apply their knowledge to the solution of real world problems. The methods for solving equations will include separation of variables, solving homogeneous and non-homogeneous linear equations of first-order and higher, homogeneous and non-homogeneous linear systems, Laplace transform. Students use software such as Maple® and Matlab to aid in their research. The text used is *Fundamentals of Differential Equations* by R. Keut Nagle et al. [2].

2 Grading Policy

A comparison of the grading policies is shown below:

Old Grading Policy	New Research-based Grading Policy
3 Exams 60% 1 Mid-term 20% 1 Final 20%	2 Exams 40% Research Project 40% 1 Final 20%
Total 100%	100%

The former course outline is shown below. The text book used was *Elementary Differential Equations*, by Earl D. Rainville [3].

Subject Matter	Hours
Definitions, Introduction	2
Equations of First-Order	5
Nonlinear Equations	2
Elementary Applications	7
Linear Differential Equations	12
Non-homogeneous Equations	4
Variation of Parameters	6
Application: Keplers Laws	3
Systems of Equations	4
The Laplace Transform	10
Inverse Transform	5
Numerical Methods	4
Electrical Applications	2
Total	66

The new research-based course outline, is shown below. The text being used is *Fundamentals of Differential Equations* by R. Kent Nagle [2].

Subject Matter	Hours
Module 1: Introduction Direction fields Isoclines	9
Module 2: First Order Differential Equations Mathematical Models Involving First-Order Equations Laplace Transform Methods Applied to First-Order Equations	21
MODULE 3: Linear Second-Order Equations Laplace Transform Methods Applied to Second-Order Equations	18
MODULE 4: Theory of Higher-Order Linear Differential Equation Laplace Methods for Linear Systems Matrix Methods for Systems	18
Total	66

Research projects were assigned in the middle of Module 2. In addition, each module was initiated with an application problem from Physics, engineering or business.

3 Final Exam Results

Final exam results for Fall 2015 and Fall 2017 Semesters are shown below. The Fall Semester starts in late August and ends in mid/late December. The research-based pedagogical approach began in Fall 2017. The Course did not run in Fall 2016.

Fall 2015	**Fall 2017**
Average: 57% Class Size: 15	Average: 77% Class Size: 16

The same exam was given to both groups.

3.1 Research Projects

Research projects for three of the Fall 2017 students follows and are meant to demonstrate the high quality of the research. The problems were posed in [2].

Differential Equations in Clinical Medicine

Aircraft Guidance in a Crosswind

Dynamics of HIV Infection

4 Differential Equations in Clinical Medicine

In this project, we model the mechanical process performed by the ventilator. In clinical medicine, mechanical ventilation can assist or replace spontaneous breathing for critically ill patients, using the ventilator device. Modern ventilation uses positive pressure to inflate the lungs of the patient. The goal of mechanical ventilation is to provide oxygen to the lungs and to remove carbon dioxide.

In this model, we assume the following process of filling the lungs with air and then letting them deflate to some rest volume.

1. The length (in seconds) of each breath is fixed (t_{tot}) and is set by the clinician, with each breath being identical to the previous breath.

2. Each breath is divided into two parts: *inspiration* (air flowing into the patient) and *expiration* (air flowing out of the patient). We assume that inspiration takes place over the interval $[0, t_i]$ and expiration over the time interval $[t_i, t_{\text{tot}}]$. The time t_i is called the *inspiratory time*.

3. During inspiration the ventilator applies a constant pressure P_{app} to the patient's air-way, and during expiration this pressure is zero, relative to atmospheric pressure. This is called *pressure-controlled ventilation*.

4. We assume that the pulmonary system (lung) is modeled by a single compartment. Hence, the action of the ventilator is similar to inflating a ballon and then releasing the pressure.

   Given the pressure balance:

   $$P_r + P_e + P_{ex} = P_{aw} \tag{1}$$

   where P_r denotes pressure losses due to resistance to flow into and out of the lung, P_e is the elastic pressure due to changes in volume of the lung, at the completion of a breath, and P_{aw} denotes the pressure applied to the airway. ($P_{\text{aw}} = P_{\text{app}}$ during inspiration and $P_{\text{aw}} = 0$ during expiration). The residual pressure P_{ex} is called the *end-expiratory pressure*.

5. Let $V(t)$ denote the volume of the lung at time t, with $V_i(t)$, $0 \le t \le t_i$, denoting its volume during inspiration and $V_e(t)$, $t_i \le t \le t_{\text{tot}}$, its volume during expiration. We assume that $V_i(0) = V_e(V_{\text{tot}}) = 0$. The number $V_i(t_i) = V_T$ is called the *tidal volume* of the breath.

6. We assume that the resistive pressure P_r is proportional to the flows into and out of the lung such that $P_r = R(dV/dt)$, and we assume that the proportionality constant R is the same for inspiration and expiration.

7. Furthermore, we assume that the elastic pressure is proportional to the instantanous volume of the lung. That is, $P_e = (1/C)V$, where the constant C is called the *compliance* of the lung.

Using the pressure balance the equation (1) together with the above assumptions, a mathematical model for the instantaneous volume in the single compartment lung is given by the following pair of first-order linear differential equations:

$$R\left(\frac{dV_i}{dt}\right) + \left(\frac{1}{C}\right) V_i + P_{\text{ex}} = P_{\text{app}}, \quad 0 \le t \le t_i, \tag{2}$$

$$R\left(\frac{dV_e}{dt}\right) + \left(\frac{1}{C}\right) V_e + P_{\text{ex}} = 0, \quad t_i \le t \le t_{\text{tot}}, \tag{3}$$

Strategy

(a) Solve equation (2) for $V_i(t)$ with the initial condition $V_i(0) = 0$.

(b) Solve equation (3) for $V_e(t)$ with the initial condition $V_e(t_i) = V_T$.

(c) Using the fact that $V_i(t_i) = V_T$, show that

$$P_{\text{ex}} = (e^{t_i/RC-1})P_{\text{app}}/(e^{t_{\text{tot}}/RC} - 1)$$

(d) For $R = 10$ cm H_2O/L/sec, $C = 0.02$ L/cm (H_2O), $P_{\text{app}} = 20$ cm (H_2O), $t_i = 1$ sec and $t_{\text{tot}} = 3$ sec, plot the graphs of $V_i(t)$ and $V_e(t)$ over the interval $[0, t_{\text{tot}}]$. Compute P_{ex} for these parameters.

(e) The *mean alveolar pressure* is the average pressure in the lung during inspiration and is given by the formula

$$P_m = \frac{1}{t_i}\int_0^{t_i}\left(\frac{V_i(t)}{C}\right)dt + P_{\text{ex}}.$$

Compute this quantity using the expression for $V_i(t)$ in part (a).

Analysis

For problems (a) and (b), we use the method studied in [2] to solve the linear first-order equation, in form $a_1(x)dy/dx + a_0(x)y = b(x)$.

Calculation

a)

$$R\frac{dV_i}{dt} + \frac{1}{C}V_i = P_{\text{app}} - P_{\text{ex}}$$
$$\frac{dV_i}{dt} + \frac{1}{CR}V_i = \frac{P_{\text{app}} - P_{\text{ex}}}{R}$$
$$\mu(x) = e^{\int \frac{1}{CR}dt} = e^{\frac{1}{CR}t}$$
$$e^{\frac{1}{CR}t}\frac{dV_i}{dt} + \frac{1}{CR}e^{\frac{1}{CR}t}V_i = e^{\frac{1}{CR}t}\left(\frac{P_{\text{app}} - P_{\text{ex}}}{R}\right)$$
$$e^{\frac{1}{CR}t}V_i = \int e^{\frac{1}{CR}t}\left(\frac{P_{\text{app}} - P_{\text{ex}}}{R}\right)dt$$
$$e^{\frac{1}{CR}t}V_i = CRe^{\frac{1}{C}t}\left(\frac{P_{\text{app}} - P_{\text{ex}}}{R}\right) + \text{constant}$$
$$0 = C(P_{\text{app}} - P_{\text{ex}}) + \text{constant}$$
$$\text{constant} = -C(P_{\text{app}} - P_{\text{ex}}),$$
$$e^{\frac{1}{CR}t}V_i = Ce^{\frac{1}{CR}t}(P_{\text{app}} - P_{\text{ex}}) - C(P_{\text{app}} - P_{\text{ex}})$$
$$= C(P_{\text{app}} - P_{\text{ex}})(e^{\frac{1}{CR}t} - 1)$$
$$V_i = C(P_{\text{app}} - P_{\text{ex}})\left(1 - e^{-\frac{1}{CR}t}\right)$$

b)

$$R\frac{dV_e}{dt} + \frac{1}{C}V_e + P_{\text{ex}} = 0$$
$$\frac{dV_e}{dt} + \frac{1}{CR}V_e = -\frac{P_{\text{ex}}}{R}$$
$$\mu(x) = e^{\int \frac{1}{CR}dt} = e^{\frac{1}{CR}t}$$
$$e^{\frac{1}{CR}t}\frac{dV_e}{dt} + \frac{1}{CR}e^{\frac{1}{CR}t}V_e = e^{\frac{1}{CR}t}\left(-\frac{P_{\text{ex}}}{R}\right)$$
$$e^{\frac{1}{CR}t}V_e = -CRe^{\frac{1}{CR}t}P_{\text{ex}} + \text{constant}$$

when $t = t_{\text{tot}}, v_e = 0,$

$$0 = -Ce^{\frac{1}{CR}t}P_{\text{ex}} + \text{constant}$$
$$\text{constant} = Ce^{\frac{1}{CR}t}P_{\text{ex}}$$
$$e^{\frac{1}{CR}t}V_e = -CR_{\text{ex}}\left(e^{\frac{1}{CR}t} - e^{\frac{1}{CR}t_{tot}}\right)$$
$$V_e == -CP_{\text{ex}}\left(1 - e^{\frac{1}{CR}(t_{\text{tot}}-t)}\right)$$

c) When $t = t_i, v_i = v_e$;

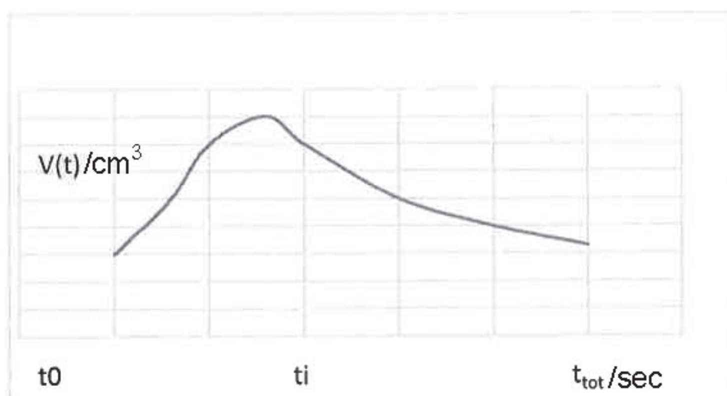
$$C(P_{app} - P_{ex})\left(e^{\frac{1}{CR}t} - 1\right) = -CP_{ex}\left(e^{\frac{1}{CR}t} - e^{\frac{1}{CR}t_{tot}}\right)$$
$$CP_{app}\left(e^{\frac{1}{CR}t} - 1\right) = -CP_{ex}\left(e^{\frac{1}{CR}t} - e^{\frac{1}{CR}t_{tot}}\right)$$
$$+ CP_{ex}\left(e^{\frac{1}{CR}t} - 1\right)$$
$$P_{app}\left(e^{\frac{1}{CR}t} - 1\right) = P_{ex}\left(e^{\frac{1}{CR}t_{tot}} - 1\right)$$
$$P_{ex} = \frac{P_{app}\left(e^{\frac{1}{CR}t} - 1\right)}{\left(e^{\frac{1}{CR}t_{tot}} - 1\right)}$$

Proved.

d) R = 10 cm H_2O/L/sec, $C = 0.02$ L/cm (H_2O), $P_{\text{app}} = 20$ cm (H_2O), $t_i = 1$ sec and $t_{tot} = 3$ sec;

$$P_{ex} = \frac{20\left(e^{\frac{1}{10+0.02}} - 1\right)}{e^{\frac{3}{10+0.02}} - 1} = 9.019 * 10^{-4}$$

For the graphs of $V_i(t)$ and $V_e(t)$ over the interval $[0, t_{tot}]$, through the result of the calculation, we find that V_i increases with time in the interval $[0, t_i]$; V_e decreases with time in the interval $[t_i, t_{tot}]$.



e)

$$P_m = \frac{1}{t_i}\int_0^{t_i}\left(\frac{V_i(t)}{C}\right)dt + P_{ex}$$
$$= \frac{1}{t_i}\int_0^{t_i}\left((P_{app} - P_{ex}(1 - e^{-\frac{1}{CR}t})\right)dt + P_{ex}$$
$$= \frac{P_{app} - P_{ex}}{t_i}\left(t + CRe^{-\frac{1}{CR}t}\right)|_0^{t_i} + P_{ex}$$
$$= \frac{P_{app} - P_{ex}}{t_i}\left(t_i + CRe^{-\frac{1}{CR}t_i} - CR\right) + P_{ex}$$

Finally plug (c) *into above, we have*

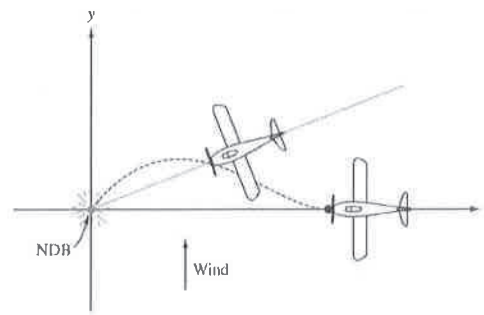
$$\begin{aligned} P_m &= P_{app} + \frac{P_{app} - P_{ex}}{t_i} CR \left(e^{-\frac{1}{CR} t_i} - 1 \right) \\ &= P_{app} + CR \frac{P_{app}}{t_i} \frac{\left(e^{\frac{1}{CR} t_{tot}} - e^{\frac{1}{CR} t_i} \right)}{\left(e^{\frac{1}{CR} t_{tot}} - 1 \right)} \end{aligned}$$

By setting up a model of the ventilator, we were able to model the breathing process with linear first-order differential equations. Using the pressure balance equation we were able to solve these equations.

5 Aircraft Guidance in a Crosswind

Problem

An aircraft flying under the guidance of a nondirectional beacon (a fixed radio transmitter, abbreviated NDB) moves so that its longitudinal axis always points toward the beacon (see Figure 3.19). A pilot sets out toward an NDB from a point at which the wind is at right angles to the initial direction of the aircraft; the wind maintains this direction. Assume that the wind speed and the speed of the aircraft through the air (its "airspeed") remain constant. (Keep in mind that the latter is different from the aircraft's speed with respect to the ground.)



a) Locate the flight in the xy-plane, placing the start of the trip at (2, 0) and the destination (0, 0). Set up the differential equation describing the aircraft's path over the ground.

b) Make an appropriate substitution and solve this equation.

c) Use the fact that $x = 2$ and $y = 0$ at $t = 0$ to determine the appropriate value of the arbitrary constant in the solution set.

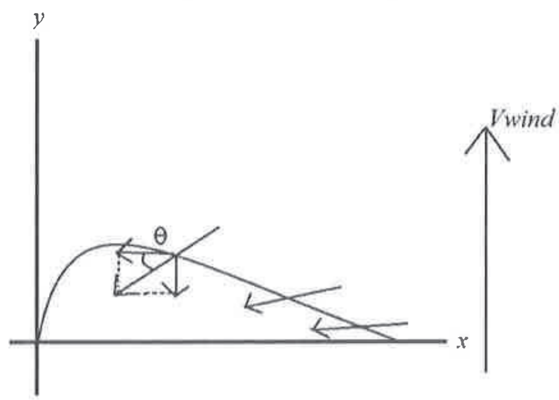
d) Solve to get y explicitly in terms of x. Write your solution in terms of a hyperbolic function.

e) Let γ be the ratio of windspeed to airspeed. Using a software package, graph the solutions for the cases $\gamma = 0.1$, 0.3, 0.5 and 0.7 all on the same set of axes. Interpret these graphs.

f) Discuss the cases $\gamma = 1$ and $\gamma > 1$.

6 Analysis

In order to understand the problem we first drew a picture. At all times the wind speed and aircraft airspeed are constant. We can make a few observations: the aircraft points to the origin at all times, thus has an angle that changes with respect to the x axis. This means the aircraft's speed can be split into two components, the vertical component which points downward and needs to overcome the wind speed, and the horizontal component, which is dependent on the angle of the airplane. These two components are then changing with respect to one another. This allows us to find both components. We are not interested in the net speed of the aircraft but rather its x-y position.



Solution

Let:

$V_w = \textit{wind speed}$
$V_a = \textit{aircraft speed}$

We express the two components of the aircraft's actual speed as follows:

$$V_{net\ y} = V_a \sin(\theta) - V_w \Rightarrow \frac{dy}{dt} = V_a \sin(\theta) - V_w$$
$$V_{net\ x} = V_a \cos(\theta) \Rightarrow \frac{dx}{dt} = V_a \cos(\theta)$$

Consequently,

$$\frac{dy}{dt} = V_a \frac{y}{\sqrt{x^2+y^2}} - V_w$$
$$\frac{dx}{dt} = V_a \frac{x}{\sqrt{x^2+y^2}}$$

We want change in y with respect to change in x, thus:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{V_a \frac{y}{\sqrt{x^2+y^2}}}{V_a \frac{x}{\sqrt{x^2+y^2}}} = \frac{dy}{dx} = \frac{y}{x} - \frac{V_w}{V_a}\frac{\sqrt{x^2+y^2}}{x}$$

We notice the equation obtained is a homogeneous equation similar to those studied in class, and we also notice that $\frac{V_w}{V_a}$ is a constant ratio.

$$\frac{dy}{dx} = \frac{y}{x} - \frac{V_w}{V_a}\frac{\sqrt{x^2+y^2}}{x}\frac{x^{-2}}{\sqrt{x^{-2}}} = \frac{y}{x} - \frac{V_w}{V_a}\sqrt{1+\left(\frac{y}{x}\right)^2}$$

let $v = \frac{y}{x}$ thus $y' = v + xv' = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = v - \frac{V_w}{V_a}\sqrt{1+\sqrt{v^2}} \Rightarrow x\frac{dv}{dx} = -\frac{V_w}{V_a}\sqrt{1+v^2}$$
$$\frac{1}{\sqrt{1+v^2}}dv = -\frac{V_w}{V_a}\frac{1}{x}dx \Rightarrow \int \frac{1}{\sqrt{1+v^2}}dv$$
$$= -\frac{V_w}{V_a}\int \frac{1}{x}dx$$
$$\sinh^{-1}\left(\frac{y}{xx}\right) = -\frac{V_w}{V_a}\ln(x) + c$$

We take the sinh of both sides and rearrange a few terms to obtain the following expression:

$$y = x\sinh\left(c - \frac{V_w}{V_a}\ln(x)\right)$$

Using the initial condition $t = 0$, $y = 0$, $x = 2$, we obtain

$$0 = 2\sinh\left(c - \frac{V_w}{V_a}\ln(2)\right)$$

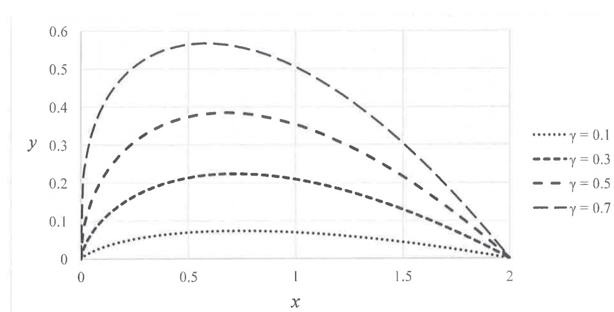
Consequently,

$$c - \frac{V_w}{V_a}\ln(2) = 0 \Rightarrow c = \frac{V_w}{V_a}\ln(2)$$

Our final expression is:

$$y = x\sinh\left(\frac{V_w}{V_a}\ln(2) - \frac{V_w}{V_a}\ln(x)\right)$$
$$y = x\sinh\left(\frac{V_w}{V_a}\ln\left(\frac{2}{x}\right)\right)$$

We plot the function $y = x\sinh\left(\frac{V_w}{V_a}\ln\left(\frac{2}{x}\right)\right)$ where the ratio $\gamma = \frac{V_w}{V_a}$ is 0.1, 0.3, 0.5, and 0.7.



If the airspeed to wind speed ratio is 1 then the airplane will not be able to make it to its destination. If the ratio is greater than one then the aircraft will be vertically displaced indefinitely.

7 Dynamics of HIV infection

Introduction

The dynamics of HIV (human immunodeficiency virus) infection within a human host involve the interaction of the HIV virions and CD4+ T lymphocytes. CD4+ T lymphocytes are long-lived white blood cells that play a major role in the defense of the human body against microbial invaders. HIV targets these very cells. When HIV first appeared as a new and major health threat, it was recognized that the disease typically exhibited a lengthy gradual progression lasting 10 or more years. It was widely believed that the dynamics of HIV destruction of the CD4+ T-cell population involved a very low rate of infection and a very slow turnover of virus and infected cells. In 1995 differential equation models of HIV-CD4+ T-cell interaction revealed that the turnover rate for the infected CD4+ T cells was very much faster than this (about 2 days)—a scientific breakthrough reported simultaneously in the papers of D. D. Ho et al., "*Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection*," *Nature*, 1995; and of G. M. Shaw et al., "*Viral Dynamics in Human Immunodeficiency Virus Type I Infection*," *Nature*, 1995. Underlying the models in these papers is the knowledge that within a person infected with HIV, the virus spends part of its existence free and part inside an infected CD4+ T cell. The time spent free was known to be very short—on the order of 30 minutes. The time spent inside an invaded CD4+ T cell was *believed* to be very long—on the order of years. When a cell was invaded, a virion (a complete viral particle, consisting of RNA surrounded by a protein shell) took over the cell's DNA and used it to replicate its own RNA, thereby creating new virions; then it budded, or burst the cell, to release multiple virus particles. [2]

Methodology

1) Definitions of parameters and variables:

T(t) = the population of uninfected CD4+ T cells at time t.

I(t) = the population of infected CD4+ T cell at time t.

V(t) = the population of virus at time t.

$\lambda(\mathbf{cells/day})$ = constant input source of uninfected cells per day (the human body produces these cells daily in the thymus).

$\delta\mathbf{day}^{-1}$ = normal loss rate constant of uninfected cells ($1/\delta$ = the average lifespan of an uninfected cell in days2).

$\beta(\mathbf{virions}^{-1} * \mathbf{day}^{-1})$ = infection rate constant of uninfected cells per infected cell (the rate is of mass action form, i.e., $\beta V(t)T(t)$).

$\mu(\mathbf{day}^{-1})$ = loss rate constant of infected cells ($1/\mu$ = the average lifespan of an infected cell in days).

$\mu\mathbf{day}^{-1}$ = loss rate of free virus ($1/\gamma$ = the average lifespan of a free virion in days).

$N(\mathbf{virions}^{-1} * \mathbf{cell}^{-1})$ = number of virions produced per day per infected cell (the burst number of an infected cell).

2) Mathematical Models and Numerical Methods Involving First-Order Equations:

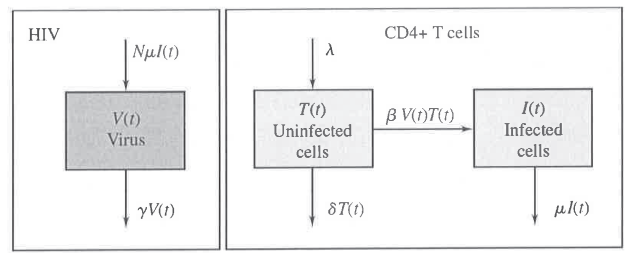


Figure 1: Compartmental views of virus, uninfected T cells, and infected I cells

The independent variable of the model is time t in days and the dependent variable of the model are $T(t)$, $I(t)$, and $V(t)$. The equations are as follows

$$\frac{d}{dt}T(t) = \lambda - \delta T(t) - \beta V(t)T(t)$$
$$\frac{d}{dt}I(t) = \beta V(t)T(t) - uI(t)$$
$$\frac{d}{dt}V(t) = N\mu I(t) - \gamma V(t)$$

The average lifespan of a free virion, $1/\gamma$, is approximately 30 minutes, which means $\gamma \approx 48$ day^{-1}. On the other hand, it was thought that 1/m, the average length of time an infected CD4+ T cell lasts before bursting to produce new virions, should be several years, implying that m must be quite small (on the order of 10^{-3} day^1). However, when drugs to treat HIV infection first became available in the mid-1990s, researchers were able to deduce a surprisingly different value from patient data and the differential equation models.

Calculations

To incorporate the effect of treatment in the differential equations model, set $\beta = 0$; that is, assume the action of the drug completely inhibits the infection process. This is a reasonable approximation and it simplifies the analysis.

Then, the equation system became,

$$\frac{d}{dt}T(t) = \lambda - \delta T(t) \tag{1}$$
$$\frac{d}{dt}I(t) = -\mu I(t) \tag{2}$$
$$\frac{d}{dt}V(t) = N\mu I(t) - \gamma V(t) \tag{3}$$

Equation (1) becomes $\frac{d}{dt}T(t) + \delta T(t) = \lambda$, δ and λ are constants.

Then let $u(x) = e^{\int \delta dt} = e^{\delta t}$, and multiply both sides by $e^{\delta t}$,

$$\Rightarrow \frac{d}{dt}[e^{\delta t} \cdot T(t)] = \lambda e^{\delta t}$$
$$\int \frac{d}{dt}[e^{\delta t} \cdot T(t)] = \int \lambda e^{\delta t}$$
$$[e^{\delta t} \cdot T(t)] = \frac{\lambda e^{\delta t}}{\delta} + C$$
$$T(t) = \frac{\lambda e^{\delta t}}{\delta e^{\delta t}} + \frac{C}{\lambda e^{\delta t}}$$
$$T(t) = \frac{\lambda}{\delta} + \frac{C}{\lambda e^{\delta t}}$$
$$T(t) = \frac{\lambda}{\delta} + C \cdot \lambda e^{-\delta t} \tag{4}$$

Move $-\mu I(t)$ to left side from equation (2) to obtain

$$\frac{d}{dt}I(t) + \mu I(t) = 0$$

Let $u(x) = e^{\int \mu dt} = e^{\mu t}$, multiply both sides by $e^{\mu t}$ to get

$$\frac{d}{dt}[e^{\mu t} \cdot I(t)] = 0$$
$$[e^{\mu t} \cdot I(t)] = C$$
$$I(t) = \frac{C}{e^{\mu t}} = C \cdot e^{-\mu t} \tag{5}$$

To get the reduced form of equation (3), substitute equation (4) into equation (3)

$$\frac{d}{dt}V(t) = N\mu C \cdot e^{-\mu t} - \gamma V(t) \tag{6}$$

Since the parameters N, μ, C, and γ are constants let $C_1 = N\mu C$
then,

$$\frac{d}{dt}V(t) = C_1 \cdot e^{-\mu t} - \gamma V(t)$$

Consequently,

$$\frac{d}{dt}V(t) + \gamma V(t) = C_1 \cdot e^{-\mu t}$$

Let $u(x) = e^{\int \gamma dt} = e^{\gamma t}$, and multiply both sides by $e^{\gamma t}$,

$$e^{\gamma t} \cdot \frac{d}{dt}V(t) + e^{\gamma t} \cdot \gamma V(t) = e^{\gamma t} \cdot C_1 \cdot e^{-\mu t}$$
$$\frac{dv}{dt}[e^{\gamma t} \cdot V(t)] = e^{\gamma t} \cdot C_1 \cdot e^{-\mu t}$$
$$\frac{dv}{dt}[e^{\gamma t} \cdot V(t)] = C_1 \cdot e^{\gamma t - \mu t} = C_1 \cdot e^{(\gamma-\mu)t}$$
$$e^{\gamma t} \cdot V(t) = C_1/(\gamma - \mu) \cdot e^{(\gamma-\mu)t} + C_2$$
$$e^{\gamma t} \cdot V(t) = N\mu C/(\gamma - \mu) \cdot e^{(\gamma-\mu)t} + C_2$$
$$V(t) = \frac{N\mu C}{\gamma - \mu}e^{-\mu t} + C_2 e^{-\gamma t} \tag{7}$$

The reduced forms for $T(t)$, $I(t)$ and $V(t)$ are,

$$T(t) = \frac{\gamma}{\delta} + C \cdot \lambda e^{-\delta t}$$
$$I(t) = \frac{c}{e^{\mu t}} = C \cdot e^{-\mu t}$$
$$V(t) = \frac{N\mu C}{\gamma - \mu}e^{-\mu t} + C_2 e^{-\gamma t} \tag{8}$$

For $T(t)$, apply the initial condition $T(0) = T_0$, $I(0) = I_0$, and $V(0) = V_0$,

$$T(0) = \frac{\lambda}{\delta} + C \cdot \lambda e^{-0 \cdot t} = T_0 \quad \Rightarrow C = T_0/\lambda - \frac{1}{\delta}$$

then,

$$T(t) = \frac{\lambda}{\delta} + T_0 \lambda e^{-\delta t} - \frac{\lambda}{\delta}e^{-\delta t}$$
$$I(0) = C \cdot e^{-\mu \cdot 0} = I_0 \Rightarrow C = I_0$$

then,

$$I(t) = I_0 e^{-\mu t}$$
$$V(0) = \frac{N\mu I}{\gamma - \mu}e^{-\mu + 0} + C_2 e^{-\gamma \cdot 0} = V_0 \Rightarrow C = V_0 - \frac{N\mu I}{\gamma - \mu}$$

Then,

$$V(t) = \frac{N\mu I}{\gamma - \mu}e^{-\mu t} + (V_0 - \frac{N\mu I}{\gamma - \mu})e^{-\gamma t},$$

where $\frac{N\mu I}{\gamma - \mu}$ is a constant. Studies ([4], [5], [6]) have shown that

$$\gamma \approx 48\text{day}^{-1}$$
$$\mu \approx 10^{-3}\ \text{day}^{-1}$$

From the formula for $V(t)$ we found that the graph of $V(t)$ on a log scale (i.e., the graph of log V) over an extended period of time (say, several weeks) will tend toward a graph of a straight line whose slope is either $-\gamma$ (the negative reciprocal of the average lifepan of a free virus) or $-\mu$ (the negative reciprocal of the average lifespan of an infected CD4+ T cell), according to whether γ or μ is smaller. See Figure 2 below and [2].

Taking $\ln$ of the function $V(t)$

$$\ln\{V(t)\} = \ln\left\{\frac{N\mu I}{\gamma-\mu}e^{-\mu t} + \left(V_0 - \frac{N\mu I}{\gamma-\mu}\right)e^{-\gamma t}\right\}$$

Since $\frac{N\mu I}{\gamma-\mu}$ and V_0 are constants, let $K_1 = \frac{N\mu I}{\gamma-\mu}$, $K_2 = V_0 - \frac{N\mu I}{\gamma-\mu}$, then the equation becomes,

$$\ln(V(t)) = \ln(K_1 e^{-ut} + K_2 e^{-\gamma t}) \qquad (9)$$

To find the asymptote, we apply the limit to $V(t)$ to obtain

$$\lim_{t\to\infty}\frac{V(t)}{t} = \lim_{t\to\infty}\frac{\ln(K_1e^{-\mu t} + K_2e^{-\gamma t})}{t}$$
$$= \lim_{t\to\infty}\ln(K_1e^{-\mu}t + K_2e^{-\gamma t})^{\frac{1}{t}}$$

Case 1: $\gamma > \mu$

$$\lim_{t\to\infty}\ln(K_1e^{-\mu t} + K_2e^{-\gamma t})^{\frac{1}{t}}$$
$$= \lim_{t\to\infty}\ln(K_1^{\frac{1}{t}}e^{-\mu})$$
$$= \ln(e^{-\mu}) = -\mu$$

Then the slope of the asymptote is $-\mu$.

Case 2: $\gamma < \mu$

$$\lim_{t\to\infty}\ln(K_1e^{-\mu t} + K_2e^{-\gamma t})^{\frac{1}{t}}$$
$$= \lim_{t\to\infty}\ln(K_2e^{-\gamma t})^{\frac{1}{t}} = \lim_{t\to\infty}\ln(K_1^{\frac{1}{t}}e^{-\gamma}) = -\gamma$$

Then the slope of the asymptote k is $-\gamma$

Case 3: $\gamma = \mu$

$$\lim_{t\to\infty}\ln(K_1e^{-\mu t} + K_2e^{-\gamma t})^{\frac{1}{t}}$$
$$= \lim_{t\to\infty}\ln((K_1+K_2) * e^{-\mu t})^{\frac{1}{t}} = -\mu = -\gamma$$

To find the y-intercept, b, of the asymptote we need to substitute the slope k into eq (9).

Since the slope of the asymptote is $-\gamma$, then

$$b = \ln(K_1e^{-ut} + K_2e^{-\gamma t}) - \ln(e^{-\gamma t})$$
$$= \ln\left(K_1\frac{e^{-ut}}{e^{-\gamma t}} + K_2\right)$$
$$= \ln(K_1e^{(\gamma-u)t} + K_2)$$

When $\gamma = \mu$,

$$\lim_{t\to\infty}\ln(K_1e^{(\gamma-u)t} + K_2) = \lim_{t\to\infty}\ln(K_1e^{0t} + K_2)$$
$$= \ln(K_1 + K_2)$$

When $\gamma < \mu$,

$$\lim_{t\to\infty}\ln(K_1e^{(\gamma-u)t} + K_2) = \ln(K_2)$$

When $\gamma > \mu$,

$$\ln(K_1e^{-ut} + K_2e^{-\gamma t}) - \ln(e^{-ut})$$
$$= \ln\left(K_1 + K_2\frac{e^{-\gamma t}}{e^{-ut}}\right)$$
$$\ln(K_2e^{(u-\gamma)t} + K_1)$$
$$\lim_{t\to\infty}\ln(K_2e^{(u-\gamma)t} + K_1)$$
$$= \ln(K_1)$$

In conclusion, the asymptote equations $y = kt + b$ are,

When $\gamma > \mu$, $y = -\gamma t + \ln(K_1)$

When $\gamma < \mu$, $y = -\gamma t + \ln(K_2)$

When $\gamma = \mu$, $y = -\gamma t + \ln(K_1 + K_2) = -\mu t + \ln(K_1 + K_2)$

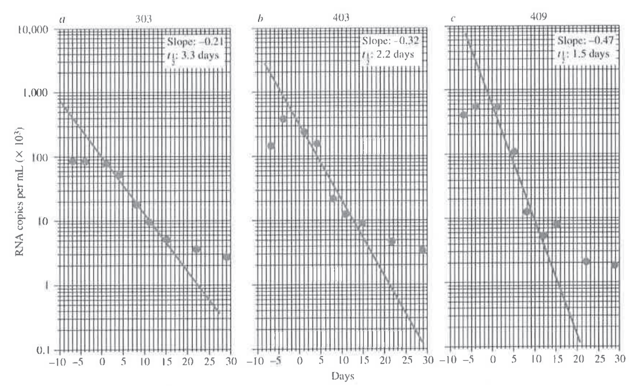


Figure 2: Viral load decrease in three HIV patients

8 Conclusion

After adopting a research-based pedagogical approach to teaching Introduction to Ordinary Differential Equations at Queensborough Community College we noticed a marked improvement in student performance on the cumulative final examination. The class average increased by about 20 percentage points.

Student attitude and engagement in lecture was observed to be more positive.

While we have limited data for comparative purposes, the results observed in data are encouraging and we plan on continuing this pedagogical approach going forward.

References:

[1] https://en.wikipedia.org/wiki/Community_college.

[2] R. Kent Nagle, Edward B. Saft, Arthur David Snider, *Fundamentals of Differential Equations*, Addison–Wesley Publishers, 9th Edition

[3] Earl D. Rainville, Philip E. Bedient, Richard E. Bedient, *Elementary Differential Equations*, Prentice Hall Publishing Co., Inc.

[4] D. D. Ho et al., "*Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection*," *Nature*, 1995

[5] Dynamics of HIV Infection, Courtesy of Glenn Webb, Vanderbilt University G. M. Shaw et al., "*Viral Dynamics in Human Immunodeficiency Virus Type I Infection*," *Nature*, 1995

[6] *Quantifying Residual HIV-1 Replication in Patients Receiving Combination Anti-retroviral Therapy.* Zhang LQ, Ramratnam B, Tenner-Racz K, He YX, Vesanen M, Lewin S, Talal A, Racz P, Perelson AS, Korber BT, Markowitz M, and Ho DD. *New England Journal of Medicine* 340:1605–1613, 1999.

[7] *The Decay of the Latent Reservoir of Replication-Competent HIV-1 is Inversely Correlated with the Extent of Residual Viral Replication During Prolonged Anti-retroviral Therapy.* Ramratnam B, Mittler JE, Zhang LQ, Boden D, Hurley A, Fang F, Macken CA, Perelson AS, Markowitz M, and Ho DD. *Nature Medicine* 6:82–85, 2000