Dominance among financial markets

SERGIO ORTOBELLI LOZZA[1,2], TOMAS TICHY [2], FILOMENA PETRONIO[2]
[1]Department SAEQM
[2]Deparment of Finance
[1]University of Bergamo
[2]VSB Technical University of Ostrava
[1]Via dei Caniana, 2 – 24127 Bergamo, ITALY
[2]Sokolská, 33 - 70121 Ostrava, CZECH REPUBLIC
sol@unibg.it; filomena.petronio@vsb.cz; tomas.tichy@vsb.cz

Abstract: - In this paper, we deal and evaluate the comparison problem among different financial markets using risk/variability measures consistent with investors' preferences. First, we recall a recent classification of multivariate stochastic orderings consistent with preferences and we properly define the selection problem among different financial markets. Secondly, we propose an empirical financial application where multivariate stochastic orderings consistent with the non-satiable and risk averse investors' preferences are applied to compare and evaluate the possible dominance among the most developed market in the world (the US stock market) and two European markets (the German stock market and the UK stock market). In this context, we propose an ex-ante and an ex-post evaluation of the dominance among country stock markets. Moreover, in both cases we evaluate the dominance, considering the "oldest" and "youngest" firms of selected countries over previous decade.

Key-Words: - Multivariate preferences, Stochastic Dominance, Financial Market comparison.

1 Introduction

At financial markets, we can identify many distinct problems, solution of which requires application of various mathematical methods. In this paper, we focus on the problem of portfolio selection and related issues.

In this framework, we introduce multivariate orderings consistent with investors' preferences and we show how they can be used in order to determine dominant sectors and markets in different financial contexts. Therefore, we define the dominance among financial markets and we propose a comparison methodology that uses probability functionals to optimize choices consistent with the investors' preferences. Then, we propose an ex-ante and an ex-post empirical application of multivariate orderings, in this context.

Thus, we first generalize the concept of univariate FORS orderings, risk and reward measures in the multivariate framework (see Ortobelli et al. in [6], [7] and [8]; FORS is an acronym derived from the name of the authors). FORS probability functionals and orderings generalize those found in the literature (see Shaked and Shanthikumar in [14], and Muller and Stoyan in [5]) and are strictly related to the theory of choice under uncertainty and to the theory of probability functionals and metrics (see Rachev in [11], Stoyanov et al. in [15] and Tversky, and Kahneman [17]). While the new orderings serve to further characterize and specify the investors' choices and preferences, the new risk measures should be used either to minimize the risk or to minimize its distance from a given benchmark. In particular, in the paper we suggest to use multivariate ordering consistent with investors' preferences to define the dominance among different financial markets/sectors.

Secondly, we propose an empirical comparison to evaluate the possible dominance among three different stock markets (US stock market, the German stock market and the UK stock market). In this framework, we preliminarily test the return distributions of each market, to understand which distributional assumption is suitable for a mean-risk comparison among stock markets. Then, we examine ex ante when one market dominates the others. Finally, we forecast and compare the ex-post dominance among markets. In this ex-post analysis we evaluate the future market evolution either using a myopic portfolio selection approach or forecasting the returns evolution with proper Markov Chains as suggested by Ortobelli et al. in [9] and Angelelli et al. in [2] and [3].

The paper is organized as follows. In Section 2 we introduce multivariate FORS orderings and the definition of orderings among markets. Section 3 introduces a preliminary ex-ante empirical analysis. In Section 4 we propose an ex-post empirical comparison among the three stocks markets. Finally, the last section summarizes and examine the main results of the paper and their potential possible applications.

2 Orderings among markets

In this section we examine FORS multivariate measures and orderings and we introduce the concept of orderings among financial markets.

Recall that the most important property that characterizes any probability functional associated with a choice problem is the consistency with a stochastic order. In terms of probability functionals, the consistency is defined as: X dominates Y with respect to a given order of preferences $\succ$ implies $\mu(X,Z)\le\mu(Y,Z)$ for a fixed arbitrary benchmark Z (where $X,Y,Z\in\Lambda$ that is a non-empty space of real valued random variables defined on $(\Omega,\Im,P)$). Since an univariate FORS measure induced by of preferences $\succ$ is any probability functional $\mu:\Lambda\times\Lambda\to R$ that is consistent with a given order of preferences $\succ$ we can similarly define multivariate FORS measures.

Definition 1 *We call FORS measure induced by a preference order* $\succ$ *any probability functional* $\mu:\Lambda\times\Lambda\to R^s$ *(where* Λ *a non-empty set of real-valued n-dimensional random vectors defined on the probability space (*$\Omega,\Im,P$*)) that is consistent with a given order of preferences* $\succ$ *(that is, if* X *dominates* Y *with respect to a given order of preferences* $\succ$ *implies* $\mu(X,Z)\le\mu(Y,Z)$ *for a fixed arbitrary benchmark* Z *where the vectorial inequality is considered for each component i.e.,* $\mu_i(X,Z)\le\mu_i(Y,Z)$ *for any i=1,...,s.*

As for the FORS measures we can easily extend the definition of multivariate FORS ordering developed in Ortobelli et al. (see [6] and [7]).

Definition 2 *Let* $\rho_X : A\to\overline{R}^s$ *(with compact and convex* $A\subseteq\overline{R}^n$*) be a bounded variation function, for every n-dimensional random vector* X *belonging to a given class* Λ*. Assume that* $\forall X,Y\in\Lambda,\rho_X=\rho_Y$ *, a.e. on* A *iff* $X\overset{d}{=}Y$*. If, for any fixed* $\lambda\in A$*,* $\rho_X(\lambda)$ *is a FORS measure induced by an ordering* $\succ$*, then*

$$\forall X,Y\in\Lambda_\alpha=\left\{X\in\Lambda\ :\ \left|\int_A\prod_{i=1}^n|t_i|^{\alpha_i-1}d\rho_X(t_1,\dots,t_n)\right|<\infty\right\}$$

for every $(\alpha_1,\dots,\alpha_n)$ *with* $\alpha_i\ge 1$*, we say that* X *dominates* Y *in the sense* α*-FORS ordering induced by* $\succ$ *, in symbols:*

$$X\ \underset{\succ,\alpha}{FORS}\ Y \text{ if and only if}$$
$$\rho_{X,\alpha}(u)\le\rho_{Y,\alpha}(u)\ \ \forall u\in A$$

where $\rho_{X,\alpha}(u_1,\dots,u_n)=\rho_X(u_1,\dots,u_n)$ *if* $\alpha_i=1$ *(*$i=1,\dots,n$ *) otherwise:*

$$\rho_{X,\alpha}(u_1,\dots,u_n)=\frac{1}{\prod_{i=1}^n\Gamma(\alpha_i)}\int_{a_1}^{u_1}\dots\int_{a_n}^{u_n}\prod_{i=1}^n(u_i-t_i)^{\alpha_i-1}d\,\rho_X(t_1,\dots,t_n)\quad .(1)$$

The integral in (1) is an s-dimensional vector where the integral is applied for each component of the vector $d\,\rho_X=[d\rho_{(1)X},\dots,d\rho_{(s)X}]$ *(whose components are the differential of the components of vector* ρ_X *).*

This expression generalizes the one proposed by Petronio et al. in [10]. Besides, we call ρ_X FORS measure associated with the FORS ordering of random vectors belonging to Λ. We say that ρ_X generates the FORS ordering.

Example 1: Consider the cumulative multivariate function associated with the vector *X*, $P_X(y)=P(X_1\le y_1,\dots,X_n\le y_n)=F_X(y_1,\dots,y_n)$.

It generates the lower orthant FORS order (see Shaked and Shanthikumar in [14]). So the measure associated to the α -FORS ordering is

$$F_X^{(\alpha)}(u)=\frac{1}{\prod_{i=1}^n\Gamma(\alpha_i)}\int_{-\infty}^{u_1}\dots\int_{-\infty}^{u_n}\prod_{i=1}^n(u_i-t_i)^{\alpha_i-1}d\,F_X(t)=$$
$$=E\left(\prod_{i=1}^n(u_i-X_i)_+^{\alpha_i-1}\right)/\prod_{i=1}^n\Gamma(\alpha_i).\qquad(2)$$

Multivariate orderings can have several applications in economics and finance. In this paper we discuss a possible application in ordering financial markets by the point of view of investors who has to choose the main market in which investing. With this aim we need to give some possible alternative definitions of orderings among financial markets/sectors.

Let us assume there are two markets: A with n assets, and B with s assets. Assume, the vector of the positions taken by an investor in the n risky assets of market A is denoted by $x=[x_1,...x_n]'$ and similarly the vector of the positions taken by an investor in the s risky assets of market B is denoted by $y=[y_1,...y_s]'$. We assume that no short sales are allowed.

Definition 3 *We say that a market/sector A with n assets strongly dominates another market/sector B with s assets with respect to a multivariate FORS ordering if for any vector of returns* Y_B *of* $t \le u = \min(s,n)$ *assets of market/sector B there exists a vector* X_A *of market/sector A such that:* $X_A FORS\, Y_B$. *Similarly we say that a market/sector A with n assets weakly dominates another market/sector B with s assets with respect to the FORS ordering if for any given portfolio of gross returns* $y'Y_B$ *of market/sector B there exists a portfolio* $x'X_A$ *of the market/sector A such that:* $x'X_A FORS\, y'Y_B$.

Example 2. Suppose that the return distributions of markets A and B are jointly elliptically distributed. Suppose the markets have the same number of assets n, vectors of averages μ_A, and μ_B and dispersion matrixes Q_A and Q_B such that $\mu_A \ge \mu_B$ and (Q_A-Q_B) is negative semi-definite. Then market A strongly dominates market B with respect to *the increasing concave multivariate order* (see Muller and Stoyan in [5]). Moreover, under these assumptions, market A weakly dominates market B with respect to the concave order since portfolio $x'\mu_A \ge x'\mu_B$ and $x'Q_A x \le x'Q_B x$ for any vector $x \ge 0$. Observe that this weakly dominance between elliptically distributed vectors is also known in ordering literature as *the increasing positive linear concave multivariate order* (see [5]).

Example 2 can be use in financial applications. In particular, if we assume that the returns of different markets are jointly elliptically distributed and they are uniquely determined by a risk measure and a reward measure, we can order the markets in a reward-risk framework. This observation is used in the following empirical analysis.

3. An ex-ante empirical comparison among the US, UK and German stock markets

In order to identify the dominance among different markets we compare the reward-risk investor's choices of three different stock markets (among the main developed ones): US (Nyse, Nasdaq), UK (London), and Germany (Frankfurt and Berlin). We consider all the returns in USD. Since it is not easy to prove the strong stochastic dominance among markets, then we try to evaluate the weakly stochastic dominance among the markets observing if one market dominates the other in a reward risk framework under the implicit assumption the returns of different markets are jointly elliptically distributed.

In particular, we first examine the statistical characteristics of the returns of each market. Secondly, we propose an ex-ante empirical analysis when we simply observe the dominance during the decade 2003-2013.

3.1 Empirical evidence

In this subsection, we analyze the stocks of NYSE, NASDAQ (US), London stock exchange (UK), Frankfurt and Berlin stock exchanges (Germany) used in the following portfolio empirical analysis.

We consider the stocks of the three countries starting from January 2003 till May 2013 and we test some distributional hypothesis. In particular, we want to know the empirical behaviour of the asset returns and if there exist an elliptical distribution that could be used to approximate the returns of each country.

We consider and test three possible distributions: Gaussian, Stable Paretian, and Student t. Recall that, the Central Limit Theorem for normalized sums of independent and identically distributed random variables determines the domain of attraction of each stable law $S_\alpha(\beta,\sigma,\delta)$, which depends on four parameters: the index of stability $\alpha \in (0,2]$, the asymmetry parameter $\beta \in [-1,1]$, the dispersion parameter $\sigma > 0$, and the location parameter δ (where $\alpha=2$ corresponds to the Gaussian law). For further details on stable distributions and their financial applications see [12] and [13]. To test whether asset returns follow a normal distribution, we compute the Jarque-Bera statistic with a 95% confidence level. Similarly, we employ Kolmogorov-Smirnov statistic with a 95% confidence level to test whether asset returns follow or a stable Paretian distribution or a Student t distribution.

Table 1 reports the results of the percentage of rejection of the statistical hypotheses and the values on average and on an annual basis for the maximum likelihood estimates of stable Paretian parameters (α, β, σ, δ), Student t parameters (mean, standard deviation, degrees of freedom) and the basic statistics of individual asset return series: mean, standard deviation, skewness, and kurtosis. These parameter estimates suggest the presence of a slight skewness (since the asymmetry parameter β and the skewness are near to zero) and of heavy tails (since the kurtosis exceeds three, the stability parameter α is less than two and the Student degrees of freedom are small). Based on these tests, we find that the stable Paretian distributional hypothesis is rejected on average for less than 20% of the cases for each country. While the Gaussian hypothesis is rejected on average for more than 75% of the cases for each country and the Student t distributional hypothesis is rejected on average for less than 25% for each country.

Table 1. *Statistics of the asset returns on average and an annual basis: mean, standard deviation, skewness, kurtosis, and Maximum Likelihood Estimates of the stable Paretian parameters (α,β,σ,δ) and of Student t (μ,σ,ν). Percentages of assets rejected with Jarque-Bera (J-B) test (95% confidence level) Kolmogorov-Smirnov (K-S) test (95% confidence level).*

Hypothesis:Gaussian distribution					
	Mean	**St. dev.**	**Skew.**	**Kurt.**	**% J-B rejected**
Germany	1.21%	16.73%	-0.016	6.841	80.21%
UK	-0.51%	13.22%	-0.0036	6.312	75.43%
USA	2.31%	11.81%	0.031	6.195	79.71%
Hypothesis:Stable Paretian Distribution					
	Alpha α	**Beta B**	**Sigma σ**	**Delta δ**	**% K-S rejected**
Germany	1.642	-0.067	17.43%	-2.1%	19.93%
UK	1.701	-0.027	12.91%	-4.9%	16.11%
USA	1.619	0.032	11.23%	3.45%	17.77%
Hypothesis:Student *t*					
	Mean μ	**St. dev. Σ**	**degrees ν freedom**	**% K-S rejected**	
Germany	1.22%	16.59%	4.56	20.33%	
UK	-0.5%	13.14%	5.91	21.71%	
USA	2.29%	11.77%	4.97	24.96%	

From these preliminary tests and analyses, it is reasonable to conclude that we can assume a joint non-Gaussian elliptical distribution for the assets returns since there is not strong evidence of skewness and we observe a strong evidence of heavy tails. Typically we could assume that the returns are jointly alpha stable sub-Gaussian distributed (see [12]) or that returns follow a multivariate Student t.

3.2 An ex-ante comparison among stock markets

In this subsection, we evaluate the weakly stochastic dominance among the US, UK and German stock markets in a reward risk framework.

Clearly, we suppose that the distributional assumptions of Example 2 are verified for all the three markets. In particular, as reward measure we use the mean, while as risk measure we use either the variance or the Conditional Value-at-Risk, *CVaR*,[1] expressed as:

$$CVaR_\alpha(X) = \frac{-1}{\alpha}\int_0^\alpha F_x^{-1}(u)du \qquad (5)$$

In the following empirical analysis we use $\alpha = 5\%$.

Let us introduce some notation. The gross returns on date $t+1$ of the n assets are denoted as $z_{t+1} = [z_{1,t+1},\ldots,z_{n,t+1}]'$. Generally, we assume the standard definition of gross return between time t and time $t+1$ of asset i, as $z_{i,t+1} = \frac{s_{i,t+1} + d_{i,[t,t+1]}}{s_{i,t}}$, where $s_{i,t}$ is the price of the *i*-th asset at time *t* and $d_{i,[t,t+1]}$ is the total amount of cash dividends paid by the asset between t and $t+1$. We distinguish the definition of gross return from the definition of return, i.e., $z_{i,t} - 1$ or the alternative definition of log returns $r_{i,t} = \log z_{i,t}$. The vector $x = [x_1,\ldots,x_n]'$ indicates the positions taken in the *n* assets, i.e., the portfolio weight x_i represents the percentage of wealth invested in the *i*-th asset. Assuming that no short sales are allowed, the vector *x* of portfolio weights belongs to the $(n-1)$-dimensional simplex $S = \{x \in \Re^n \mid \Sigma_{i=1}^n x_i = 1; x_i \geq 0\}$.

We consider the stocks of the three markets starting from January 2003 till May 2013. Every three months (60 daily observations) we estimate the reward-risk efficient frontiers of the three markets using:

a) the first 150 most traded (in average) assets which were active during the last 12 years (3000 daily historical observations);

b) the first 350 most traded (in average) assets which were active during the last 4 years (1000 daily historical observations).

Therefore, every three months we use a moving window either of 12 years or of 4 years. In this analysis we consider a dynamic dataset whose data are taken from DataStream. We identify the most traded assets of each market computing the mean of

[1]See Szegö in [16] and the references therein.

the daily average of traded value of each asset that is given by:

Daily average of traded value=
=Closing price $\times$ *Daily volume.*

Once the mean of the daily average of traded value is computed over the historical period of observation (that is either 12 years or 4 years) we order them and we select the most traded in each market. Therefore, every trimester (60 trading days), starting from January 1, 2003, we fit the mean risk efficient frontiers of the three different markets for their oldest and youngest firms. With this double comparison we evaluate the dynamicity of each market comparing the contributions of the recent firms and of the oldest ones.

Thus, at the *k*-th recalibration time (k = 1, 2,…,45), the following steps are performed:

Step 1 Preselect the most traded assets for each market and for each class of firms (old and young).

Step 2 Fit the mean risk efficient frontier solving the optimization problem for 30 levels of mean *m*:

$$\min_x \rho(x'z)$$
$$s.t.$$
$$\sum_{i=1}^{n} x_i = 1; \quad x'E(z) = m$$
$$x_i \geq 0; i = 1,..., n \tag{3}$$

where $\rho(x'z)$ is the risk measure (variance or CVaR) associated to the portfolio $x'z$.

The two steps are repeated for the three markets the two different class of firms and until the observations are available. The results of this empirical analysis are reported in Table 2 and Figures 1 and 2.

Table 2. Number of trimesters (January 2003- May 2013) there exist a reward-risk dominance among markets

	Analysis that uses the first 150 most traded assets active during the last 12 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	41	42	0	0	22	15
Mean-CVaR	14	0	0	15	21	15
	Analysis that uses the first 350 most traded assets active during the last 4 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	0	2	10	31	3	19
Mean-CVaR	0	2	10	17	2	11

Table 2 reports the number of trimesters a market dominates another one in terms of reward risk analysis during the decade January 2003–May 2013.

We point out with:

1) UG the number of times the US market dominates the German one;
2) UL the number of times the US market dominates the London stock exchange market;
3) LU the number of times the London stock exchange market dominates the US market;
4) GU the number of times the German market dominates the US market;
5) LG the number of times the London stock exchange market dominates the German market;
6) GL the number of times the German market dominates the London stock exchange market.

First of all, we observe that there exists a strong difference between the comparison which uses the oldest firms with respect to the youngest of the markets. Considering the oldest firms we observe that generally US market dominates the other two in the mean variance framework but not always in the mean-CVaR framework. Moreover, we observe a different behavior before the crisis (2003- half 2008) and during the crisis (half 2008-2013). Before the crisis several times the oldest firms of the London stock exchange market present a much better behavior in terms of reward-risk than the analogous firms of the German market. By contrast, during the crisis it happened exactly the vice versa, since the German stock market sometimes provided better performance even than the US market.

This is also confirmed by Figures 1 and 2, which report the mean-risk efficient frontiers of some cases of observed dominance before and during the crisis, considering the firms existing during the last twelve years before the examination.

It is useful to observe that when we consider the first 350 most traded assets active during the last 4 years for each market the obtained results are completely different. Table 2, Figure 3 and Figure 4 show that the youngest firms of the German market present the best performance in particular during the crisis, while before the crisis (2003 – half 2008) the London stock exchange market sometimes dominates the US market and the German one.

Moreover, using the youngest firms we observe that the dominance results in terms of mean variance or mean – CVaR are not too different.

4. An ex-post empirical comparison among the US, UK and German stock markets

In this section, we propose an ex-post empirical analysis where we forecast the dominance at a given time and we verify if the forecasted dominance holds. In this analysis, we use the same dataset of the previous section, splitting the 150 oldest most traded firms from the 350 youngest of each market. Moreover, we examine two alternative ways to evaluate the ex-post dominance.

In the first case, we use a *myopic* methodology (often applied in the financial practice, see, for a discussion, Angelelli and Ortobelli in [1]) that does not use the time evolution of the wealth process.

With this approach we assume that investors recalibrate their portfolios every T periods considering a predictable wealth process. Thus, the efficient frontiers obtained with the myotic approach to forecast the future dominance among markets are the same we get solving optimization problems of type (3).

In the second approach, we assume that the portfolio returns follow a Markov process and thus we approximate the future wealth distribution using a proper Markov chain.

In the following section we briefly summarize this assumption that has been widely used in finance (see, among others, Ortobelli et al. in [9] and Angelelli et al. in [2] and [3]).

4.1 A non parametric Markovian framework

In this subsection, we describe the behaviour of portfolios through a homogeneous Markov chain.

We show how to determine the future wealth distribution considering a discrete sequence of investor wealth W_k equally spaced in time $k=0,1,\dots,T$ (e.g. days). The initial wealth (i.e. $W_0=1$) is invested at time $k=0$ in *n* risky assets. In a dynamic framework the percentage of wealth invested in each asset could change over time.

However, for sake of simplicity, in this paper we study and describe all admissible wealth processes $W(x)=\{W_t(x)\}_{t\ge 0}$ depending on an initial portfolio of weights $x\in S$ that is assumed constant over time.

Moreover, we assume that these wealth processes are adapted processes defined on a filtered probability space $(\Omega,\phi,(\phi_t)_{0\le t\le\infty},\Pr)$. Thus, the gross return of a portfolio x during a period $[t,t+1]$ is given by $z_{(x),t+1}=x' z_{t+1}=\Sigma_{i=1}^n x_i z_{i,t+1}$. From a financial model point of view we assume that the gross returns have a Markovian behavior and can be modeled with an homogeneous Markov chain. Thus, we have to discretize the support of any portfolio.

Given a set $\Upsilon(x)=\{z_{(x),[-h]} \mid h=0,\dots,H-1\}$ of H past observations of the portfolio gross returns, we define N states denoted as $Z(x)=[z_{(x)}^{(1)},\dots,z_{(x)}^{(N)}]'$ in the interval $(\min\Upsilon(x);\max\Upsilon(x))$ where w.l.o.g. we assume $z_{(x)}^{(s)}>z_{(x)}^{(s+1)}$ for $s=1,\dots,N-1$. In general, the wealth obtained with the portfolio $x\in S$ at time $k=1,2,\dots$ is a random variable $W_k(x)$ with a number of possible values increasing as a polynomial of order N in variable k. In order to keep the complexity of the computation reasonable, we first divide the portfolio support $(\min\Upsilon(x);\max\Upsilon(x))$ in N intervals $(a_{(x),i},a_{(x),i-1})$, where $a_{(x),i}$ (decreasing with index *i*) is given by: $a_{(x),i}=\left(\frac{\min\Upsilon(x)}{\max\Upsilon(x)}\right)^{i/N}\cdot\max\Upsilon(x)\ i=0,1,\dots,N$; then, we compute the return associated to each state as the geometric average of the extremes of the interval $(a_{(x),i},a_{(x),i-1})$ that is

$$z_{(x)}^{(s)}=\sqrt{a_{(x),s}a_{(x),s-1}}=\max Y(x)\left(\frac{\max\Upsilon(x)}{\min\Upsilon(x)}\right)^{\frac{1-2s}{2N}},\quad s=1,2,\dots,N$$

As a consequence, $z_{(x)}^{(s)}=z_{(x)}^{(1)}u^{1-s}$, where $u=\sqrt[N]{\frac{\max\Upsilon(x)}{\max\Upsilon(x)}}>1$ and the wealth $W_k(x)$ obtained along a path after k steps (i.e. at time k) can only assume $1+(N-1)k$ distinct values instead of $O(k^N)$. We denote such property as the *recombining effect*. Thanks to the recombining effect of the wealth $W(x)$, the possible values of $W_k(x)$ up to time T $(k=1,\dots,T)$ can be stored in a matrix with T columns and $1+(N-1)T$ rows resulting in $O(NT^2)$ memory space requirement.

The transition matrix $P(x)_k=[p_{i,j;k}(x)]_{1\le i,j\le N}$ valued at time k measures the probabilities $p_{i,j;k}(x)$ (valued at time k) of the transition process from state $z_{(x)}^{(i)}$ at time k to state $z_{(x)}^{(j)}$ at time $k+1$. In this paper we only consider homogeneous Markov chains, so transition matrix does not depend on time and it can be simply denoted by $P(x)$. In order to simplify the notation, when the choice of the portfolio can be tacitly understood, we omit the reference to the portfolio x.

Thus, the transition matrix will be denoted simply as P and similarly we get the probability $p_{i,j}$, the wealth W_k, the state $z^{(s)}$ and so on. Moreover with a little abuse of notation we will use the terms " s-th state" or "state s" of the Markov chain to point both the return $z^{(s)}$ and the index s itself; context will make clear the meaning of the term. The entries $p_{i,j}$ of matrix P are estimated using the maximum likelihood estimates $\hat{p}_{i,j} = \frac{\pi_{ij}}{\pi_i}$ where π_{ij} is the number of historical observations that transit from the state i to the state j (i.e. from $z^{(i)}$ to $z^{(j)}$) and π_i is the number of historical observations in state i. The $(N-1)k+1$ values of the wealth $W_k = [w^{(l,k)}]_{1\le l\le (N-1)k+1}$ after k periods can be computed by the formula:

$$w^{(l,k)} = \left(z^{(1)}\right)^k \cdot u^{(1,l)}, l = 1,\ldots,(N-1)k+1 .$$

Thus, the *l*-th node at time *k* of the wealth-tree corresponds to wealth $w^{(l,k)}$. The procedure to compute the distribution function of the future wealth is strictly connected to the recombining feature of the wealth-tree. Under these assumptions Iaquinta and Ortobelli in [4], have shown how to compute the unconditional and conditional (conditional on the initial state s_0, i.e. $z^{(s_0)}$) probability of each node of the future wealth.

4.2 An ex-post comparison among stock markets

In this subsection, we apply the myopic and Markovian approaches to evaluate the ex-post dominance among US, UK and German stock markets. For both approaches (myopic and Markovian) we propose an algorithm (very similar to the one proposed in Section 3.2) to test if there exist dominance among markets.

Therefore, every trimester (60 trading days), starting from the first January 2003, we fit the mean risk efficient frontiers of the three different markets for their oldest and the youngest firms. For the Markovian approach we use N=9 states. Thus the final wealth after 60 days is described by 481 nodes with the Markov approximation.

Then we verify when the observed dominance applies in the future trimester. Thus, at the *k*-th recalibration time (*k* = 1, 2,…,45), the following steps are performed:

Step 1 Preselect the most traded assets for each market and for each class of firms (old and young).

Step 2 Fit the mean risk efficient frontier solving the optimization problem for 30 levels of mean *m*:

$$\begin{aligned} &\min_x \rho(W_{60}(x)) \\ &s.t. \\ &W_0 = x'e = 1; \quad E(W_{60}(x)) = m \\ &x_i \ge 0;\ i = 1,\ldots,n \end{aligned} \tag{4}$$

where $\rho(W_{60}(x))$ is the risk measure (variance or CVaR) associated to the forecasted wealth $W_{60}(x)$ obtained after 3 months (60 trading days) with the portfolio weights $x = [x_1,\ldots x_n]'$ (the initial wealth is equal to 1, i.e., $W_0 = \sum_{i=1}^n x_i = 1$). Observe that the efficient frontier we get with the myotic approach is the same we get solving optimization problem (3).

While to solve the optimization problem under the Markovian hypothesis we use the heuristic for global optimization proposed by Angelelli and Ortobelli in [1].

Step 3 Once we observe a dominance among two markets as a solution of problems (4) we verify after 3 months if the dominance holds.

The three steps are repeated for the three markets the two different class of firms, the two different approaches and until the observations are available.

The results of this empirical analysis are reported in Tables 3 and 4.

Table 3. Number of trimesters (January 2003–May 2013) we forecast a reward-risk dominance among markets using the myopic methodology.

	Analysis that uses the first 150 most traded assets active during the last 12 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	31	32	0	0	12	8
Mean-CVaR	8	0	0	7	11	7
	Analysis that uses the first 350 most traded assets active during the last 4 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	0	0	6	17	1	11
Mean-CVaR	0	0	4	8	0	7

Table 3 reports the number of trimesters we are able to predict the dominance of a market with respect to another one using the myopic approach during the decade January 2003- May 2013. Obviously the number of trimesters observed in Table 3 are always lower than the analogous observed in Table 2.

Practically, with the myopic approach we estimate “today” the future dominance of a market assuming that should apply “tomorrow” anytime is verified today. Thus Table 3 measures, in some sense, the persistency of market dominance results.

Obviously, even from Table 2, we know that the persistence of dominance is not true in absolute terms. However, Table 3 suggests that there exists this persistence of dominance results for several periods.

Table 4 reports the number of trimesters we are able to predict the dominance of a market respect to another one using the Markovian approach during the decade January 2003- May 2013. It is interesting to observe that the Markovian approach is able to predict the dominance among markets a number of times greater than the analogous observed in Table 3.

Table 4. Number of trimesters (January 2003- May 2013) we forecast a reward-risk dominance among markets using the Markovian methodology.

	Analysis that uses the first 150 most traded assets active during the last 12 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	34	34	0	0	17	10
Mean-CVaR	9	0	0	8	16	9
	Analysis that uses the first 350 most traded assets active during the last 4 years					
	UG	UL	LU	GU	LG	GL
Mean-Var	0	1	6	23	2	15
Mean-CVaR	0	1	4	11	1	7

Therefore, Table 4 suggests that the Markovian approach is able to predict better the dominance among markets with respect to the myopic approach.

5 Conclusions

FORS orderings can be used to extend several results of the theory of integral stochastic orderings that can be used to solve many financial problems. In this paper, we propose an extension of the concept of multivariate FORS stochastic orderings and then we compare the reward risk behaviour of three developed countries.

In this framework, we propose a possible application where multivariate preferences are applied to order three financial stock markets (US, German and UK). In particular, we identify the concept of dominance among different markets and we propose an ex-ante and an ex-post empirical comparison to evaluate their dominance relationships when we assume the returns are elliptically distributed.

With the ex-ante empirical analysis we observe that several times there exists reward risk dominance among the financial stock markets of different countries. Moreover, we also evaluate the dominance of the “oldest” and “youngest” firms of the different countries. Considering the US oldest firms generally dominates the ones of the other two countries in the mean variance framework but not always in the mean-CVaR framework.

This aspect suggests that the big losses observed during the crisis have a stronger impact in the US stock market than in the UK and in the German ones. This is also confirmed by the youngest German firms which present better performance in the analysed decade (2003-2013). In particular, we observe a different behaviour before the crisis (2003- half 2008) and during the crisis (half 2008-2013). Before the crisis several times the oldest and youngest firms of the London stock exchange market present a much better behaviour in terms of reward-risk than the analogous firms of the German market. While during the crisis exactly vice versa happens – the German stock market presents better performance even than the US market.

With the ex-post empirical analysis we evaluate with different models if we are able to forecast the dominance among the financial stock markets of different countries. In this context we observe that the dominance results are often persistent during the decade (2003-2013). Moreover, we show that predicting the wealth evolution with an approximating Markov process we are often able to forecast the dominance between markets.

In this analysis, we also evaluate the dominance of the “oldest” and “youngest” firms of the different countries. Considering the US oldest firms generally dominates the ones of the other two countries in the mean variance framework but not always in the mean-CVaR framework. On the one hand, the methodology presented in this paper could be very useful for investors who want to optimize their international portfolio. In particular, this analysis can be generally applied to preselect the “best” markets where to invest. On the other hand, the strong differences observed between the two reward-risk approaches suggest that the optimal choices cannot be easily described by only two parameters. Thus, further analyses and comparisons that account the return distributional behaviour seem to be necessary to better describe orderings among markets.

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Fig. 1: Mean-Variance dominance considering the firms existing during the last twelve years before the examination.

Example of Case 2003-2008 Example of Case 2008-2013

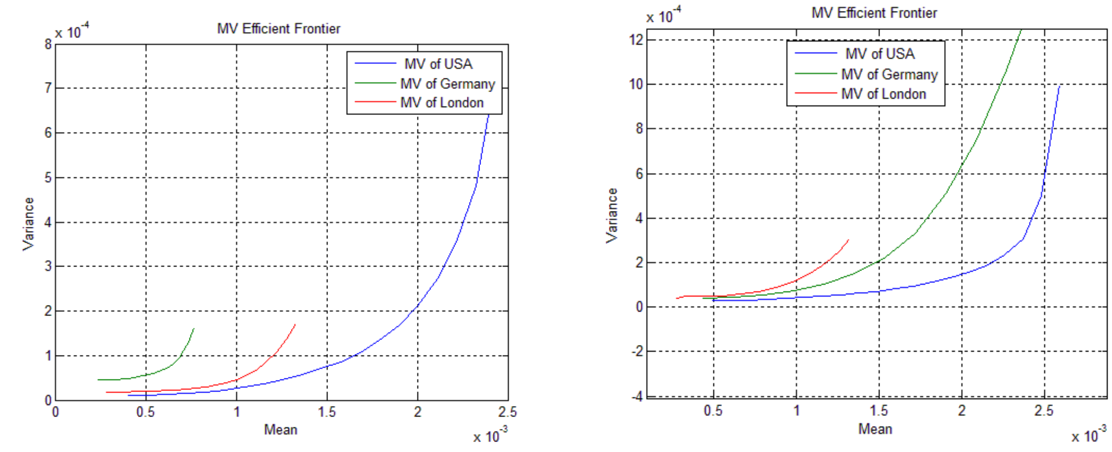


Fig. 2: Mean-CVaR dominance considering the firms existing during the last twelve years before the examination.

Example of Case 2003-2008 Example of Case 2008-2013

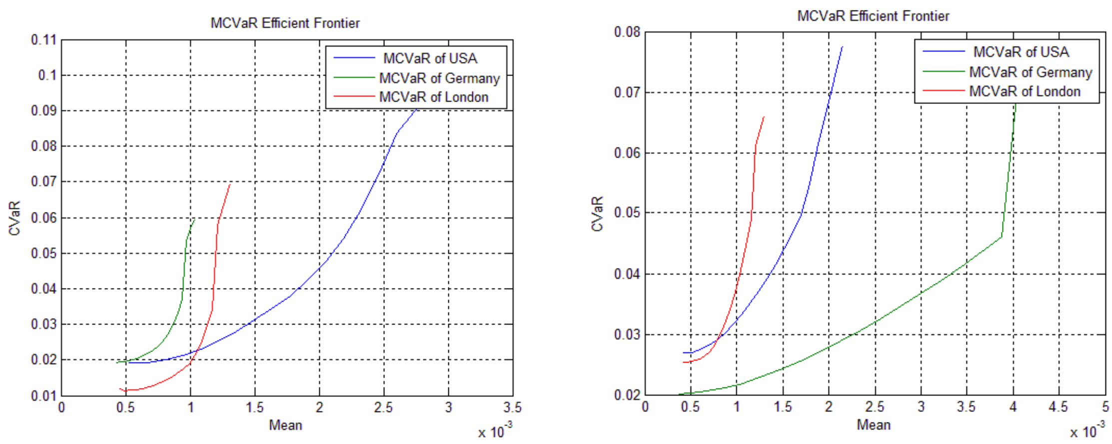
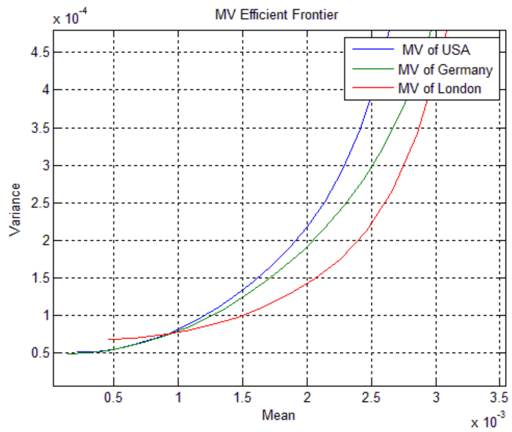


Fig. 3: Mean-Variance dominance considering the firms existing during the last four years before the examination.

Example of Case 2003-2008



Example of Case 2008-2013

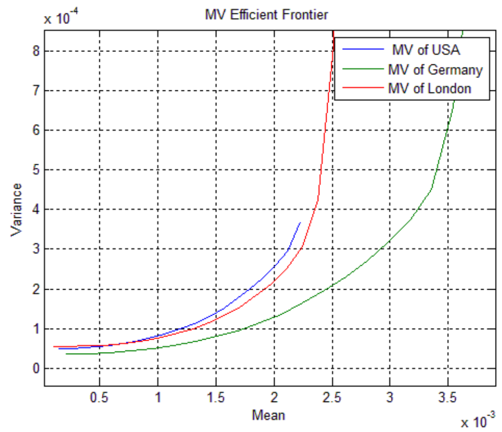
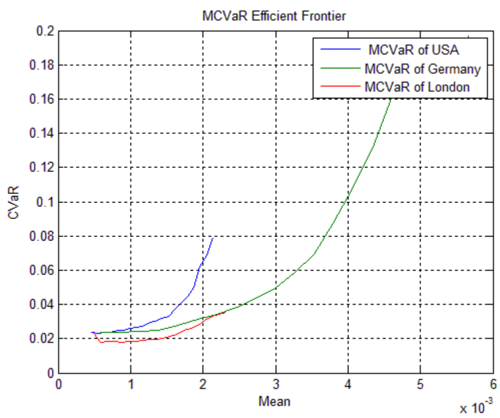


Fig. 4: Mean-CVaR dominance considering the firms existing during the last four years before the examination.

Example of Case 2003-2008



Example of Case 2008-2013