Optimizing a Collaborative two Layer Supply Chain with Probabilistic Quality

NOURA YASSINE
Mathematics and Computer Science Department, Faculty of Science
Beirut Arab University
P.O. Box 11-5020, Beirut
LEBANON
noura.yassein@bau.edu.lb

Abdullah AlSagheer
Higher College of Technology
Abu Dhabi Men's College
P.O BOX: 25035, Abu Dhabi
UAE
aalsagheer@hct.ac.ae

Abstract: - Two mathematical models for a two-layer supply chain are developed. The models describe the cases of collaborative and non-collaborative supply chains. In each case, explicit expressions for the optimal solution are derived. Moreover, the uniqueness of the optimal solution is demonstrated. The model incorporate the effects of quality and shortages into the classical production model with planned shortages. Numerical examples are presented to illustrate the calculations of the optimal solution and to examine the differences between the two models.

Key-Words: - Collaborative supply chain, existence and uniqueness of the optimal solution, probabilistic quality, planned shortages

1 Introduction

A supply chain is a network of players which coordinates a s eries of interrelated processes. Modelling in the context of supply chain has become very critical to the success of all players within the chain. Collaboration among the players is necessary to optimize the total chain and has been shown to surpass the outcome of the individual decision making strategy. In a t wo-layer supply chain consisting of a supplier and a producer, collaborative modelling must be considered in order to optimize the flow of raw materials as well as the finished product [1]. The quality of items of raw material significantly influence the coordination among the supply chain's players [2]. To address this issue, the classical production and inventory models have been extended in many directions to take into account the quality of the raw materials into their development of the mathematical model [2].

Salameh and Jaber [3] introduced a model that triggered this significant and rapidly growing area of research. One particular direction is the modifications and extensions of this model and of the classical production quantity (EPQ) model to the context of supply chain.

This classical model EPQ, which dates back to the early 20th century which dates back to the early 20th century [4], is based on v arious underlying assumptions. Since then, this model has been heavily researched by relaxing the underlying assumptions and introducing factors that resemble real life situation into this model. Such factors include: planned shortages, monetary consideration, and the quality of the finished product as well as the raw materials used in the production process.

In the classical EPQ model, a c ertain item is produced in order to meet its demand. The production rate α and the demand rate β are assumed to be known and constant with $\alpha > \beta$. Let C_0 denote the setup cost of production, h_p the holding cost per item produced per unit time h_p, and C_p unit production cost. Then, the total cost per unit time function is

$$TCU(Y) = C_p\beta + \frac{C_0\beta}{Y} + \frac{h_p}{2}\left(1 - \frac{\beta}{\alpha}\right)Y, \qquad (1)$$

where Y is any production quantity. The optimal production quantity Y^* is the minimizer of the TCU. It is given by

$$Y^* = \sqrt{\frac{2C_0\beta}{h_p(1-\beta/\alpha)}}. \quad (2)$$

An underlying assumptions of the classical EPQ model is that shortages are not allowed. Relaxing this assumption by allowing for planned shortages is an extension that can be found in books that deals with inventory models; for instance, see [5]. Another assumption is that the costs of the raw materials used in the production process are not considered in the classical model.

Recently, a great deal of research work has been done on extending the classical EPQ to account for the costs of raw materials. Salameh and El-Kassar incorporated the cost of raw material used in the production process into the classical EPQ model [6]. Several research papers have extended this model in various directions. In particular, a number of papers studied the effects of quality of the raw materials [2], [7], [8], [9], and [10]. In another direction, the effect of raw material on the EPQ model was considered in the context of supply chain [2], [11], [12] and [13].

A mathematical model for production process with shortages and raw materials was recently presented in [14]. A closed form formula for the optimal solution was obtained and the uniqueness of the solution was demonstrated. However, the model assumed that the raw materials used are of perfect quality and considered the producer as the decision maker.

The purpose of this paper is to extend the model presented in [14] to incorporate the effect of the quality of raw materials as well as the planned shortages. Moreover, the mathematical models are developed in a two-layer supply chain context consisting of a supplier and a producer. Two models are presented, one for a collaborative chain and another for non-collaborative one. Without loss of generality, it is assumed that a single type of raw material is used in the production process and each unit of the finished product requires one single unit of a raw material. The rest of this paper is organized as follows. In section 2, the mathematical model is developed and expressions for the expected total profit functions for the producer, supplier and the supply chain are derived. In section 3, the non-collaborative supply chain case i s considered and explicit expressions for the optimal solution are obtained. The uniqueness of the optimal solution is demonstrated. The collaborative supply chain case is presented in section 4. Numerical examples are presented in section 5 to illustrate the calculations of the optimal solution and to examine the differences between the two models. In section 6 a conclusion is presented and several suggestions for future research are stated.

2 The Mathematical Model

The following notation is used throughout the rest of this paper:

Y	Order size of raw material
S	Finished product shortage size
Z	Finished product maximum inventory level
α	Production rate
β	Demand rate
γ	Screening rate of raw materials
δ	Percentage of imperfect quality items of raw materials
$f(\delta)$	Probability density function of δ
T_p	Length of the production period
T	Length of the inventory cycle
T_1	Time to fulfil the backorder of size S
T_2	Time to build the maximum inventory level
T_3	Time to deplete the maximum inventory
T_4	Time to build a backorder of size S
T_s	Raw material screening time
Crs	Cost per unit of raw material to supplier
Crp	Cost per unit of raw material to producer
Cp	Production cost per unit
Cb	Administrative cost per item short
Cs	Cost per item short per unit time
C_0s	Ordering cost of raw materials to supplier
C_0p	Ordering cost of raw materials to producer
C_1	setup cost for production
Cm	Screening cost per unit of raw material
R	Selling price per unit of finished product
Cd	Discounted selling price per unit of raw material ($Cd < Crp$)
hrs	Holding cost per unit of raw material per unit time to supplier
hrp	Holding cost per unit of raw material per unit time to producer
hp	Holding cost due to production per unit per time
$TPSU$	Supplier total profit per unit function
$TPPU$	Producer total profit per unit function
$TPCU$	Supply chain total profit per unit function
N	Number of production cycles in one inventory cycle of supplier
i	inventory holding cost rate

The raw materials acquired from a supplier are processed into a finished product at a production rate α. Let Y be the order size of items of raw materials, an unknown to be determined and let Crp be the cost per unit of raw material to producer. The raw materials received at the beginning of the inventory cycle are screened at a rate γ so that the screening time is

$T_s = Y/\gamma$. (3)

At the end of this period, δY imperfect quality items of raw material are sold at a discounted price of Cd. The $(1-\delta)Y$ items of raw materials are stored and processed at a rate α until it is depleted at the end of the production period. The length of the production period is

$T_p = (1-\delta)Y/\alpha$. (4)

The inventory level for raw materials is shown in figure 1. To determine the producer optimal order quantity Y^* and the optimal shortage quantity S^*, we first calculate the producer total profit per cycle function and then the producer total profit per unit time function $TPPU$. The cost and revenue components per inventory cycle consists of:

- Revenues from selling the finished product and the imperfect quality raw material;
- ordering, purchasing, screening and holding costs of raw materials;
- setup cost of production as well as the production and holding costs of finished product;
- shortage and backorder costs.

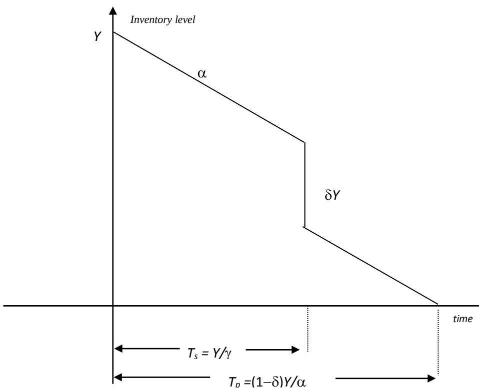

Fig. 1 : Inventory Level of Raw Material

The inventory cycle begins with the production period during which the finished product is produced at a rate α and consumed at the demand rate β. From the start of the cycle and until time T_1, the excess amount of the finished product is used to fulfil the S units of backorders at a rate of $\alpha-\beta$. Hence,

$T_1 = S/(\alpha-\beta)$. (5)

After such time and until the end of the production period, the excess amount of the finished product is used to accumulate inventory of the finished product at a rate of $\alpha-\beta$. This occurs during a time period of T_2, where $T_p = T_1+T_2$. Using (4), we have

$T_2 = T_p - T_1 = (1-\delta)Y/\alpha - S/(\alpha-\beta)$. (6)

At the end of this period, a maximum inventory level Z is reached. Then,

$$Z = T_2(\alpha-\beta) = \left(\frac{(1-\delta)Y}{\alpha} - \frac{S}{\alpha-\beta}\right).(\alpha-\beta) \\ = Y(1-\delta)(1-\beta/\alpha) - S. \tag{7}$$

This maximum level will be used to meet the demand at a rate β until time T_3 when the inventory level of the finished product reaches zero. Hence,

$$T_3 = \frac{Z}{\beta} = \frac{Y(1-\delta)(1-\beta/\alpha)-S}{\beta} \\ = \frac{Y}{\beta}(1-\delta)(1-\beta/\alpha) - \frac{S}{\beta}. \tag{8}$$

Throughout the remainder of the inventory cycle, a planned shortage of size S is built up at the demand rate β until time T_4, where

$$T_4 = \frac{S}{\beta}. \tag{9}$$

The inventory level for the finished product is shown in figure 2. Note that the length of the inventory cycle is $T = T_1 + T_2 + T_3 + T_4$; that is,

$$T = \frac{(1-\delta)Y}{\beta}. \tag{10}$$

The ordering cost of raw materials to producer is C_0p and the purchasing cost is $CrpY$. The raw materials holding cost is the holding cost per unit of material per unit time, h_{rp}, multiply by the average on hand inventory of raw materials times the cycle length. From figure 2, we have

Producer Raw Material Holding Cost

$$= hrp\left(\frac{(1-\delta)YT_p}{2} + \frac{\delta Y^2}{\gamma}\right) \\ = hrp\left(\frac{(1-\delta)^2}{2\alpha} + \frac{\delta}{\gamma}\right)Y^2. \tag{11}$$

Note that, the holding cost per unit of material per unit time, hrp, is the product of the inventory holding cost rate i and the unit purchasing cost C_r. That is,

$$hrp = iC_{rp}. \tag{12}$$

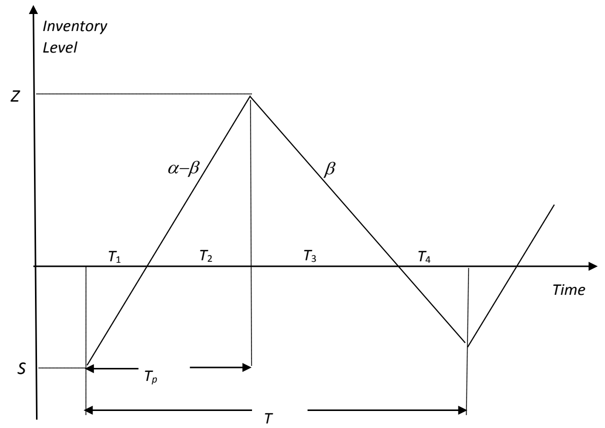

Fig. 2: Finished Product Inventory level

The cost of producing the Y units of the finished product is the sum of the setup is C_1 and the production cost is C_pY. The holding cost per unit of the finished product is the sum of h_r and h_p, where $h_p = iC_p$. This is due to the fact that a single unit of the finished product incur both the cost of production as well as the cost of raw material. Thus the finished product holding cost is the average inventory of on hand finished product times the inventory cycle length times the holding cost per unit of a finished product per unit time. From (5)-(7), we have

Producer Holding Cost of Finished Product

$$= (h_r + h_p).\frac{1}{2}.Z.\frac{(T_2 + T_3)}{T}.T$$
$$= \frac{(h_r + h_p)}{2\beta}\left(Y(1-\delta)\left(1-\frac{\beta}{\alpha}\right) - S\right).\left(Y(1-\delta) - \frac{S}{1-\beta/\alpha}\right). \quad (13)$$

The producer's shortage cost is the sum of the time independent administrative cost given by C_bS, and the time dependent shortage cost per unit short per unit time C_s multiplied by the area under the curve in figure 2 representing the inventory level of units short. From (4) and (8), the shortage cost per inventory cycle is

Producer Shortage Cost

$$= C_bS + C_s\frac{1}{2}S\frac{(T_1 + T_4)}{T}.T \quad (14)$$
$$= C_bS + C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)}.$$

The producer total inventory cost per cycle function $TCP(Y, S)$ is the sum of all cost components. Hence,

$$TCP(Y,S) = \left(C_{0p} + C_1\right) + (C_{rp} + C_m)Y + C_p(1-\delta)Y + h_{rp}\left(\frac{(1-\delta)^2}{2\alpha} + \frac{\delta}{\gamma}\right)Y^2 + C_bS + C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)} + (h_{rp} + h_p)\frac{1}{2\beta}\left(Y(1-\delta)\left(1-\frac{\beta}{\alpha}\right) - S\right).\left(Y(1-\delta) - \frac{S}{1-\beta/\alpha}\right). \quad (15)$$

The producer's revenues consist of:

- revenues obtained from selling finished product $R(1\text{-}\delta)Y$; and
- selling the imperfect quality raw materials at a discounted price $Cd\,\delta Y$

So, the total profit function for a producer is given by

$$TPP(Y,S) = C_d\delta Y + RY(1-\delta) - \left(C_{0p} + C_1\right) - (C_{rp} + C_m + C_p(1-\delta))Y - h_{rp}\left(\frac{(1-\delta)^2}{2\alpha} + \frac{\delta}{\gamma}\right)Y^2 - C_bS - C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)} - (h_{rp} + h_p)\frac{1}{2\beta}\left(Y(1-\delta)\left(1-\frac{\beta}{\alpha}\right) - S\right).\left(Y(1-\delta) - \frac{S}{1-\beta/\alpha}\right). \quad (16)$$

Simplifying the last term in (16), we get

$$TPP(Y,S) = C_d\delta Y + RY(1-\delta) - \left(C_{0p} + C_1\right) - (C_{rp} + C_m + C_p(1-\delta))Y - C_bS - C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)} - h_{rp}\left(\frac{(1-\delta)^2}{2\alpha} + \frac{\delta}{\gamma}\right)Y^2 \cdot - (h_{rp} + h_p)\frac{1}{2\beta}.\left(Y^2(1-\delta)^2\left(1-\frac{\beta}{\alpha}\right) - 2SY(1-\delta) + \frac{S^2}{1-\beta/\alpha}\right) \quad (17)$$

The expected value of the producer's total revenue function is

$$\begin{aligned}E[TPP] &= C_d \mu Y + RY(1-\mu) \\ &-\left(C_{0p}+C_1\right)-(C_{rp}+C_m+C_p(1-\mu))Y \\ &-C_b S - C_s \frac{\alpha S^2}{2\beta(\alpha-\beta)} - h_{rp}\left(\frac{E[(1-\delta)^2]}{2\alpha}+\frac{\mu}{\gamma}\right)Y^2 \\ &-(h_{rp}+h_p)\frac{1}{2\beta} \\ &.\left(Y^2 E[(1-\delta)^2]\left(1-\frac{\beta}{\alpha}\right)-2SY(1-\mu)+\frac{S^2}{1-\beta/\alpha}\right).\end{aligned} \quad (18)$$

Since expected value $E[(1-\delta)^2] = (1-\mu)^2 + \sigma^2$, hence

$$\begin{aligned}E[TPP] &= C_d \mu Y + RY(1-\mu) \\ &-\left(C_{0p}+C_1\right)-(C_{rp}+C_m+C_p(1-\mu))Y \\ &-C_b S - C_s \frac{\alpha S^2}{2\beta(\alpha-\beta)} - h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{2\alpha}+\frac{\mu}{\gamma}\right)Y^2 \\ &-(h_{rp}+h_p)\frac{1}{2\beta}\left(Y^2\left((1-\mu)^2+\sigma^2\right)\left(1-\frac{\beta}{\alpha}\right)\right. \\ &\left.-2SY(1-\mu)+\frac{S^2}{1-\beta/\alpha}\right).\end{aligned} \quad (19)$$

The Renewal Reward theorem can be used to approximate the expected value of the total profit of a producer's per unit time function E[*TPPU*], see [15] for instance. The approximation is obtained by dividing (16) by the expected value of the cycle length $E[T] = (1-\mu)Y/\beta$. Hence,

$$\begin{aligned}E[TPPU] &= \frac{\beta\mu}{(1-\mu)}C_d + \beta R - \beta C_p \\ &-\frac{\beta}{(1-\mu)Y}\left(C_{0p}+C_1\right)-\frac{\beta}{(1-\mu)}(C_{rp}+C_m) \\ &-h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{2\alpha}+\frac{\mu}{\gamma}\right)\frac{\beta Y}{(1-\mu)} \\ &-\left(C_b S + C_s\frac{S^2}{2\beta(1-\beta/\alpha)}\right)\frac{\beta}{(1-\mu)Y} \\ &-\frac{h_{rp}+h_p}{2(1-\mu)}\left(Y\left((1-\mu)^2+\sigma^2\right)\left(1-\frac{\beta}{\alpha}\right)\right. \\ &\left.-2S(1-\mu)+\frac{S^2}{(1-\beta/\alpha)Y}\right).\end{aligned} \quad (20)$$

The total profit per unit time function for the supplier is determined by assuming that the supplier inventory cycle coincides with a producer's cycle and its length is a multiple N of the length of the producer cycle T. Hence, the supplier's inventory cycle begins with having an inventory level of NY units of raw materials obtained at an ordering cost C_{0s} and a unit cost of C_{rs}. This inventory is used to provide the producer with N batches each of size Y. The first batch is assumed to be delivered at the start of the supplier cycle. Hence, the supplier inventory starts at a maximum level of $(N-1)Y$ units and drops by Y units after each T units of time. The behavior of the supplier inventory is depicted in Figure 3.

The cost components per supplier inventory cycle are:

Cost of raw materials	$= C_{rs} NY$
Ordering cost of raw materials	$= C_{0s}$
Holding cost	$= hrs\ YT\ N(N-1)/2$

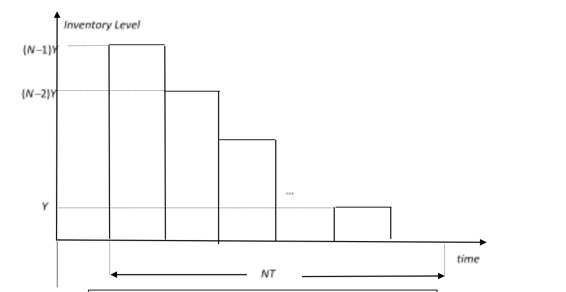

Figure 3: Raw Materials Inventory Level at Supplier

The supplier's total cost function is:

$$TCS(Y,S) = C_{0s} + C_{rs}NY + h_{rs}\left(\frac{N(N-1)}{2}\right)TY. \quad (21)$$

Revenues for the supplier consist of selling the raw materials to the producer. Thus, the supplier's total profit function is

$$\begin{aligned}TPS(Y,S) &= C_{rp}NY - C_{0s} - C_{rs}NY \\ &-h_{rs}\left(\frac{N(N-1)}{2}\right)TY.\end{aligned} \quad (22)$$

The expected value of the total profit function is:

$$\begin{aligned}E[TPS] &= C_{rp}NY - C_{0s} - C_{rs}NY \\ &-h_{rs}\left(\frac{N(N-1)}{2}\right)E[T]Y.\end{aligned} \quad (23)$$

Using the Renewal Reward Theorem, the expected value of the producer's total profit function per unit time can be obtained by dividing (23) by the expected value of the supplier inventory cycle length $E[NT]$. From (10), we have

$$E[NT] = N\,E[T] = N(1-\mu)Y/\beta. \quad (24)$$

Hence,

$$E[TPSU] = \frac{C_{rp}\beta}{1-\mu} - \frac{C_{0s}\beta}{N(1-\mu)Y} - \frac{C_{rs}\beta}{1-\mu} - \frac{1}{2}h_{rs}(N-1)Y. \tag{25}$$

The supply chain expected total profit function per unit time is obtained by adding (20) and (25). That is

$$\begin{aligned} &E[TPCU] = E[TPPU] + E[TPSU] \\ &= \frac{\beta\mu}{(1-\mu)}C_d + \beta R - \frac{\beta}{(1-\mu)Y}\left(C_{0p} + C_1\right) \\ &- \frac{\beta}{(1-\mu)}(C_{rp} + C_m) - \beta C_p \\ &- h_{rp}\left(\frac{(1-\mu)^2 + \sigma^2}{2\alpha} + \frac{\mu}{\gamma}\right)\frac{\beta Y}{(1-\mu)} \\ &- \left(C_b S + C_s \frac{\alpha S^2}{2\beta(\alpha-\beta)}\right)\frac{\beta}{(1-\mu)Y} \\ &- (h_{rp} + h_p)\frac{1}{2(1-\mu)}\left(Y\left((1-\mu)^2 + \sigma^2\right)\left(1 - \frac{\beta}{\alpha}\right)\right. \\ &\left. - 2S(1-\mu) + \frac{S^2}{(1-\beta/\alpha)Y}\right) \\ &+ \frac{C_{rp}\beta}{1-\mu} - \frac{C_{0s}\beta}{N(1-\mu)Y} - \frac{C_{rs}\beta}{1-\mu} - \frac{1}{2}h_{rs}(N-1)Y. \end{aligned} \tag{26}$$

3 Non-Collaborative Supply Chain

In the case when there is no coordination between supplier and producer, we assume that the inventory decision is made by the producer. The optimal solution is obtained by differentiating (17) with respect to S and Y and equating to zero. That is

$$\begin{aligned} \frac{\partial E[TPPU]}{\partial Y} &= \frac{\beta}{(1-\mu)Y^2}\left(C_{0p} + C_1\right) \\ &- h_{rp}\left(\frac{(1-\mu)^2 + \sigma^2}{2\alpha} + \frac{\mu}{\gamma}\right)\frac{\beta}{(1-\mu)} \\ &+ \left(C_b S + C_s \frac{S^2}{2\beta(1-\beta/\alpha)}\right)\frac{\beta}{(1-\mu)Y^2} \\ &- \frac{h_{rp} + h_p}{2(1-\mu)}\left(\left((1-\mu)^2 + \sigma^2\right)\left(1 - \frac{\beta}{\alpha}\right)\right. \\ &\left. - \frac{S^2}{(1-\beta/\alpha)Y^2}\right). \end{aligned} \tag{27}$$

and

$$\begin{aligned} \frac{\partial E[TPPU]}{\partial S} &= -\left(C_b + C_s \frac{S}{\beta(1-\beta/\alpha)}\right)\frac{\beta}{(1-\mu)Y} \\ &+ (h_{rp} + h_p)\left(1 - \frac{S}{(1-\beta/\alpha)(1-\mu)Y}\right). \end{aligned} \tag{28}$$

Setting $\partial E[TPPU]/\partial S$ equal to zero and solving for Y, we have

$$\begin{aligned} Y &= \frac{S}{(1-\beta/\alpha)(1-\mu)} \\ &+ \left(C_b\beta + C_s \frac{S}{(1-\beta/\alpha)}\right)\frac{1}{(1-\mu)(h_{rp} + h_p)}. \end{aligned} \tag{29}$$

Simplifying the expression in (29), we have

$$\begin{aligned} Y &= \frac{\beta C_b}{(1-\mu)(h_{rp} + h_p)} \\ &+ \frac{S}{(1-\mu)(1-\beta/\alpha)}\left(1 + \frac{C_s}{h_{rp} + h_p}\right). \end{aligned} \tag{30}$$

Setting (16) equal to zero and solving for Y, we have

$$\begin{aligned} Y = &\left(\left(2(1-\beta/\alpha)\beta\left(C_{0p} + C_1 + C_b S\right)\right.\right. \\ &\left.+ S^2\left(C_s + h_{rp} + h_p\right)\right) \\ &/\left((\beta h_{rp}\left(\frac{(1-\mu)^2 + \sigma^2}{\alpha} + \frac{2\mu}{\gamma}\right)(1-\beta/\alpha)\right. \\ &\left.\left. + \left(h_{rp} + h_p\right)\left((1-\mu)^2 + \sigma^2\right)(1-\beta/\alpha)^2\right)\right)^{1/2}. \end{aligned} \tag{31}$$

Setting the expressions for Y in (19) and (20) equal to each other, squaring, cross multiplying and rearranging, we get

$$\begin{aligned} &2(1-\beta/\alpha)\beta\left(C_{0p} + C_1 + C_b S\right) \\ &+ S^2\left(C_s + h_{rp} + h_p\right) = \\ &\frac{K\beta^2 C_b{}^2}{(1-\mu)^2(h_{rp} + h_p)^2} \\ &+ \frac{KS^2}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1 + \frac{C_s}{h_{rp} + h_p}\right)^2 \\ &+ \frac{2\beta C_b KS}{(1-\mu)^2(h_{rp} + h_p)(1-\beta/\alpha)} \\ &.\left(1 + \frac{C_s}{h_{rp} + h_p}\right) \end{aligned} \tag{32}$$

where K is the denominator in (20); that is

$$K = \beta h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{\alpha}+\frac{2\mu}{\gamma}\right)(1-\beta/\alpha) + \left(h_{rp}+h_p\right)\left((1-\mu)^2+\sigma^2\right)(1-\beta/\alpha)^2. \quad (33)$$

Rearranging the terms of (21), the following quadratic equation is obtained

$$\left(\frac{K}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1+\frac{C_s}{h_{rp}+h_p}\right)^2 - \left(C_s+h_{rp}+h_p\right)\right)S^2 + \left(\frac{2K\beta C_b}{(1-\mu)^2(h_{rp}+h_p)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right) - 2(1-\beta/\alpha)\beta C_b\right)S + \frac{K\beta^2 {C_b}^2}{(1-\mu)^2(h_{rp}+h_p)^2} - 2(1-\beta/\alpha)\beta\left(C_{0p}+C_1\right)=0 \quad (34)$$

Define

$$A = \frac{K}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1+\frac{C_s}{h_{rp}+h_p}\right)^2 - \left(C_s+h_{rp}+h_p\right)$$
$$B = \frac{2K\beta C_b}{(1-\mu)^2(h_{rp}+h_p)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right) - 2(1-\beta/\alpha)\beta C_b \quad (35)$$
$$C = \frac{K\beta^2 {C_b}^2}{(1-\mu)^2(h_{rp}+h_p)^2} - 2(1-\beta/\alpha)\beta\left(C_{0p}+C_1\right)$$

Then the optimal shortage size is

$$S^* = \frac{-B \pm \sqrt{B^2-4AC}}{2A}. \quad (36)$$

The optimal production quantity is obtained by substituting the value of S^* in (19) so that

$$Y^* = \frac{\beta C_b}{(1-\mu)(h_{rp}+h_p)} + \frac{S^*}{(1-\mu)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right). \quad (37)$$

To examine the nature of the optimal solution, the determinant of the Jacobian matrix with entries equal to the second partial derivatives should be calculated:

$$\frac{\partial^2 E[TPPU]}{\partial Y^2} = \frac{-2\beta}{(1-\mu)Y^3} \cdot \left(C_{0p}+C_1+C_bS+\frac{(C_s+h_{rp}+h_p)S^2}{2\beta(1-\beta/\alpha)}\right), \quad (38)$$

$$\frac{\partial^2 E[TPPU]}{\partial S^2} = -\frac{C_s+h_{rp}+h_p}{(1-\beta/\alpha)(1-\mu)Y}, \quad (39)$$

and

$$\frac{\partial^2 E[TPPU]}{\partial S \partial Y} = \frac{C_b\beta(1-\beta/\alpha)+\left(C_s+h_{rp}+h_p\right)S}{(1-\mu)(1-\beta/\alpha)Y^2}. \quad (40)$$

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial^2 E[TPPU]}{\partial Y^2} & \frac{\partial^2 E[TPPU]}{\partial S \partial Y} \\ \frac{\partial^2 E[TPPU]}{\partial S \partial Y} & \frac{\partial^2 E[TPPU]}{\partial S^2} \end{pmatrix}.$$

The determinant of the Jacobian matrix is

$$|J| = \frac{\left(C_s+h_{rp}+h_p\right)}{(1-\beta/\alpha)^2(1-\mu)^2Y^4} \left(2\beta\left(C_{0p}+C_1+C_bS\right)(1-\beta/\alpha)+(C_s+h_{rp}+h_p)S^2\right). - \frac{\left(C_b\beta(1-\beta/\alpha)+\left(C_s+h_{rp}+h_p\right)S\right)^2}{(1-\mu)^2(1-\beta/\alpha)^2Y^4}.$$

Hence,

$$|J| = \frac{\beta\left(2\left(C_s+h_{rp}+h_p\right)\left(C_{0p}+C_1\right)-{C_b}^2\beta(1-\beta/\alpha)\right)}{(1-\beta/\alpha)(1-\mu)^2Y^4} \quad (41)$$

In order for the solution (Y^*, S^*) to be the maximizer of the $E[TPPU]$, $|J|$ must be positive and both $\frac{\partial^2 ETPPU}{\partial Y^2}$ and $\frac{\partial^2 ETPPU}{\partial S^2}$ must be negative. Hence, the optimal solution (Y^*, S^*) given in (24) and (25) is the maximizer of the $E[TPPU(Y, S)]$ function whenever the following condition is

$$C_b \le \sqrt{\frac{2\left(C_s + h_{rp} + h_p\right)\left(C_{0p} + C_1\right)}{\beta(1-\beta/\alpha)}} \tag{42}$$

4 Collaborative Supply Chain

In the case when there is coordination between the supplier and produce, the optimal solution is obtained by maximizing the expected total profit function for the supply chain. From (26), we have

$$\begin{aligned}\frac{\partial E[TPCU]}{\partial Y} &= \frac{\partial E[TPPU]}{\partial Y} + \frac{\partial E[TPSU]}{\partial Y} \\ &= \frac{\beta}{(1-\mu)Y^2}\left(C_{0p} + C_1\right) - h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{2\alpha} + \frac{\mu}{\gamma}\right)\frac{\beta}{(1-\mu)} \\ &+ \left(C_b S + C_s \frac{S^2}{2\beta(1-\beta/\alpha)}\right)\frac{\beta}{(1-\mu)Y^2} \\ &- \frac{h_{rp}+h_p}{2(1-\mu)}\left(\left((1-\mu)^2+\sigma^2\right)\left(1-\frac{\beta}{\alpha}\right) - \frac{S^2}{(1-\beta/\alpha)Y^2}\right) \\ &+ \frac{C_{0s}\beta}{N(1-\mu)Y^2} - \frac{1}{2}h_{rs}(N-1).\end{aligned} \tag{43}$$

and

$$\begin{aligned}\frac{\partial E[TPCU]}{\partial S} &= \frac{\partial E[TPPU]}{\partial S} + \frac{\partial E[TPSU]}{\partial S} \\ &= -\left(C_b + C_s\frac{S}{\beta(1-\beta/\alpha)}\right)\frac{\beta}{(1-\mu)Y} \\ &+ (h_{rp}+h_p)\left(1 - \frac{S}{(1-\beta/\alpha)(1-\mu)Y}\right).\end{aligned} \tag{44}$$

Setting (44) and (43) equal to zero and solving each for Y, we have

$$\begin{aligned}Y &= \frac{\beta C_b}{(1-\mu)(h_{rp}+h_p)} \\ &+ \frac{S}{(1-\mu)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right),\end{aligned} \tag{45}$$

and

$$\begin{aligned}Y = \Bigg(&\Big(2\beta\left(C_{0s}/N + C_{0p} + C_1 + C_b S\right) \\ &+ \frac{(C_s+h_{rp}+h_p)S^2}{(1-\beta/\alpha)}\Big) \\ &\Big/\Big(h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{\alpha} + \frac{2\mu}{\gamma}\right)\beta \\ &+ (h_{rp}+h_p)\left((1-\mu)^2+\sigma^2\right)\left(1-\frac{\beta}{\alpha}\right) \\ &+ h_{rs}(N-1)(1-\mu)\Big)\Bigg)^{1/2}.\end{aligned} \tag{46}$$

Substituting (45) in (46), squaring both sides and cross multiplying, we obtain

$$\begin{aligned}&2\beta\left(C_{0s}/N + C_{0p} + C_1 + C_b S\right) + \frac{(C_s+h_{rp}+h_p)S^2}{(1-\beta/\alpha)} = \\ &\frac{K\beta^2 C_b{}^2}{(1-\mu)^2(h_{rp}+h_p)^2} + \frac{KS^2}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1+\frac{C_s}{h_{rp}+h_p}\right)^2 \\ &+ \frac{2\beta C_b K S}{(1-\mu)^2(h_{rp}+h_p)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right),\end{aligned} \tag{47}$$

where K is the denominator in 46; that is

$$\begin{aligned}K &= h_{rp}\left(\frac{(1-\mu)^2+\sigma^2}{\alpha} + \frac{2\mu}{\gamma}\right)\beta + h_{rs}(N-1)(1-\mu) \\ &+ (h_{rp}+h_p)\left((1-\mu)^2+\sigma^2\right)\left(1-\frac{\beta}{\alpha}\right)\end{aligned} \tag{48}$$

Rearranging the terms in the above equation we get

$$\begin{aligned}&\left(\frac{K}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1+\frac{C_s}{h_{rp}+h_p}\right)^2 - \frac{(C_s+h_{rp}+h_p)}{(1-\beta/\alpha)}\right)S^2 \\ &+ \left(\frac{2\beta C_b K}{(1-\mu)^2(h_{rp}+h_p)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right) - 2\beta C_b\right)S \\ &+ \frac{K\beta^2 C_b{}^2}{(1-\mu)^2(h_{rp}+h_p)^2} - 2\beta\left(C_{0s}/N + C_{0p} + C_1\right) = 0.\end{aligned} \tag{49}$$

Define the coefficients A, B and C by

$$A = \frac{K}{(1-\mu)^2(1-\beta/\alpha)^2}\left(1+\frac{C_s}{h_{rp}+h_p}\right)^2 - \frac{(C_s+h_{rp}+h_p)}{(1-\beta/\alpha)}$$
$$B = \frac{2\beta C_b K}{(1-\mu)^2(h_{rp}+h_p)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right) - 2\beta C_b$$
$$C = \frac{K\beta^2 C_b{}^2}{(1-\mu)^2(h_{rp}+h_p)^2} - 2\beta\left(C_{0s}/N + C_{0p} + C_1\right). \quad (50)$$

The solution of (49) is the optimal shortage size is

$$S^* = \frac{-B \pm \sqrt{B^2-4AC}}{2A}. \quad (51)$$

The optimal production quantity is obtained by substituting the value of S^* in (45) so that

$$Y^* = \frac{\beta C_b}{(1-\mu)(h_{rp}+h_p)} + \frac{S^*}{(1-\mu)(1-\beta/\alpha)}\left(1+\frac{C_s}{h_{rp}+h_p}\right). \quad (52)$$

Similar to the discussion from (38) to (41), (Y^*, S^*) given in (51) and (52) is the maximizer of the $E[TPCU(Y, S)]$ function for a certain condition on C_b.

In the following section, a numerical example is provided to illustrate the calculations of the optimal solution for both scenarios and to show the advantage of collaboration within the supply chain.

5 Numerical Examples

The daily demand and production rates for a certain item are β = 100 and α = 200. The producer's ordering cost of the raw materials is C_{0p} = \$100 with a unit purchasing cost C_{rp} = \$5. The unit production and setup costs are C_p = \$10 and C_1 = \$100. The producer holding cost rate is 2% per day so that h_{rp} = 0.02 (5) = \$0.1 per item per day, and h_p = 0.02 (10) = \$0.2 per item per day. The shortage cost is C_s = \$1 per unit per day and the administrative shortage cost is C_b = \$0. The percentage of imperfect quality items of raw materials is uniformly distributed between 10% and 30%. Hence, the expected percentage of imperfect quality items is μ = (0.1+0.3)/2 = 0.2 and the standard deviation is σ = (0.3–0.1)/√12 = 0.0577. The screening rate of imperfect quality items is γ = 1000 units per day and the unit screening cost is C_m = \$0.2. The imperfect quality items are sold at a discounted price of C_d = \$3 per unit and the finished product is sold at a unit price of R = \$20. The supplier ordering cost is C_{0s} = \$500 and unit purchasing cost of C_{rs} = \$3. The supplier holding cost rate is 4% per day so that h_{rs} = 0.04 (3) = \$0.12 per item per day.

In the case when the producer is the decision maker, K = 0.0663, A = 6.4849, B = 0, and C = –20,000, so that the optimal solution is to produce Y^* = 601.62 ≈ 602 units in each production run and to plan for S^* = 55.53 ≈ 56 units of shortages. The producer expected daily total profit is $E[TPPU(Y^*, S^*)]$ = \$341.89. The optimal policy calls for an inventory cycle length quantity of T^* = 4.8 days, and production period of $T_p{}^*$ = 2.4 days. Under this scenario, the supplier cycle length is determined by calculating $E[TPSU(Y^*, S^*)]$ for various values of N. Then, $E[TPSU(Y^*, S^*)]$ = \$146.11 for N = 1, $E[TPSU(Y^*, S^*)]$ = \$ 161.96 for N = 2, and $E[TPSU(Y^*, S^*)]$ = \$143.18 for N = 3. The supplier expected profit decreases for higher values of N. Hence, the supplier expected cycle length is two times that of the producer; that is, $E[NT]$ = 2 $E[T]$ = 9.6 days. The supply chain expected total profit is $E[TPCU(Y^*, S^*)]$ = \$503.85.

In a collaborative supply chain, $K1$ = 0.132667, $A1$ = 12.9699, B = 0, and C = –140,000, so that the optimal solution is to produce Y^* = 1125.53 ≈ 1126 units in each production run and to plan for S^* = 103.89 ≈ 104 units of shortages. The producer expected daily total profit is $E[TPPU(Y^*, S^*)]$ = \$693.64. The optimal policy calls for an inventory cycle length quantity of T^* = 9 days, and production period of $T_p{}^*$ = 4.5 days. Under this scenario, the supplier cycle length is again N = 1, and a corresponding expected total daily profit of $E[TPSU(Y^*, S^*)]$ = \$ 194.47. The supply chain expected total profit is $E[TPCU(Y^*, S^*)]$ = \$519.52.

6 Conclusion

An economic production quantity model with raw materials and quality within a supply chain is presented. Two scenarios were examined. The first considers the case when the producer is the decision maker. In the second scenario, the decision is made under a collaborative supply chain. The model accounts for the costs and quality of raw materials as well as the effects of planned shortages on the optimal solution. The mathematical models was derived and explicit expressions for the optimal production and shortage quantities were obtained. The uniqueness of the optimal solution was presented. Numerical examples were given to illustrate the models.

For future work, we suggest extending the models to the case of more than one type of raw materials is used in the production process. Also, we suggest studying the effects of quality of the finished product where on t hese models where imperfect quality finished items can be reworked or scraped. In other direction, we suggest extending the model to a t hree layer supply chain consisting of a supplier, a producer and retailer.

References:

[1] S. Pal, G. S. Mahapatra, G. P. Samanta, A Three-Layer Supply Chain EPQ Model for Price-and Stock-Dependent Stochastic Demand with Imperfect Item Under Rework, *Journal of Uncertainty Analysis and Applications*, 2016, Vol. 4, No. 1, 10.

[2] M. Khan, M. Y. Jaber, A. L. Guiffrida, S. Zolfaghari, A review of the extensions of a modified EOQ model for imperfect quality items, *International Journal of Production Economics,* 2011, Vol. 132, No. 1, pp. 1-12.

[3] M. K. Salameh, M. Y.,Jaber, Economic production quantity model for items with imperfect quality, *International Journal of Production Economics,* 2000, Vol. 64, N o. 1, pp. 59–64.

[4] F.W. Harris, What quantity to make at once. In: The Library of Factory Management. Operation and Costs, *The Factory Management Series*, Vol. 5, pp. 4 7–52, A.W. Shaw Co., Chicago, 1913.

[5] L. A. Johnson, D. C. Montgomery, *Operations research in production planning, scheduling, and inventory control,* Vol. 6, New York: Wiley, 1974.

[6] M. K. Salameh, A. N. El-Kassar, Accounting for the Holding Cost of Raw Material in the Production Model. *In Proceeding of BIMA Inaugural Conference,* March 2007, Sharjah, pp. 72-81.

[7] N. Yassine, Joint Probability Distribution and the Minimum of a Set of Normalized Random Variables, *Procedia-Social and Behavioral Sciences*, 2016, Vol. 230, pp. 235-239.

[8] A. N. El-Kassar, M. K. Salameh, M. Bitar, EPQ model with imperfect quality raw material, *Mathematica Balkanica*, 2012, V ol. 26, pp. 123-132.

[9] A. N. El-Kassar, M. K. Salameh, M. Bitar, Effects of time value of money on t he EPQ Model with the imperfect quality items of Raw material, *Proceedings of Academy of Information and Management Sciences*, New Orleans, 2012, Vol. 16, No. 1, pp. 11-18.

[10] T. Mikdashi, A.N. El-Kassar, A. Joudi, Effects of Probabilistic Imperfect Quality Items of Raw Material on the Optimal Production Lot Size, *Journal of the Academy of Business & Economics*, 2012, Vol. 12, No. 3, pp. 75-84.

[11] S. Pal, G. S. Mahapatra, A manufacturing-oriented supply chain model for imperfect quality with inspection errors, stochastic demand under rework and shortages, *Computers & Industrial Engineering*, 2017, 106, 299-314.

[12] M. Khan, M. Y. Jaber, Optimal inventory cycle in a two-stage supply chain incorporating imperfect items from suppliers, 2011, *International Journal of Operational Research*, Vol. 10, No. 4, pp. 442-457.

[13] S. S. Sana, A production-inventory model of imperfect quality products in a three-layer supply chain, *Decision Support Systems*, 2011, Vol. 50, No. 2, pp. 539-547.

[14] N. Yassine, A. Alsagheer, The Optimal Solution of a Production Model with Shortages and Raw Materials, *International Journal of Mathematical and Computational Methods*, 2017, Vol. 2, pp. 13-18.

[15] N. Yassine, The effects of approximating a probabilistic process using the renewal reward theorem and the optimal solution of an EOQ model with quality, *International Journal of Mathematics and Computers In Simulation*, 2016, Vol. 10, pp. 354-360.