Possibilities and Limitations of Deterministic Nonlinear Dynamic Model of the Stock Market

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Abstract: - This paper proposes a nonlinear dynamical model of stock market. Dynamical variables of the model are the variation of ask and bid price relative to equilibrium values and difference between numbers of market agents in a-state and p-state. A particular market agent being in a-state has maximum amount of valuable information about financial asset and has minimum information being in p-state. This model explains the impossibility of existence of an equilibrium state of the market, shows the presence of deterministic chaos in a stock market and fractal financial time series. The results of the nonlinear dynamical analysis and statistical analysis of the empirical financial time series are presented. We show the results of nonlinear analysis for the model as an open nonequilibrium system, as well as comparison with empirical results.

Key-Words: - stock market model, stock market indexes, deterministic chaos, financial time series, correlation dimension, fractal dimension, 1/f noise, long memory, q-Gaussian distribution.

1 Introduction

From the second half of the XX century the general trend of science development is the penetration of the ideas and methods of physics into other natural and humanitarian disciplines. Methods of physical modelling are often used in sciences such as demography, sociology and linguistics.

An interdisciplinary research field, known as econophysics, was formed in the middle 1990s as an approach to solve various problems in economics, such as uncertainty or stochastic processes and nonlinear dynamics, by applying theories and methods originally developed by physicists. The term “econophysics” was coined by H. Eugene Stanley in order to describe the large number of papers written by physicists in the problems of (stock and other) markets (for econophysics reviews see refs. [1-4]).

Current state of theoretical economics allows one to effectively use advanced methods of physico-mathematical modelling for economical system. A remarkable example is applying nonlinear dynamics to analysis of financial time series [5,6]. Moreover, in 1963 Benoit B. Mandelbrot[7] during his research of cotton prices found out that the prices follows a scaled distribution in time. That discovery originated a new approach in market research called fractal market analysis [8]. A systematic research of deterministic chaos in financial markets started from works of Robert Savit [9].

By the end of 20^{th} century there were formed two lines of research of deterministic chaos in financial markets. The first one is related to discovery and analysis of deterministic chaos in the structure of financial markets. Studies of that kind are usually based on qualitative characteristic and quantitative measures of chaos [10], and their results show conclusively that deterministic chaos exists in financial markets [11-19]. The second line connected to retrieval of explicit form of such dynamical systems. Definitely, construction of such models is far more complex problem. That is why the number of relevant publications on this topic is relatively small. The most comprehensive survey of mathematical models of financial markets can be found in the book of R.J. Elliott and P.E. Kopp [20]. Although the book and other relevant publications contain numerous conceptual models, we have not found any econophysical model of a stock market that can explain its fundamental functioning mechanisms.

Thus, the purpose of this work is building of econophysical model of a stock market using parallels between market functioning and physical principles of laser operation and is defining possibilities and limitations of deterministic nonlinear dynamic model of the stock market.

This paper is organized as follows. In section 2 we present the simple econophysical model of stock market as an open nonequilibrium system, including the definition of the dynamic variables and parameters, and the results and discussions, including the definition of the regular and the chaotic states of the stock market. In section 3 we present the results of nonlinear dynamical and statistical analysis of empirical financial time series (stock market indexes). In section 4 we conclude this paper, defining possibilities and limitations of deterministic nonlinear dynamic model of the stock market.

2 Construction of the Model

2.1 Class of thermodynamical systems

In general, theoretical economics consider the problem of constructing mathematical concepts for modeling of economic processes, which consist of numerous internal elements and incorporates several influential forces. Therefore, theoretical economics provide basic ideas, propositions and methods as a ground for creating a mathematical model of economic dynamics that clearly reflects the evolution of complex economic systems according to specific economical principles. The key property of such systems is synergy that makes oneself evident in the fact that the whole system obtains some characteristics that its elements do not originally have.

Although the economic dynamics is a new and developing field of study, it originates from natural sciences where the methods of dynamical modeling have already been successfully applied to variety of problems. However, mathematical concepts, which work fine in electrodynamics or physics of liquids and gas, cannot be applied to economic objects "as is". On the contrary, the best results can be achieved only when the model is created with due consideration of the properties of the object being modeled.

In physics, there is a class of the models that was constructed according to the guidelines explained above. These systems are called thermodynamical systems. Originally, the systems of this class were developed for modeling aggregated gas and fluid characteristics like temperature or pressure, but based on characteristics of gas (or fluid) particles, which are simply single molecules that do not have these characteristics. That is why we do not propose to use already built physical models "of the box", but to use the same principles in constructing custom economic dynamical models.

Complex thermodynamical systems are the systems that consist of numerous elements, similar to each other ("atoms"). So, these atoms interaction follows some determined rules. As a result, the evolution of the whole system depends in a complicated way from the evolution of every atom.

In thermodynamical systems, there are three levels of detail:

Local micro-dynamics S_0**:** at this level interaction of an atom with others is considered. It is an ontogenetic level of the system, which forms all other dynamical effects.

Meso-dynamics S_1**:** this "convoluted" level considers the averaged motion characteristics of each atom.

Macro-dynamics S_2**:** observed indicators as a function of aggregated system states from two previous levels.

Let us denote an observed variable with index α at the moment of observation $t \in T$ as x_t^α. Therefore, the tuple of simultaneously observed variables which specify the global macro state of the system is

$$\mathbf{x}_t \equiv \left(x_t^1, x_t^2, ..., x_t^\alpha, ... \right). \qquad (1)$$

Assume that every single atom a_i in Λ belongs to basic set of system elements and has its own space of internal states. We denote the internal state of element a_i; $i = 1,..., N$ at the moment t as ω_t^i. By gathering all ω_t^i, we have phase micro states for the whole system at the moment t:

$$\boldsymbol{\omega}_t \equiv \left(\omega_t^1, \omega_t^2, ..., \omega_t^N \right). \qquad (2)$$

Important remark: here we consider big, but finite systems, so the set of atoms Λ is large enough.

Let Φ be the union of all possible macro states and Ω union of all micro states.

$$\Phi \equiv \mathbf{X}_t, \boldsymbol{\omega} \Omega \in \quad {}_t . \qquad (3)$$

Then Ω is the underlying space of the system.

It is worth to notice, that Ω is an abstract mathematical construct, since, in general, one cannot observe the micro states of the system. So, one can only make an assumption on Ω and on kind of interaction between the atoms, from which the abstract stochastic dynamics of micro level is derived. Further, the global dynamics on macro level can be reconstructed by applying the convolution techniques to micro-dynamical information of the system, which is a typical task of stochastic dynamics.

It is fact that every stochastic dynamical system being without external influence tends to the nearest local equilibrium state, both on micro and macro levels. The process of system transition to equilibrium is called relaxation. From the research transitional processes in big stochastic dynamical systems, the effect of self–organization and creation of ordered structures was discovered. The necessary conditions for new dynamical information are strong deviation from equilibrium states and nonlinear interaction between the elements. This effect is called synergy, and it works in various complex thermodynamical systems. Therefore, synergy shows itself in economics due to the fact, that the big economical systems like global markets can also be modeled with this kind of dynamical systems.

Back to the micro level of the system, assume, that for every atom $a_t^i \in \Lambda, A^i = \omega_t^i$ is an internal phase space for a^i. Then

$$A = \xi^i \equiv \left(a^i, A^i\right), i = 1, 2, ..., N \ . \tag{4}$$

is a structure of the micro level. It includes all system atoms and their possible internal states.

Moving further, let us consider the micro-dynamics of the system. Again, assume that at moment t the system is in state $\boldsymbol{\omega}_t$ (2). Then

$$\boldsymbol{\omega}^0 \xrightarrow{\hat{L}_1} \boldsymbol{\omega}^1 \xrightarrow{\hat{L}_2} ...\boldsymbol{\omega}^{f-1} \xrightarrow{\hat{L}_f} \boldsymbol{\omega}^f \tag{5}$$

is a generalized trajectory of system evolution. Here the trajectory starts with initial system state $\boldsymbol{\omega}^0$ at time $t = 0$ and reaches the state $\boldsymbol{\omega}^f$ at time $t = f$. In this model, local micro movements $\boldsymbol{\omega}^{k-1} \xrightarrow{\hat{L}_k} \boldsymbol{\omega}^k$ are determined by stochastic dynamical operators $\hat{L}_k$. Thus, the problem of finding the micro dynamics of the system reduces to definition of such operators. In general, for complex dynamical systems, operators $\hat{L}_k$ are nonlinear and not local. Moreover, they have functional dependence on the previous system states or, in other words, are functions with “memory”. Due to these characteristics, these functions “enlace” the trajectories of possible system evolutions.

Therefore, the structure

$$\tilde{S} \equiv \left(A, \hat{L}(\xi)\right) \tag{6}$$

describes the global information dynamics in the system.

$$\boldsymbol{\omega}^0 \xrightarrow{\hat{L}_1} \boldsymbol{\omega}^1 ... \xrightarrow{\hat{L}_f} \boldsymbol{\omega}^f \equiv \left[\boldsymbol{\omega}^k\right]_{k=0}^{k=f} \tag{7}$$

$$\boldsymbol{\omega}_i^0 \xrightarrow{\varphi_1} \boldsymbol{\omega}_i^1 ... \xrightarrow{\varphi_f} \boldsymbol{\omega}_i^f \equiv \left[\boldsymbol{\omega}_i^k\right]_{k=0}^{k=f} \tag{8}$$

Equations (7) and (8) represent the segments of global system trajectory and trajectories of single atoms respectively. Here we introduce the evolution operators $\hat{T}\left(\Delta t, \boldsymbol{\omega}^0\right) : \Delta t = t_f - t_0$ that turn the system from initial state $\boldsymbol{\omega}^0$ into $\boldsymbol{\omega}^f$. Finally, the set of such operators form the informational and “material” flows in the system. Then, taking this set of evolution operators with $\Delta t \to 0$ and system phase space Ω, we get the Liouville flows.

The Liouville flows is well known mathematical abstraction, that allows us to connect the micro level of the system with its meso- and macro-levels, by convolving and aggregating dynamical information.

The key concept of the Liouville flows is probability density function of the system $\rho(\mathbf{x}, t)$ being in some state near the point $\mathbf{x} \equiv \left(x^1 = \omega^1, ..., x^N = \omega^N\right)$ at some moment of time $t \in T$. Assume, that

$$dx^N = \prod_{i=1}^{N} dx^i$$

is an element of volume in variable space x^i. Then

$$dW(\mathbf{x}, t) = \rho(\mathbf{x}, t) dx^N \tag{9}$$

is the probability that system will be in some state near the point $\mathbf{x}$. To convolve the information in function $\rho(\mathbf{x}, t)$ we integrate it using all variables except x_s. As a result, we have an atomic state function for atom a_s

$$\varphi_s(x, t) = \frac{1}{K} \int \rho(x_s, t) dx_s \ , \tag{10}$$

where $1/K$ is a normalizing constant. Thus, functions $\varphi_s(x, t)$ become the generalized variables that describe system state on meso-dynamical level. In fact, they are the result of aggregated influence on element a_s from the rest of the system. Let us denote the set of atomic state function $\varphi_s(x, t)$ as $\boldsymbol{\Phi}(\mathbf{x}, t)$.

Together atomic state functions defines meso-dynamics of the system.

$$\frac{\partial}{\partial t} \boldsymbol{\Phi}(\mathbf{x}_t, t) = \tilde{F}\left(\boldsymbol{\Phi}(\mathbf{x}_t, t)\right) \tag{11}$$

The equations (11) in thermodynamical systems are called kinetic equations.

Next step is to move from meso-dynamics to macro dynamics by convolving aggregated atomic state functions into aggregated global system state functions and binding them with observed indicators. No need to say, that these global system state functions depend in a particular way on global dynamical variables, which, in turn, are defined by atomic state functions of meso-dynamical level.

2.2 Model assumptions

2.2.1 Stock market is a macroscopic system
Assume that the stock market is a dynamical system that consists of numerous market agents (investors) ($N >> 1$) (fig. 1). Modeling of such systems does not require detailed analysis of interactions between the agents on the micro-level. For macrosystem description we use macroscopic parameters and dynamical variables of the system. For macroscopic dynamical variables we have chosen aggregated flows of ask and bid price changes and dynamical difference of market agents in specific states.

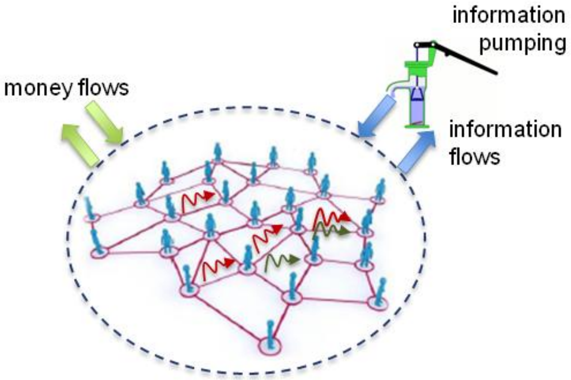

Figure 1. The stock market as a nonequilibrium open system.

2.2.2 Stock market is a point autonomous dynamical system

$$\dot{\mathbf{x}}_t = \Phi(\mathbf{x}, \boldsymbol{\beta}_t), \tag{12}$$

where $\boldsymbol{\beta} \in R^m$ is a m-dimensional vector of parameters, $\dot{\mathbf{x}}_t = \frac{d\mathbf{x}}{dt}$.

This statement goes without saying, as it depends only on chosen modelling approach and objectives. However, the choice of (12) as the base model is made for reason. It is based on tests that the constructed mathematical model agrees with empirical (observed) data, for which we used available financial time series of ask/bid stock prices.

2.2.3 Every market agent can be in one of two possible states: active ($|a\rangle$-state) or passive ($|p\rangle$-state)

A particular market agent being in $|a\rangle$-state has maximum amount of valuable information about financial asset ($I_{|a\rangle}$) and has minimum information ($I_{|p\rangle}$) being in $|p\rangle$-state.

The agent being in $|a\rangle$-state is able to generate local demand on deal with the asset and send an "ask-quantum" to other agents. If the agent is in $|p\rangle$-state (he/she does not have enough valuable information about the asset), then the agent's rational decision is do not generate demand on deal ("bid-quantum"). Moreover, for the agent in $|p\rangle$-state generating of a deal offer depends on the agent's reaction on received "ask-quantum" (fig. 2a) or can be his or her own decision (fig. 2b). General pattern in stock markets is that local "ask" waves ("quanta") induce local "bid" waves ("quanta").

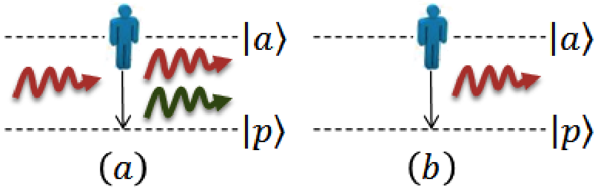

Figure 2. Agents generate «ask-quanta» (red) and «bid-quanta» (green). (a) Forsed generation of an «bid-quantum». (b) Spontaneous generation of an «ask-quantum».

2.2.4 Stock market is a nonequilibrium open system
Indeed, stock market is an open system that continuously exchanges information and money flows with the external world. Sources of external information include corporate financial reports, financial news feeds, stock-ticker data and others. This information flow, in some sense, "pump up" the stock market, making inverse population of market agents: $N_{|a\rangle} >> N_{|p\rangle}$, where $N_{|a\rangle}$ is the number of agents being in $|a\rangle$-state, $N_{|p\rangle}$ is the number of agents being in $|p\rangle$-state.

With acceptable accuracy, the distribution of number of agents by their states can be represented as follows:

$$N_{|a\rangle} = N_{|p\rangle} \exp\left(-\frac{I_{|a\rangle} - I_{|p\rangle}}{\theta} \right), \tag{13}$$

where θ is average intensity of stochastic interactions between market agents Simple analysis of equation (13) allows to identify two macroscopic states of the market: stable equilibrium state and nonequilibrium state. If $I_{|a\rangle} - I_{|p\rangle} >> \theta$, then

$N_{|a\rangle} << N_{|p\rangle}$. In this case, the system is in stable equilibrium state. Otherwise, if $I_{|p\rangle} - I_{|a\rangle} >> \theta$, then $N_{|a\rangle} >> N_{|p\rangle}$. This case corresponds to nonequilibrium state of the system.

Taking into account continuous information pumping, stock market is always functioning in nonequilibrium state, making "avalanches" of ask and bid "quanta". Due to information pumping, the equilibrium state is almost unreachable. It is crucially important, that existence of chaotic states is a fundamental property of nonequilibrium open systems [24, 25].

2.3 Dynamical variables of the model and relationship between them

2.3.1 Dynamical variables of the models

Let us define dynamical variables in equation (12) for constructing nonlinear dynamical model of stock market: $x_t \equiv a_t - a_{eq}$ is the variation of "ask" price (a_t) relative to equilibrium value (a_{eq}) is the "ask" price in equilibrium state); $y_t \equiv b_t - b_{eq}$ is the variation of "bid" price (b_t) relative to equilibrium value (b_{eq}) is the "bid" price in equilibrium state); $z_t \equiv N_{|a\rangle}(t) - N_{|p\rangle}(t)$ instantaneous difference between numbers of agents in $|a\rangle$-state and $|p\rangle$-state.

The choice of these dynamical variables responds to possibility to test whether the constructed dynamical system agrees with empirical data, for which we have chosen available time series of ask and bid prices from real stock markets. However, it is impossible to compare the third dynamical variable with actual data, due to available datasets does not contain these values. Thus, the fit test can be performed only with two dynamical variables.

Let us establish connections between dynamical variables and their change rates.

2.3.2 Ask price dynamics

Variation rate of the ask prices is defined by concurrency of two factors: rate decrease due to market relaxation ($-\alpha x_t$) and rate increase due to growing variation of bid prices ($+\beta y_t$):

$$\dot{x}_t = -\alpha x_t + \beta y_t \tag{14}$$

Term $-\alpha x_t$ in (3) is necessary due to relaxation of nonequilibrium system. According to Le Chatelier's principle [26], when the system at equilibrium is subjected to change by external force, then the system readjusts itself to counteract (partially) the effect of the applied change. Indeed, without term $+\beta y_t$ the equation (14) has the following form:

$$\dot{x}_t = -\alpha x_t\,. \tag{15}$$

A solution of differential equation (15) is a function of form $x_t = A\exp(-\alpha t)$. Therefore, $a_t \to a_{eq}$ when $t \to \infty$ (stock market tend to stable equilibrium). In equation (15) α – relaxation parameter, related to relaxation time (τ_1) according to: $\alpha = 1/\tau_1$. Term $+\beta y_t$ in (14) refers to the fact, that increase of bid price variation leads to increase of variation rate of ask prices.

2.3.3 Bid price dynamics

Variation rate of the bid prices is defined by concurrency of two factors: rate decrease due to market relaxation ($-\gamma y_t$) and rate increase due to $+cx_t z_t$.

$$\dot{y}_t = -\gamma y_t + cx_t z_t \tag{16}$$

Presence of the first term in (16) is explained by Le Chatelier's principle. Term $+cx_t z_t$ is explained as follows: "bid quantum", on which every market agent reacts considering "ask quanta" flow, is proportional to ask price variation and depends on the agent's current state ($|a\rangle$-state or $|p\rangle$-state).

2.3.4 Dynamics of difference between numbers of market agents in $|a\rangle$-state and $|p\rangle$-state

$$\dot{z}_t = \varepsilon(I_0 - z_t) + kx_t y_t \tag{17}$$

Again, term $-\varepsilon z_t$ is in equation (17) due to Le Chatelier's principle. Parameter I_0 refers to intensity of external information pumping, so instantaneous difference between numbers of agents in $|a\rangle$-state and $|p\rangle$-state grows with increase of I_0. Term $+kx_t y_t$ represents the power that the aggregated ask price variation spends on creation of the aggregated bid price variation.

2.4 Modeling results and their interpretation

The system of differential equations (14), (16) и (17) represents the well-known Lorenz–Haken equations [27]:

$$\begin{cases} \dot{x}_t = -\alpha x_t + \beta y_t \\ \dot{y}_t = -\gamma y_t + c x_t z_t \\ \dot{z}_t = \varepsilon\left(I_0 - z_t\right) + k x_t y_t \end{cases} \quad (18)$$

System (18) is one of the most studied 3-dimensional dynamical systems. General properties of (18) are presented in works [28, 29]. Let us consider (18) as a system with one control parameter I_0. From changing control parameter's value, we can make two important conclusions about system (7).

If $\beta c/\alpha\gamma \cong 1$ and $0 < I_0 < 1$, then $a_t \to a_{eq}$, $b_t \to b_{eq}$ and $N_{|a\rangle}(t) \to N_{|p\rangle}(t)$ as $t \to \infty$. In case of relatively small intensity of external information pumping, the stock market tends to stable equilibrium (fig. 3). However, practically this stable equilibrium state cannot be reached, since the market is an open system with permanent eternal information pumping.

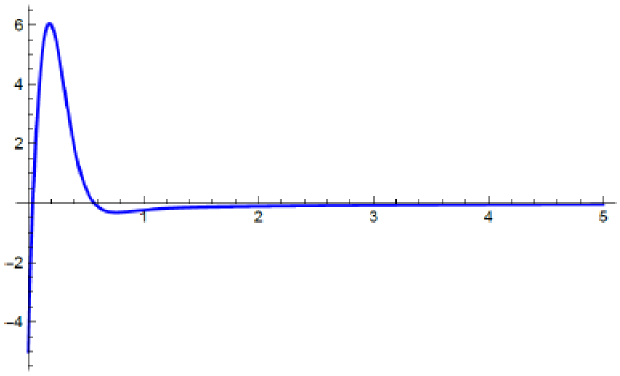

Figure 3. Asymptotically stability solution of (7).

If $\beta c/\alpha\gamma \cong 1$ and $I_0 \cong 28$, then the stock market functions as an open nonequilibrium system with deterministic chaos (fig. 4). It is worth to mention that such behaviour is typical for a financial market with considerably intense external information.

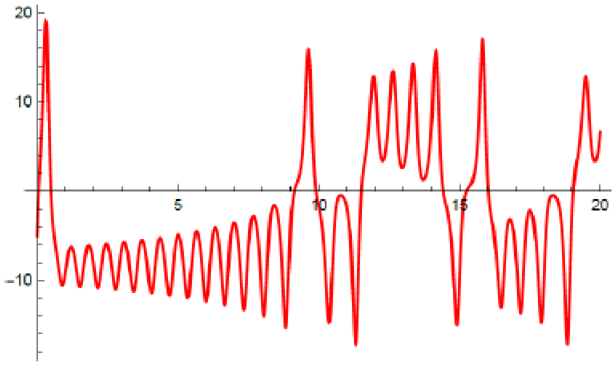

Figure 4. Chaotic solution of (7).

3-dimensional dynamical model (7) explains some properties of the stock market functioning such as fractality (fractal dimension (D_F) equals 1.497), chaotic nature (correlation dimension (D_C) equals 1.896) and absence of memory (Hurst exponent (H) equals 0.5028) of financial time series (FTS) [28].

Generally, a dynamical system can be defined as follows [8]: its state at time t is a point $\sigma(t)$ in some E-dimensional phase space R^E, and the trajectory between the times t and $t+\Delta t$ is determined by rules in which t does not enter explicitly. Each point in phase space can be taken as the initial state $S(0)$ at $t=0$, and is followed by an orbit defined by the $S(t), \forall t > 0$.

A dynamical system is said to have an attractor if there exists a proper subset $A \in R^E$, such that almost all starting points $S(0)$ and t large enough, $S(t)$ is close to some point of A. An attractor which has fractal (not integer) dimension is called strange attractor. So, the term chaotic dynamical system usually refers to a dynamical system with strange attractor. However, sometimes the word "chaotic" may be used just as opposite to "stochastic", in order to emphasize the difference between random and determined systems.

The weakness of this model lies in significant discrepancy between empirical and theoretical trajectories of FTS. Moreover, it is impossible to fit theoretical trajectories to observed data by varying control parameters (in a range of chaotic state) of the dynamical system. The dynamical system (7) has 3 equilibrium points for any values of control parameters in a range of chaotic state. Therefore, theoretical probability density function (PDF) is a three-modal distribution (three maxima of the PDF) (fig. 5) and x_t is a white random process. This PDF is not fat-tailed distribution [30]. The white noise signal is not a signal of catastrophic events.

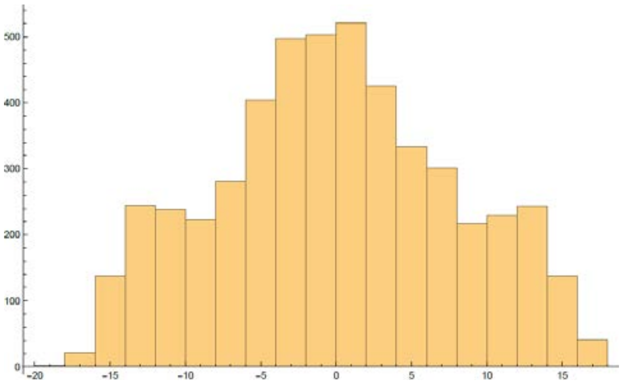
Figure 5. Probability density function of the theoretical FTS.

3 Analysis of Empirical Financial Time Series

For analysis of empirical FTS, we chose the following stock market indexes: 30Y T-Bond INT Rates (1), ASX Australia (2), BOVESPA Brazil (3), CAC 40 (4), DAX 30 (5), DJ Industrial (6), FTSE 100 (7), ISEX Spain (8), NASDAQ Comp (9), Nikkei 225 (10), S&P 500 (11), Shanghai Composite (12), USD Index (13), VIX Volatility Index (14).

3.1 Nonlinear analysis

The nonlinear analysis was conducted for all chosen TTS. Such measures as correlation dimension (D_C), Hurst exponents (H), power of power spectral density ($\beta = 2H + 1$) and fractal dimension (D_F) were calculated (table 1).

Table 1. Results of nonlinear FTS analysis

Stock market index	D_C	H	β	D_F
(1)	2.448	0.665	2.330	1.335
(2)	2.300	0.617	2.234	1.383
(3)	2.329	0.683	2.366	1.317
(4)	3.895	0.634	2.268	1.366
(5)	4.045	0.702	2.404	1.298
(6)	3.234	0.694	2.388	1.306
(7)	3.433	0.622	2.244	1.378
(8)	2.560	0.689	2.378	1.311
(9)	3.483	0.637	2.274	1.363
(10)	2.004	0.682	2.364	1.318
(11)	3.717	0.701	2.402	1.299
(12)	3.440	0.708	2.416	1.292
(13)	2.761	0.646	2.292	1.354
(14)	3.471	0.681	2.362	1.319

Determination of the correlation dimension [31] for a supposed chaotic process directly from experimental time series is often used to get information about the nature of the underlying dynamics (see, for example, contributions in ref. [32]). In particular, such analysis has been made to support the hypothesis that the time series are generated from the inherently low-dimensional chaotic process [32]. The geometry of chaotic attractors can be complex and difficult to describe. It is therefore useful to understand quantitative characterizations of such geometrical objects.

One of these characterizations is D_C. D_C of the attractor of dynamical system can be estimated using the Grassberger–Procaccia algorithm [31].

D_C has several advantages in comparison to the other dimensional measures:

1. if D_C is finite, then a FTS is a chaotic time series (generated by a dynamical system);
2. if D_C is infinite, then a FTS is a stochastic time series (generated by a purely random process).

For calculation of D_F we used the algorithm described in a paper [33]. If $D_F > D_T$ (D_T is a topological dimension of the FTS, that equals 1 for all time series), then the FTS is a random fractal.

A value of $H = 2 - D_F$ allows to give a noise classification ($1/f$-classification, where f is a signal frequency) of the FTS [34]:

1. if $0 < H < 0.5$, then the FTS is characterized by anti-persistence (the time series changes the tendency more often, than a series of independent random variables) and represents a process with $1/f$ noise or a pink noise;
2. if $0.5 < H < 1$, then the FTS is characterized by persistence (the time series is characterized by the effect of the long memory and has an inclination to follow the trends) and represents a process with $1/f^{\beta}$ ($\beta > 2$) noise or a black noise;
3. if $H = 0.5$, then FTS represents a process with the absence of memory or a white noise.

Empirical FTS is a black random process. Most of the time series, which can be observed in existence, can usually be related to one of the above-mentioned classes [35,36]. Thus, the time series observed in turbulence processes, show the best correlation with the pink noise. The black noises can be registered in floods, a solar activity, statistics of the natural and induced catastrophes. The black noise indicates long term persistence and long memory.

There is a simple scaling relation, connecting β and H [34]: $\beta = 2H + 1$. The results for β are shown in the table 1.

3.2 Statistical analysis

Point and interval estimates of the parameters of empirical PDF are the questions of statistical analysis.

The PDFs of returns and volatilities of many financial time series have power low tails at $t \to \infty$ [37]:

$$p(x) \sim x^{-(1+\gamma)}, \gamma > 0 . \quad (19)$$

The PDF (8) belongs to the class of the power law PDFs or the fat-tailed PDFs. The fundamental difference of the PDF (19) from the compact distributions is the fact, that those events, which fall on the distribution tail area, take place not so rarely to be neglected.

Figure 6 provides PDF for empirical NASDAQ time series of returns. The PDFs of other stock market indexes have a similar form.

Visually, these PDFs correspond to the Gaussian (compact distribution) or to the generalized Gaussian fat-tailed PDF (19), for example, the *q*-Gaussian distribution [38-40]:

$$\rho(x) = \frac{\sqrt{\chi}}{C_q} \exp_q\left(-\chi x^2\right), \quad (20)$$

where $\exp_q(x) \equiv \left[1 + (1-q)x\right]^{1/(1-q)}$, C_q – normalization factor.

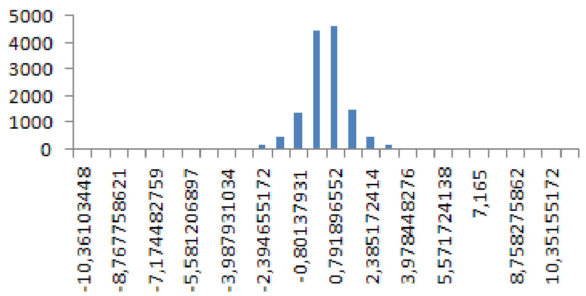

Figure 6. PDF of the NASDAQ time series of returns.

The distribution (20) is a two-parameter generalization ($q < 3$ is a shape parameter, $\chi > 0$ is a rate parameter) of the one-parameter Gaussian distribution.

In the limit $q \to 1$, PDF (20) recovers the usual Gaussian distribution, so $q \neq 1$ indicates a departure from Gaussian statistics. For $q > 1$, the tails of q-Gaussian decrease as power laws [40],

$$\rho(|x|) \sim |x|^{-2/(q-1)} . \quad (21)$$

Table 2 contains the estimated values for parameters of PDF (20) obtained by the maximum likelihood method [41].

Table 2. Point and interval estimations of the PDF (20) parameters and γ-parameters of the PDF (19)

Stock market index	q	χ	γ
(1)	2.505±0.005	1.329±0.074	0.329
(2)	2.675±0.025	1.194±0.086	0.194
(3)	2.734±0.038	1.153±0.069	0.153
(4)	2.305±0.015	1.533±0.072	0.533
(5)	2.630±0.022	1.227±0.076	0.227
(6)	2.532±0.036	1.305±0.084	0.305
(7)	2.665±0.032	1.201±0.072	0.201
(8)	2.689±0.033	1.184±0.076	0.184
(9)	2.432±0.022	1.397±0.084	0.397
(10)	2.638±0.012	1.221±0.042	0.221
(11)	2.531±0.021	1.306±0.032	0.306
(12)	2.682±0.038	1.189±0.036	0.189
(13)	2.331±0.024	1.503±0.025	0.503
(14)	2.834±0.015	1.091±0.018	0.091

Thus, according to the point and interval values of the PDF (20), shown in the table 1, the following conclusion can be made: PDF is a fat-tailed PDF (($1+\gamma$)-values vary from 1.091 to 1.533).

q-Gaussian distribution takes place by the maximization of the Tsallis entropy [42] considering definite limitations. Tsallis entropy as a non-additive generalization of the Boltzmann–Gibbs entropy has the following form:

$$T_q = \frac{1}{q-1}\left(1 - \sum_{i=1}^{q} p_i^q\right). \quad (22)$$

The probability $p_i = N_i / N(\varepsilon)$ can be estimated in much the same way as that one used in the Renyi entropy: N_i is a number of system elements for the i-element of the ε-partition; $N(\varepsilon)$ – is a full number of elements of the given ε-cover. In contrast to all entropy types, the Tsallis entropy is nonadditive. Being applied to the financial market (such as, for example, stock market) it gives a possibility to correctly describe a financial market, where any market agent interacts not only with the nearest market agent or several nearest market agents, but also with the whole market or some of its parts. Besides, from (22) it follows that T_q is concave by $q > 0$ and convex by $q < 0$.

Thus, the entropy description of the stock market, based on Tsallis statistics is appropriate for studying of evolution of the stock market that contains a large number of market agents who interact with each other in a particular way and specifically every market agent can interact not only with his or her nearest neighbors but also with remote market agents (for example, see refs [43,44]).

The ability of the system to have a “long” memory for its past, and the ability of the system elements to "feel" each other wide-apart, can be considered by the emergent behavior. At the statistical level the emergent behaviors are usually related to the long-range time and space correlations. The matter concerns the time-series with the long memory, or those time-series, autocorrelation function (ACF) of which decreases slowly. The expression "a long-range dependence", which is sometimes used to refer to noise, has also been used in the other contexts with somewhat different meanings. "Long memory" and other variants are also sometimes used in the same way.

Figure 7 provides the ACF of theoretical FTS.

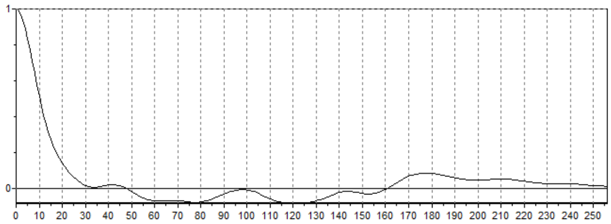

Figure 7. ACF of the theoretical FTS.

Figure 8 provides the ACF of NASDAQ FTS. The ACFs of other stock market indexes have a similar form.

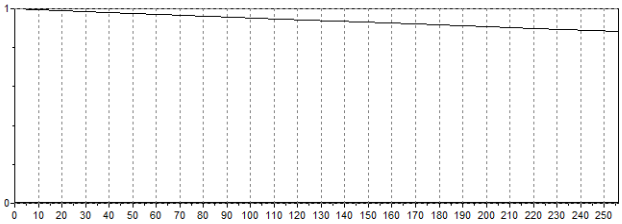

Figure 7. ACF of the NASDAQ FTS.

Thus, theoretical FTS represents a process with the absence of memory. The empirical FTS represents processes with the long memory.

4 Conclusion

The constructed simple econophysical model of a stock market as an open nonequilibrium system allows to explain the following phenomena:

1. Unrealizability of equilibrium state of the market as well.

2. Appearance of deterministic chaos in the market (D_C is finite).

3. There exists a fractal financial time series ($D_F > D_T$).

These phenomena are explained only by quantitative characteristics of external information pumping as a control parameter of the system.

However, this simple model cannot explain several other important phenomena, such as

– heavy-tailed distribution of financial time series ($(1+\gamma)$-values vary from 1.091 to 1.533, see table 2);
– financial time series as a black random process ($1/f^{\beta}$ noise, where β varies from 2.404 to 2.292, see table 1);
– financial time series as a random process with long memory (see fig. 7).

We suppose, that explanation of heavy-tailed distribution of financial time series requires modification of dynamical system by introducing the noise of specific kind (such as parametric noise, for example, intensity of external information pumping is a random variable with q-exponential distribution).

Particularly, the form of time series with heavy-tailed distribution can be achieved by including a power-law multiplicative noise [43-45], what is the subject of our further research.

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