Numerical Solution of Integral Equation for the Early Exercise Boundary of American Put Option

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Abstract: The paper is focused on numerical approximation of early exercise boundary within American put option pricing problem. Assuming non-dividend paying, American put option leads to two disjunctive regions, a continuation one and a stopping one, which are separated by an early exercise boundary. We present variational formulation of American option problem with special attention to early exercise action effect. Next, we discuss financially motivated additive decomposition of American option price into a European option price and another part due to the extra premium required by early exercising the option contract. As the optimal exercise boundary is a free boundary, its determination is coupled with the determination of the option price. However, the integral equation is known for determination of early exercise boundary. We propose an iterative procedure for numerical solution of that integral equation. We discuss the construction of initial approximations, and we also describe the steps of our submitted procedure in details. Finally, we present some numerical results of determination of free boundary based upon this approach. All computations are performed by the sw Mathematica, version 11.1.

Key-Words: American put option, early exercise premium, early exercise boundary, pricing problem, integral equation, numerical method

1 Introduction

An American put option is a financial contract between the writer and the holder of the option. It gives the holder the right, but not an obligation, to sell the amount specified in the contract of the underlying asset at the prescribed strike price and at any time between the writing and the expiration of the option contract given. The problem of efficient and accurate valuation of American options has a large literature. In accordance with our desires, we selected the books [5], [6], [9], and [10], in particular, discussing American option pricing problems with different levels of abstraction. In general, we know that in contrast to European options.

The main difficulty to valuate American options is caused by the exercising flexibility. Such exercising flexibility gives American option more profiteering opportunity than European option offers, and it means that payoff functions of American options take the following general form, where a subscript *C*, and *P* denotes an option type, i.e. *call*, and *put*, respectively

$$V_c(S,t) \geq \max(S - K, 0), \qquad (1)$$

$$V_p(S,t) \geq \max(K - S, 0), \qquad (2)$$

here $S \geq 0$, is an underlying risky asset price at the time instant t, e.g. stock price, $0 \leq t \leq T$, T is the option expiration date, and K is the option exercise price, called strike price, too. From here on, we also write simply S instead of precise S_t, unless stated otherwise.

It is worth to note, that payoff functions of European options take the same forms formally, except that S is replaced by S_T, and $t = T$, exclusively.

Now, let us focus on non-dividend paying American put option provided the standard assumptions made on the underlying asset and completeness of the financial markets as discussed in [5] and [9], in detail.

In fact, when S falls below a certain point, one should exercise the option immediately to avoid loss. For example, if at time t, the current stock price S_t fulfils relation $S_t < K(1 - e^{-r(T-t)})$, then the option holder should exercise the option immediately. Here, r denotes a risk-free interest rate. In fact, the payoff at the option expiration data T will never exceed K in any case.

So, if the option is exercised at the time t, the immediate gain is

$$K - S_t > K - K(1 - e^{-r(T-t)}) = K\, e^{-r(T-t)},$$

and by depositing this gain in a saving account, assuming a regular compounding process applies, the total payoff will exceed K at $t = T$, clearly.

Hence, a thorough inspection of relation (2) provides an idea that for American put option there exist two disjunctive subregions, denoted Σ_1 and Σ_2, covering $\Sigma = [0,+\infty) \times [0,T]$, i.e. $\Sigma_1 \cup \Sigma_2, = \Sigma$.

Let $\partial\Sigma_i$, denotes boundary of Σ_i, $i = 1, 2$, being defined as $\partial\Sigma_i, = \mathrm{cl}(\Sigma_i)\backslash\Sigma_i$, where $\mathrm{cl}(\Sigma_i)$ is usual pointwise closure of set Σ_i. Hence, a common piece of their boundaries, denoted $\Gamma = \partial\Sigma_1 \cap \partial\Sigma_2$, is called optimal exercise boundary.

The optimal exercise boundary is reasonably defined by mapping Γ: $[0,T] \to \Gamma(t)$, thus specifying the underlying asset price at Γ, denoted $S_\eta = \Gamma(t_e)$, precisely. Here, we adopt the subscript η instead of usual t in order to point out the specific asset price at the optimal exercise time t_e, in particular.

The optimal exercise boundary is not given a priori, and has to be determined together with option pricing function $V(S,t)$ by solving the corresponding option pricing problem. Here, we already drop the subscript P for simplicity, as we are handling the American put option in the paper exclusively.

Using optimal exercise boundary $\Gamma(t)$, the subregions Σ_1 and Σ_2 are defined in following way

$$V(S,t) > \max(K - S, 0), \ (S,t) \in \Sigma_1, \quad (3)$$

$$\Sigma_1 = \{(S,t) \mid \Gamma(t) \le S < +\infty\}, \ t \in [0,T],$$

$$V(S,t) = \max(K - S, 0), \ (S,t) \in \Sigma_2, \quad (4)$$

$$\Sigma_2 = \{(S,t) \mid 0 \le S \le \Gamma(t) \}, \ t \in [0,T],$$

where Σ_1 is called the continuation subregion, since the payoff is zero when $S > S_\eta$, and the holder should continue to keep the option, whereas Σ_2 is called the stoping subregion.

Noting, the continuation subregion Σ_1 is sometimes called retained subregion alternatively, whereas the stopping subregion Σ_2 is called selling subregion, or exercise subregion, as well.

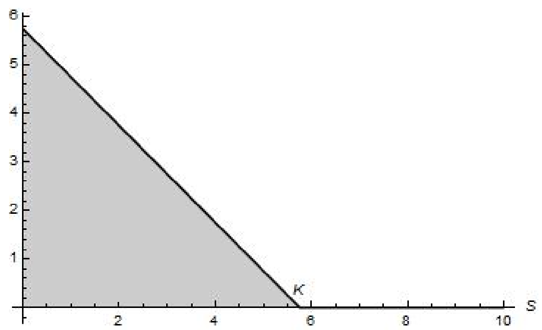

Fig. 1 Payoff function of American put option.

In Figure 1, there is depicted payoff function of American put option $V(S_t,t) = \max(K - S_t,0)$, where precise writing S_t just emphasizes the situation at time t, a stock price S_t spans [0,10], and the strike price $K = 5.75$, both being expressed in an unspecified monetary unit.

In Figure 2, there are depicted both subregions Σ_1, Σ_2, the expiration date T, and the option exercise price, all schematically. The horizontal axis represents an underlying asset price S, and the vertical axis carries the time t elapsed from entering the option contract into power.

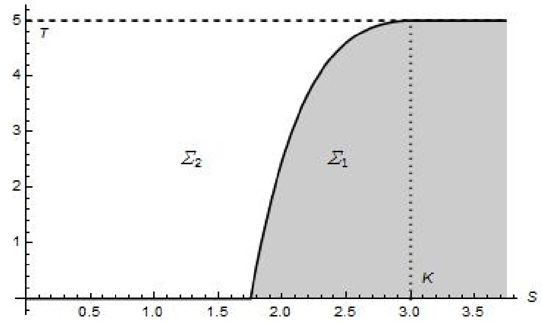

Fig. 2 Subregions Σ_1, Σ_2, and early exercise boundary.

In general, typical derivation of almost all models within the field of financial option pricing problems follows classical procedure which is based upon construction of self-financing portfolio, Δ-hedging principle, and application of Itô's formula. For corresponding details, we refer to [5] and [9], in particular.

Using such procedure, we can infer that function $V(S,t)$ satisfies the Black-Scholes partial differential equation in the subregion Σ_1

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \ \ (S,t) \in \Sigma_1, \ (5)$$

provided that pricing function $V(S,t)$ belongs to $C^{2,1}(\Sigma_1)$ class of functions, in classical formulation framework.

The boundary conditions on $\partial\Sigma_1$ are specified in following way. First, on the optimal exercise boundary Γ, it holds

$$V(S,t) = \max(K - S, 0) \ \Rightarrow \ \frac{\partial V(S,t)}{\partial S} = -1,$$

$$S = S_\eta \in \Gamma(t), \ \ t \in [0,T). \quad (6)$$

Further, the terminal condition at $t = T$, and the condition when $S \to +\infty$, are following

$$V(S_T,T) = \max(K - S_T, 0),$$

$$\lim V(S,t) = 0,\ S \to +\infty,\ t \in [0,T]. \qquad (7)$$

Concluding, we see that American put option pricing problem leads to finding a function pair $\{V(S,t), \Gamma(t)\}$ in subregion Σ_1 satisfying the partial differential equation (5) and boundary-terminal conditions (6) and (7).

Following [10], we are able to get more compact form of the problem and to elucidate its setting, too. First, it is reasonable to introduce so called Black-Scholes differential operator $\mathcal{L}_{\mathrm{BS}}$ as follows

$$\mathcal{L}_{\mathrm{BS}} F(S,t) = \frac{\partial G}{\partial t} + rS\frac{\partial G}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} - rG, \qquad (8)$$

for any function $F(S,t)$ belonging to class of functions $C^{2,1}[(0,+\infty) \times (0,T)]$.

Using (3), (4), and the operator $\mathcal{L}_{\mathrm{BS}}$, we can rewrite properties of function $V(S,t)$ in subregions Σ_1, Σ_2, in compact form

$$\mathcal{L}_{\mathrm{BS}} V(S,t) = 0,\ \ V(S,t) > \max(K - S,0),\ \ (S,t) \in \Sigma_1, \quad (9)$$

$$\mathcal{L}_{\mathrm{BS}} V(S,t) = -rK < 0,$$

$$V(S,t)=\max(K-S,0),\ \ (S,t) \in \Sigma_2. \qquad (10)$$

Since (10) holds on Γ, being tackled as a part of $\partial\Sigma_2$, we can conclude by direct computation that both the pricing function $V(S,t)$ and its first derivative $\partial V/\partial S$ are continuous on the optimal exercise boundary Γ.

Now, combining relations (9) and (10) together, we get the variational formulation of American put option pricing problem in strong case, see [6], Chapter 6.2, for more details

$$\min\{-\mathcal{L}_{\mathrm{BS}} V(S,t), V(S,t)-\max(K-S,0)\} = 0, \quad (S,t) \in \Sigma,$$

$$V(S_T,T) = \max(K - S_T, 0),\ S_T \in [0, +\infty),$$

$$\lim V(S,t) = 0,\ S \to +\infty,\ t \in [0,T]. \qquad (11)$$

Variational formulation of American put option pricing in weak form is discussed in [10], where all theoretical details are presented within a functional theoretic framework. Finally, the free boundary problem for determination of pricing function $V(S,t)$ leads to variational inequality problem. In shorter form, it is also discussed in paper [7].

2 Decomposition formula for American put option

Early exercise premium is discussed from several viewpoints in [3], and [4]. However, we follow an approach given in [6], Chapter 6.3, here. Using that, we present an interesting decomposition formula for American put option price $V(S,t)$ which gives a link to European put option price $V^{E}(S,t)$ in following way

$$V(S,t) = V^{E}(S,t) + e(S,t), \qquad (12)$$

here $V^{E}(S,t)$ is the well-known European put option price on the same underlying asset, and $e(S,t)$ is the early exercise premium.

The $V^{E}(S,t)$ is computed by well-known Black-Scholes formula, see for example [4], [5], and [9],

$$V^E(S,t) = Ke^{-r(T-t)} N(-d_2) - S\,N(-d_1). \quad (13)$$

Where d_1 and d_2 are given by following expressions

$$d_1 = (\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t))/(\sigma\sqrt{(T-t)}),$$

$$d_2 = d_1 - \sigma(T-t).$$

Function $N(x)$ is the cumulative probability of a standard normal distribution N(0,1). It is given by following expression

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi\ .$$

The early exercise premium $e(S,t)$ is given in integral representation by following expression

$$e(S,t) = rK \int_t^T d\eta \int_0^{S_\eta} G(S,t;\xi,\eta) d\xi. \qquad (14)$$

The function $G(S,t;\xi,\eta)$ is fundamental solution, sometimes called Green function, too, of the well-known Black-Scholes partial differential equation, being expressed explicitly in relation (5).

Inspecting (14) thoroughly, we reckon the upper bound S_η of the inner integral to cause direct dependence of early exercise premium $e(S,t)$ upon the optimal exercise boundary Γ, evidently.

Now, we refer to [6], Chapter 6.3, and Theorems 6.3 and 6.4 therein, for more technical details regarding the function $G(S,t;\xi,\eta)$. However, we ought to present here at least the function $G(S,t;\xi,T)$ in explicit form

$$G(S,t;\xi,T) = \frac{e^{-r(T-t)}}{\xi\sigma\sqrt{2\pi(T-t)}}\ e^{\varphi(S,t;\xi,T)},$$

$$\varphi(S,t;\xi,T) = -\frac{1}{2\sigma^2(T-t)}[\ln\frac{S}{\xi} + (r - \frac{\sigma^2}{2})(T-t)]^2. \quad (15)$$

In general, the function $G(S,t;\xi,T)$ represents a solution $U(S,t)$ of the Black-Scholes partial differential equations with special terminal condition expressing concentrated effect at an arbitrary point ξ, which is emphasized by including parameters T and ξ into (15) explicitly. The corresponding terminal value problem for determination of function $U(S,t)$ takes the following form

$$\mathcal{L}_{\mathrm{BS}} U(S,t) = 0, \quad U(S,T) = \delta(S-\xi), \quad (16)$$

where $0 < S < +\infty$, $0 < \xi < +\infty$, $0 < t < T$, and $\delta(x)$ is the well-known Dirac function being given by its properties $\delta(x) = +\infty$, if $x = 0$, otherwise $\delta(x) = 0$, and $\int_{-\infty}^{+\infty}\delta(x)dx = 1$.

3 Early exercise boundary equation

As the optimal early exercise boundary Γ is not known a priori but has to be determined together with pricing function $V(S,t)$, therefore, in general, the problem is called free-boundary value problem for parabolic partial differential equation.

Albeit we have already formulated the problem by expressions (5)-(7) explicitly, the main difficulty therein is that one needs to solve for both unknown function $V(S,t)$ and unknown boundary $\Gamma(t)$, simultaneously.

Since any American option has early exercise term in the contract, the determination of $\Gamma(t)$ is of special importance to the holder of American options, put ones in particular. In general, we remark that it is applicable in any case of American options when early exercise is interesting, i.e. in case of American put options, and in case of American call options paying dividends.

In [1], [2] and [6], Chapters 6.3-6.5, there are several interesting results related to both qualitative and quantitative properties of the optimal exercise boundary.

Except other ones, as for example position of $\Gamma(T)$, the monotonicity of $\Gamma(t)$, upper and lower bounds of $\Gamma(t)$, and convexity of $\Gamma(t)$, as well, the most important are following two ones. First, the asymptotic expression of $\Gamma(t)$ near $t = T$, and at second, the integral equation for determination of $\Gamma(t)$, which is given in [6], Chapter 6.3, Theorem 6.5, in particular.

However, we need to point out that the thorough and clear discussion of derivation of the corresponding integral equation for determination of $\Gamma(t)$ based upon solution of optimal stopping problem for Brownian motion is given in [1] and [2], precisely.

Following [6], Theorem 6.5, the equation for determination of optimal exercise boundary $\Gamma(t)$ for American put options, assuming payment of dividend, in general, is given by following expression

$$\begin{aligned}\Gamma(t) = K + \Gamma(t)e^{-q(T-t)}N[-g_2(t;K,T,\sigma,\beta_2)]\\ - Ke^{-q(T-t)}N[-g_1(t;K,T,\sigma,\beta_1)]\\ -rK\int_t^T e^{-r(\eta-t)}\{1 - N[h_1(t;\Gamma(\eta),\eta,\sigma,\beta_1)]\\ +rK\int_t^T e^{-r(\eta-t)}\{1 - N[h_2(t;\Gamma(\eta),\eta,\sigma,\beta_2)]\}d\eta,\end{aligned} \quad (17)$$

here q is dividend rate, $N(x)$ is the cumulative distribution function of N(0,1), again, and parameters β_1, β_2, are defined as follows

$$\beta_1 = r - q - \sigma^2/2, \quad \beta_2 = r - q + \sigma^2/2,$$

The functions $g_i(t;K,T,\sigma,\beta_i)$, $h_i(t;\Gamma(\eta),\eta,\sigma,\beta_i)$, $i = 1, 2$, are given by following expressions

$$g_i(t;K,T,\sigma,\beta_i) = \frac{-\ln\frac{\Gamma(t)}{K} + \beta_i(T-t)}{\sigma\sqrt{T-t}}, \quad i = 1,2, \quad (18)$$

$$h_i(t;\Gamma(\eta),\eta,\sigma,\beta_i) = \frac{\ln\frac{\Gamma(t)}{\Gamma(\eta)} + \beta_i(\eta-t)}{\sigma\sqrt{\eta-t}}, \quad i = 1,2. \quad (19)$$

However, the equation (17) is a nonlinear Volterra integral equation of the second kind.

In general, solving such type of equation is a rather challenging problem for numerical mathematics.

Interesting approaches are presented in [8], and [11], too. Here, we mention the procedure given in [8], in particular.

The numerical approximation of early exercise boundary is constructed by following procedure. First, one assumes the function $\Gamma(t)$ to be expressible by exponential function with an exponent $\gamma(\tau)$, as follows

$$\Gamma(t) = Ke^{\gamma(\tau)}, \quad \tau = T - t,$$

$\gamma(\tau) = a_0 + a_1\tau^{1/2} + a_2\tau + a_3\tau^{3/2} + a_4\tau^2 + a_5\tau^{5/2} + a_6\tau^3 + \cdots$, (20)

where τ is the time remaining to expiration date. Second, one truncates the series representing $\gamma(\tau)$ to finite number of terms, which yields an approximation containing a finite number of unknowns coefficients a_j, $j = 0,\ldots,m$

$$\gamma_m(\tau) = \sum_{j=0}^{m} a_j \tau^{j/2} . \quad (21)$$

Third, substituting an approximation $\Gamma_m(t) = Ke^{\gamma_m(\tau)}$, into the integral equation (18), one tries to calculate the corresponding integrals in order to get a set of nonlinear algebraic equations for the coefficients a_j, $j = 0,\ldots,m$. Sure, various approaches are applicable, for example some numerical integration schemes, collocation methods, etc.

In the next chapter, we propose another approach for solution of the integral equation (17), which is based upon successive approximations of early exercise boundary Γ using an adaptive iteration algorithm. The algorithm is implemented in sw package Mathematica, and it is the core of our paper. The Mathematica notebook is available upon request.

4 Early exercise boundary equation

The first step is time discretization. Let t_i, $i = 0, \ldots, n$, such that $t_0 = 0$, $t_n = T$, denote regular mesh of time points with a mesh size $\Delta t = T/n$. As the $\Gamma(t)$ represents a uncountable set of points (t,S_t) laying on the optimal early exercise boundary Γ, then adopting time discretization turns the problem to solve integral equation (17) directly to looking for a finite set of points $\{t_i,S_i\}$, $i = 0, \ldots, n$, satisfying a discrete version of (17), where we just introduce shorter notation S_i instead of cumbersome $S_t|_{t=ti}$.

Time discretization being applied upon (17) yields the following set of fixed-point looking problems

$$S_i = \Psi(S_i;\, t_i, \Gamma(\eta)),\ i = 0, \ldots, n, \quad (22)$$

here symbol Ψ expresses formally the right-hand side of (17) after discretization thus denoting its proper dependence upon variable S_i, but also upon parameters t_i and $\Gamma(\eta)$ thereby making the numerical solution of (22) rather involved.

In order to overcome numerically a complication caused by function $\Gamma(\eta)$ appearing in (22), which is itself a solution of equation (17) being seeked, we propose the following algorithm based upon successive approximation of $\Gamma(\eta)$ starting with some initial guess.

Adaptive iterative algorithm – steps:

1. Set initial approximation $\Gamma_0(\eta)$, $\eta \in [t_i,T]$, and set index $k = 0$,
2. Increment current k by 1, for loop t_i, $i = 0, \ldots, n$, solve $S_{i,k} = \Psi(S_{i,k};t_i,\Gamma_{k\text{-}1}(\eta))$, $k \geq 1$, until conver-gence of $S_{i,k}$ appears with given tolerance ε_1,
3. Build $\Gamma_k(\eta)$, $\eta \in [0,T]$ using available set $\{t_i,S_{i,k}\}$, $i = 0, \ldots, n$, by spline approximation,
4. Check convergence,
   if $\| \Gamma_k - \Gamma_{k-1} \|_{L_2(0,T)} \leq \varepsilon_2$ then *Stop*
   else *GoTo* step 2.

Our first attempt to solve (17) for a given t_i, which is a core of the step 2 within the proposed algorithm, was simply lured by computational power of sw Mathematica ver.11.1 to be able to solve it just by two following commands

```
eq6318 = S[t] == eqSt := K + # Exp[-q (T - t)] CDF[

NormalDistribution[0,

1], -(-Log[#/K] + \[Beta]2 (T - t))/(\[Sigma] Sqrt[T - t])]

- K Exp[-r (T - t)] CDF[NormalDistribution[0,

1], -(-Log[#/K] + \[Beta]1 (T - t))/(\[Sigma] Sqrt[T - t])]

- K r Integrate[Exp[-r (\[Eta] - t)] (1 -

CDF[NormalDistribution[0, 1], (Log[#/S\[Eta][\[Eta]]]

+ \[Beta]1 (\[Eta] - t))/(\[Sigma] Sqrt[\[Eta] - t])]), {\[Eta],
t, T}]

+q # Integrate[Exp[-q (\[Eta] - t)] (1 -

CDF[NormalDistribution[0, 1], (Log[#/S\[Eta][\[Eta]]]

+ \[Beta]2 (\[Eta] - t))/(\[Sigma] Sqrt[\[Eta] - t])]),

{\[Eta], t, T}] &

FindRoot[eqSt[s] == s, {s, s1}]
```

However, it failed with a Mathematica ver. 11.1 process control message:

SystemException["MemoryAllocationFailure"]

thus signalizing an extreme demand of internal Mathematica procedures upon memory allocation when executing such complex commands. It was simply over the memory capacity of our Dell Latitude E6510 computer with 4GB RAM running under MS-Windows 10.

Hence, we turned to break the given algorithm into a sequence of less complex steps and control manually their performance and convergence in adaptive way, too.

Following [6], Chapter 6.5, we also know some advanced information about optimal exercise boundary. First, the location point at $t = T$

$$\Gamma(T) = \min(rK/q, K), \qquad (23)$$

and second, an asymptotic expression of $\Gamma(t)$ near $t = T$, in case of $q = 0$, i.e. for non-dividend paying American put option, which is known in following form

$$(K - \Gamma(t))/K \approx \sigma \sqrt{[(T - t)\, |\ln(T - t)|]}. \qquad (24)$$

As already stated above, we know that $\Gamma : t \rightarrow \Gamma(t)$ is a convex and monotone increasing function on $(0,T)$.

We also note that various other asymptotic expansions are discussed in [11], in particular.

Numerical example:

For numerical determination of the optimal early exercise boundary $\Gamma(t)$, we select the American put option with following data: strike price $K = 40$, risk-free rate $r = 0.05$, underlying asset price $S \in (0,65]$, with volatility $\sigma = 0.3$, option expiration date $T = 1$ [year], and dividend rate $q = 0$, i.e. non-dividend paying case.

In Figures 3 and 4, we show two different initial variants of $\Gamma_0(t)$, constructed either by using asymptotic expression (24) with a proper prolongation, or by ad-hoc way using a cubic polynomial with proper interpolation conditions maintaining $\Gamma(T) = K$ as required by (23), and both qualitative properties, i.e. monotonicity and convexity of $\Gamma_0(t)$, on (0,T), as well.

In Figure 3, we present an idea of construction of $\Gamma_0(t)$ using (24) with proper prolongation.

A curve given by (24) is convex on (0,T), but it violates the monotonicity. Hence, to fix that defect, we propose the following procedure.

First, we find a global minimum point $t_{\min}$ of function (24) on (0,T) issuing the minimum value $\Gamma_{\min}$ of that function.

Second, we make a smooth prolongation from $t = 0$ to $t = t_{\min}$ by a horizontal line keeping the calculated minimal value $\Gamma_{\min}$, as it is depicted in Figure 3.

Finally, we join both both branches together to get a feasible guess for $\Gamma_0(t)$, $0 \leq t \leq T$.

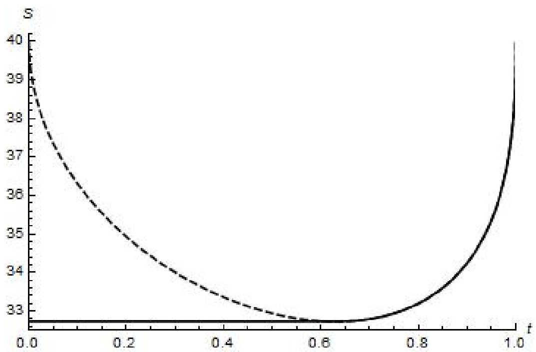

Fig. 3 $\Gamma_0(t)$ – initial early exercise boundary asymptotic-derived with proper prolongation (full line); $\Gamma(t)_{|\text{asymp}}$ – curve given by (24) on $(0,T)$ (dashed line).

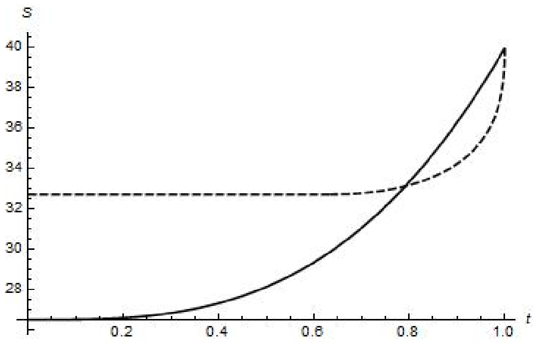

Fig. 4 $\Gamma_0(t)$ – variants: 1) ad-hoc guess cubic curve (full line); 2) asymptotic-derived with proper prolongation one (dashed line).

Assumption $q = 0$, i.e. the case of non-dividend paying option, simplifies the integral equation (17) for determination of optimal early exercise boundary $\Gamma(t)$ so that the first term is modified and the last one is dropped completely.

Hence, we obtain another set of fixed-point looking problems, written formally in the same form (22), however, with a slightly different form of Ψ. It is defined as the right-hand side of the following equation

$$\Gamma(t) = K + \Gamma(t) N[-g_2(t; K, T, \sigma, \beta_2)]$$

$$-Ke^{-q(T-t)}N[-g_1(t;K,T,\sigma,\beta_1)]$$

$$-rK\int_t^T e^{-r(\eta-t)}\{1-N[\,h_1(t;\Gamma(\eta),\text{ç},\sigma,\beta_1)], \quad (25)$$

here parameters β_1, β_2, are defined as follows

$$\beta_1 = r - \sigma^2/2, \quad \beta_2 = r + \sigma^2/2,$$

and functions $g_i(t;K,T,\sigma,\beta_i)$, $i = 1, 2$, are given by (18), and function $h_1(t;\Gamma(\eta),\eta,\sigma,\beta_1)$ by (19), correspondingly.

Classical trapeziodal rule is used for numerical approximation of integral $\int_t^T \omega(\eta)d\eta$, which appears in equation (25), as the third term on its right-hand side. Because of the time discretization, we point out that the lower bound $t = t_i$, in particular case, whereas the upper bound T is fixed in any case.

Using function $h_1(t;\Gamma(\eta),\eta,\sigma,\beta_1)$, and $N(x)$, we can define the function $\omega(\eta)$ in following way

$$\omega(\eta) = e^{-r(\eta-t)}(1-N[h_1(t;\Gamma(\eta),\eta,\sigma,\beta_1)\,]). \quad (26)$$

For trapezoidal rule, let m is a number if intervals covering $[t_i,T]$ with a step $\Delta\eta = (T - t_i)\,/m$, thus providing

$$\int_t^T \omega(\eta)d\eta \approx$$

$$\frac{\Delta\eta}{2}\left\{\omega(t_i) + 2\left[\sum_{j=1}^{m-1}\omega(t_i + j\Delta\eta)\right] + \omega(T)\right\} = Q_i. \quad (27)$$

Now, we formulate the inner-most problem. For given set of points $\{t_i, S_{i,0}\}$, find $\{t_i, S_{i,k}\}$, $k \geq 1$, such that equation (25) holds, with given tolerance appearing in convergence criterion (29)

$$S_{i,k} = \Psi(S_{i,k};\, t_i, \Gamma_{k-1}(\eta))$$

$$= \mathrm{K} + S_{i,k}\Phi\Big[-\frac{-\ln\frac{S_{i,k}}{K}+\beta_2(T-t_i)}{\sigma\sqrt{T-t_i}}\Big]$$

$$-Ke^{-r(T-t_i)}\Phi\Big[-\frac{-\ln\frac{S_{i,k}}{K}+\beta_1(T-t_i)}{\sigma\sqrt{T-t_i}}\Big] - r\,KQ_{i,k-1}, \quad (28)$$

and

$$|\, S_{i,k} - \Psi(S_{i,k};\, t_i, \Gamma_{k-1}(\eta))\,| \leq \varepsilon_1. \quad (29)$$

Now, we present some snippets of our Mathematica ver. 11.1 code being developed for numerical solution of the problem to find optimal early exercise boundary for American put option.

First, there is a construction of the initial early exercise boundary $\Gamma_0(t)$ based upon asymptotic expression of $\Gamma(t)$ near $t = T$, as given by (24)

```
eebAsymp:=K (1-σ Sqrt[(T-#)Abs[Log[T-#]]])&

Plot[eebAsymp[t],{t,0,T}]
FindMinimum[{eebAsymp[t], t > 0, t < 1}, {t,.5}]
tm=t/.Last[%]

eebm=%%[[1]]
eeb1:=If[#<tm,eebm,eebAsymp[#]]&;

Plot[{eeb1[t],eebAsymp[t]},{t,0,T},PlotStyle-
>{Automatic,Dashed},PlotRange->{{0,1},{32.5,40.2}}]
eeb2[t_]:=Piecewise[{{eebm,t<tm},{eebAsymp[t],t>=tm}}
] Plot[Evaluate[eeb2[t]],{t,0,T},
PlotStyle->{Red},PlotRange->{{0,1},{32.5,40.2}}]
```

Second, we present a main part of code computing the right-hand side of (28)

```
Δt=(T-t)/(n-1);
ψ3:=With[{s=sLast,t=tLast},Exp[-r(#-t)](1-
CDF[NormalDistribution[0,1],(Log[s/Evaluate[Sη[#]]]+β1(
#-t))/(σ Sqrt[#-t])])&];

fstη=Table[ηη=t+j Δt;ψ3[ηη]//Evaluate,{j,n-1}];
fL=PadLeft[fstη,n,fstη[[1]]];
fR=PadRight[fstη,n,fstη[[n-1]]];

f3is=Δt (Total[fL]+Total[fR])/2   (* <Trapezoidal Rule *)

(* <Compute 1-st & 2-nd terms *)
ψ1:=With[{s=39.181,t=.98},K+# Exp[-q(T-
t)]CDF[NormalDistribution[0,1],-(-Log[#/K]+β2(T-t))/(σ
Sqrt[T-t])]&];

ψ2:=With[{s=sLast,t=tLast},-K Exp[-r(T-
t)]CDF[NormalDistribution[0,1],-(-Log[#/K]+β1(T-t))/(σ
Sqrt[T-t])]&];
sRhs=ψ1[s]+ψ2[s]+f3is
```

Now, setting $\varepsilon_1 = 10^{-3}$, we compute two variants of optimal early exercise boundary using two initial functions $\Gamma_0(t)$, as discussed above and shown in Figure 4, respectively.

In Figures 5 and 6, we show also the selected set of points for time discretization, i.e. $\{t_i\}$, $i = 0, ..., n$, which are used for basic time discretization of nonlinear Volterra integral equation (17), in general.

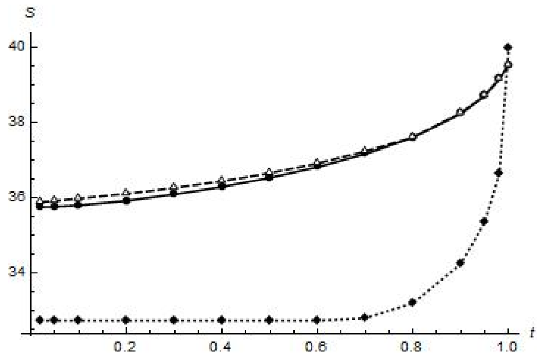

Fig. 5 Local comparison of computed optimal early exercise boundaries $\Gamma(t)$ computed from two initial curves $\Gamma_0(t)$ depicted in Figure 4 (full line and dashed one), and location of asymptotic-derived with proper prolongation curve $\Gamma_0(t)$ (doted line), with set of nodal points $\{t_i,S_i\}$.

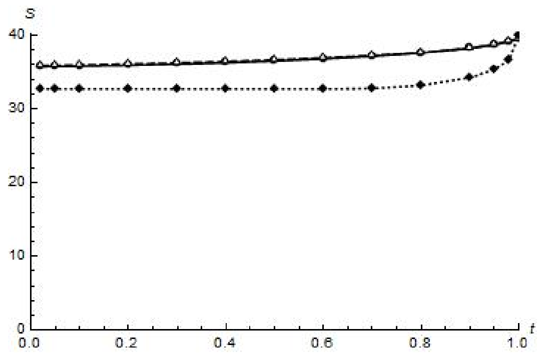

Fig. 6 Global comparison of computed optimal early exercise boundaries $\Gamma(t)$, initial $\Gamma_0(t)$, and set of nodal points $\{t_i,S_i\}$, just similar to the results depicted in Figure 5.

Inspecting the results presented in Figures 5 and 6 , we may conclude that different initial functions $\Gamma_0(t)$ cause almost negligible differences between our computed approximations of the optimal early exercise boundary $\Gamma(t)$.

The proposed numerical procedure for approximation of optimal early exercise boundary for American put option seems to work well, which sounds rather promising for further research.

5 Conclusion

In the paper, we discuss briefly a variational formulation of American put option problem, in strong sense. Next, we are focused upon the early exercise premium, which is an integral part of pricing function of American put option.

However, the most attention is devoted to problem of finding optimal early exercise boundary for American put option, which is formulated by nonlinear Volterra integral equation of the second kind.

We propose the adaptive iteration algorithm for solving that problem. The numerical procedure implementing our algorithm is coded in sw Mathematica ver. 11.1, and it seems to work satisfactory. So, it presents a promising platform for our future research.

Sure, the further and more complicated numerical experiments are necessary to investigate the computational behaviour and performance of the proposed algorithm in more details.

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References:

[1] AitSahlia, F., and Lai, T. L., A canonical optimal stopping problem for American options and its numerical solution, Names of the Authors, Title of the Paper, *Journal of Computational Finance*, Vol.3, No.2, 1999, pp. 33-52.

[2] AitSahlia, F., and Lai, T. L., Exercise boundaries and efficient approximations to American option prices and hedge parameters, *Journal of Computational Finance*, Vol.4, No.4, 2001, pp. 87-103.

[3] Doffou, A., Estimating the early exercise premium of American put index option, *Int. Journal of Banking and Finance*, Vol.6, No. 1, 2008, pp.31-47.

[4] Dueker, M., Directly Measuring Early Exercise Premiums Using American and European S&P 500 Index Option, Working Paper Series, on-line, https://research.stlouisfed.org/wp/2002/2002-016.pdf.

[5] Fries, Ch., *Mathematical Finance – theory, modeling, implementation*, John Wiley & Sons, Hoboken, New Jersey, 2007.

[6] Jiang, L., *Mathematical Modeling and Methods of Option Pricing*, World Scientific Publ. Co., Singapore, 2005.

[7] Lukáš, L., American Option Pricing Problem Formulated as Variational Inequality Problem, In: *Conf. Proc., 34-th Int. Conf. Math. Methods in Economics 2016*, Tech. Univ. of Liberec, Liberec, Czech Republic, 2016, pp.512-517.
[8] Mallier, R., Evaluating approximations to the optimal exercise boundary for American options, *Journal of Applied Mathematics*, Vol.2, No.2, 2002, pp. 71-92. Hindawi Publishing Corporation, on-line, http://dx.doi.org/10.1155/S1110757X02000268
[9] Neftci, S. N., *An Introduction to the Mathematics of Financial Derivatives*, 2.ed.,Academic Press, London, 2000.
[10] Pironneau, O., and Achdou, Y., *Partial Differential Equations for Option Pricing*, In: *Mathematical Modeling and Numerical Methods in Finance*, Special Volume (Bensoussan, A., and Zhang, Q., eds) of *Handbook of Numerical Analysis*, Vol.XV, (Ciarlet, P. G., ed), North-Holland, Elsevier, Amsterdam, 2007.
[11] Stamicar, R., Ševčovič, D., and Chadam, J., The early exercise boundary for the American put near empiry: numerical approximation, *Canadian Applied Math. Quarterly*, Vol.7, No.4, 1999, pp. 427-444.