Pricing and service decisions in a dual-channel supply chain with manufacturer’s direct channel service and retail service

LISHA WANG and JING ZHAO
School of Science
Tianjin Polytechnic University
399, Binshui West Road, Xiqing District, Tianjin 300387
CHINA
wanglisha391@163.com; tjbentley2005@163.com

Abstract: This paper studies pricing decisions and service decisions of a product in a dual-channel (i.e., the traditional retail channel and direct online channel) supply chain consisting of one manufacturer and one retailer. We consider two types of channel pricing decisions (i.e., consistent pricing and inconsistent pricing) in the scenario which the manufacturer holds more bargaining power than the retailer and thus as the Stackelberg leader. By applying a game-theoretical approach, corresponding analytic equilibrium solutions are obtained. At last, we use numerical examples to compare the analytical results and illustrate the impacts of the degree of customer loyalty to the retail channel on the manufacturer and the retailer’s optimal prices, optimal service levels, maximal demands and maximal profits.

Key–Words: Dual-channel, Direct channel service, retail service, Consistent pricing, Inconsistent pricing, Pricing and service decision

1 Introduction

Recently, with the rapid growth of science and technology, and the improvement of the standard of living, consumers aim for higher-quality products and services, preferring manufacturers who are able to provide diverse selling modes. An increasing number of people has begun to shop through the Internet, although some still prefer to shop in stores personally. Then more and more manufacturers redesign their traditional channel structures by engaging in direct online sales to reach different customer segments that cannot be reached by the traditional retail channel. For instance, Sony, Dell Computer and Cisco System have begun selling to end customers through direct channels. These are giving birth to the dual-channel (i.e., a combination of the traditional retail channel and the direct online channel).

Modern supply chain managers are required to possess a set of competencies or multiple intelligences in order to meet the pressing business challenges[1]. Lee[2] proposes a favorable method combining multiple intelligences theory with ANP (Analytic Network Process) approach to help companies that need to select competent supply chain managers. There have been a lot of researchers that consider the dual-channel problems in supply chains. Previous much research on the dual-channel supply chain tended to focus on pricing decisions, and showed that a direct online channel plays a role of exerting potential competition pressure on the existing retailer by increasing the manufacturer’s negotiation power and reducing the double marginalization in the retail market even though its profit is non-positive (Dan et al. [3]). Chun and Kim [4] analyzed why the price differences between a conventional retail channel and a direct online channel occur. Ahn et al. [5] studied the pricing decisions of a manufacturer’s direct online channel and traditional retail channel, when the two channels have spatially separated markets. Chiang et al. [6] used a game theory model to study the price competition between a manufacturer’s direct online channel and its traditional channel partner. They argued that the direct online channel allows a manufacturer to constrain the partner retailer’s pricing behavior, but this may not always be detrimental to the retailer because the constraints on pricing may be accompanied by wholesale price reduction. Considering the market power balance between the channel members, researchers investigated three game models in the supply chain. Wenjing Si and Junhai Ma [7] present a theoretical model for remanufacturing system which is a closed-loop supply chain consists of one manufacturer, one dealer and one recycler, and put forward some coordination suggestions. Wei Jie et al. [8] establish five pricing models under decentralized decision cases, including the MS-Bertrand, MS-Stackelberg, RS-
Bertrand, RS-Stackelberg, and NG models, with consideration of different market power structures among channel members. In a dual-channel supply chain, Keenan [9] had given examples where the manufacturers took steps to explain to the retailers that their direct channel is targeted to a different market segment. Cai et al. [10] evaluated the impact of price discount contracts and pricing schemes on the dual-channel supply chain competition, meanwhile, from supplier-Stackelberg, retailer-Stackelberg, and Nash game theoretic perspectives, also showed that the scenarios with price discount contracts can outperform the non-contract scenarios. Zhang et al. [11] investigated three game models to analyze the influences of pricing decisions on product substitutability and channel power with the view of manufacturers, retailers, the entire supply chain and customers respectively. In addition, Lippmand and Macardle [12] considered a competitive version of the classical newsvendor problem in which a firm must choose an inventory or production level for a perishable good with random demand, with the optimal solution being a fractal of the demand distribution. Karjalainen [13] investigated the impact of competition upon industry inventory, and the firm demands in his models are independent. They analyzed the case where independent firm demands are aggregated to form industry demand. Furthermore, some researchers studied coordination in the dual-channel supply chain, such as Taylor [14], Cachon and Lariviere [15], Bernstein et al. [16], and so on.

With the current dynamic and competitive environment, the market must compete with more complicated strategies than simply lowering their price. For example, in auto industry, financial services such as auto loan, insurance, and maintenance service play an important role in selecting a brand for customers. IBM and HP are famous for their customer support. One of the major concerns for end customers is not only how low the price is, but also how good the after-sale and repair service is. In addition, the manufacturer can offer direct online channel service, which including presale service (technical service, counseling service, product advertising, etc.), in-sale service (on-time product delivery, etc.) and after sale service (subsequent tracking service, enjoy all-around technical support and counseling all the time, etc.). Early research focusing on service can be found in the economics literature, e.g., Bernstein [17], Zarei et al. [18] and Dixit [19]. In marketing literature, Perry and Porter [20] focused on a type of service that has a positive externality effect across the retailers. Lederer and Li [21] considered a more general problem by considering multiple classes of customers with general service time that is class-specific. In addition, these retailers usually offer more service as a dimension of competition in the dual-channel supply chain because they directly serve the customers. Xia and Gilbert [22] showed that a manufacturer can benefit from offering traditional retail channel service. Yan and Pei [23] focused on the strategic role played by the retail services in a dual-channel competitive market. Yao and Liu [24] showed that introducing a direct online channel not only generates competitive pricing and payoffs, but also encourages cost effective retail services. However, the above papers just focused on the traditional retail channel services or the manufacturer’s direct channel service. Many models have also addressed service competition, such as Ba et al. [25], De Borger and Van Dender [26], Darian et al. [27], and Kurata and Nam [28].

Based on the above literature, we present an analytical framework for both retail channel services and manufacturer’s channel services in a decentralized dual-channel competitive market. For a decentralized dual-channel supply chain, we formulate a Stackelberg game model, with the manufacturer as the leader, determining the direct sale price in the direct channel as well as the wholesale price, the retailer is represented as the follower, determining his own retail service and retail price given the manufacturer’s direct channel price and wholesale price. We consider two types of channel pricing decisions - consistent pricing and inconsistent pricing. A consistent pricing decision exists if the direct online channel and the retail channel are priced the same; otherwise, it is inconsistent pricing. By using the two-stage optimization technique and Stackelberg game, we evaluate the impacts of the manufacturer’s direct channel service and the traditional retail service, the degree of customer loyalty to the retail channel on the manufacturer and the retailer’s pricing behaviors, and illustrate the effectiveness of the price-sensitive parameter on the optimal prices, optimal service levels, and maximal expected profits. Based on our results, we derive optimal market strategies and managerial insights for the chain members.

The rest of this paper is organized as follows. Section 2 introduces the notation and formulates the decision models for the manufacturer and the retailer in consistent pricing and inconsistent pricing. Section 3 discusses the decision models and obtains the corresponding analytical equilibrium solutions by solving the established models. In section 4, we report the results of numerical experiments and explore the effects of the price-sensitive parameter and the degree of customer loyalty to the retail channel on the optimal prices, optimal service levels, and maximal expected profits. We conclude the results and suggest topics for future research in Section 5.
2 Problem Description

We consider a supply chain consisting of a manufacturer and a retailer. The manufacturer produces a single product at a unit cost \( c \) and distributes it through the direct channel at direct channel price \( p_m \) and the traditional retail channel at wholesale price \( w \). The retailer will resell the product at retail price \( p_r \). The manufacturer’s direct channel service level is denoted as \( s_m \) and the retail service level is denoted as \( s_r \). The manufacturer’s decision variables are \( w, p_m \) and \( s_m \), the retailer’s decision variable is \( p_r \) and \( s_r \). Similar to Dan et al. [3], we assume that customers are either retail channel loyal or direct channel loyal. Let \( \delta \) (\( 0 < \delta < 1 \)) be the degree of customer loyalty to the retail channel. Correspondingly, \( 1-\delta \) is the degree of customer loyalty to the direct channel. The retailer loyal market size is denoted by \( a_r \) and the manufacturer loyal market size is \( a_m = (1-\delta)a \). We model the decision process as a Stackelberg game, where the manufacturer is the leader and the retailer is the follower. The manufacturer and the retailer make their decisions in order to achieve their own maximal profits, respectively. Let parameter \( a \) represents the market base of the product. The schematic representation of this model is shown in Fig.1.

![Main model diagram](image)

Fig. 1: Main model.

Let \( D_m \) denote the consumer demand to the manufacturer and \( D_r \) the consumer demand to the retailer. The demand function for the percent market size has a linear form with slope of \( \beta \). If the price is consistent, the channel price is denoted by \( p \) (\( p = p_r = p_m \)). If the price is inconsistent, (i.e., \( p_r \neq p_m \)), some customers might switch from one channel to the other due to the price gap with the price-sensitive parameter \( \lambda \). The corresponding channel demand functions to the manufacturer and the retailer are described as follows, which have been adopted in Cai et al. [10], Ingene and Parry [29], and many others.

The demand functions under consistent pricing decision:

\[
D^*_m = a_r(1 - \beta p) + \beta_1 s_r - \beta_2 s_m
\]

\[
D^*_r = a_m(1 - \beta p) + \beta_1 s_m - \beta_2 s_r
\]

The demand for the product is increasing in its own channel service and decreasing in the other channel service. Moreover, increasing service level will trigger two phenomena: (a) a group of customers will decide to switch from the other channel to purchase product; (b) a group of customers who otherwise would not purchase product in any channel, then, we assume \( \beta_1 > \beta_2 \) is reasonable.

The demand functions under inconsistent pricing decision:

\[
D^*_m = a_r(1 - \beta p_r) - a\lambda(p_r - p_m) + \beta_1 s_r - \beta_2 s_m
\]

\[
D^*_r = a_m(1 - \beta p_m) + a\lambda(p_r - p_m) + \beta_1 s_m - \beta_2 s_r
\]

where the parameter \( a, c, \delta, \beta, \beta_1, \beta_2 \) and \( \lambda \) are nonnegative.

Cost of providing service has decreasing-return property, and the diminishing return of service can be captured by quadratic form of service cost. Similar to Tsay and Agrawal [30] and Lu et al. [31], we assume that the cost of providing the retail service \( s_r \) and the manufacturer’s direct channel service \( s_m \) is \( \eta s_r^2 \) and \( \eta_m s_m^2 \), respectively, where \( \eta_r \) and \( \eta_m \) denote service cost coefficient.

Thus, the profit functions of the manufacturer and the retailer are expressed as follows.

The profit functions under consistent pricing decision:

\[
\Pi^*_m(p, s_r) = (p - w)[a_r(1 - \beta p) + \beta_1 s_r] - \beta_2 s_m - \frac{\eta s_r^2}{2}
\]

\[
\Pi^*_r(p, s_r) = (p - c)[a_r(1 - \beta p) + \beta_1 s_r] - \beta_2 s_m + (p - c)[a_m(1 - \beta p)]
\]

\[
+ \beta_1 s_m - \beta_2 s_r - \frac{\eta_m s_m^2}{2}
\]

The profit functions under inconsistent pricing decision:

\[
\Pi^*_m(p_r, s_r) = (p_r - w)[a_r(1 - \beta p_r) - a\lambda(p_r - p_m) + \beta_1 s_r] - \beta_2 s_m - \frac{\eta s_r^2}{2}
\]

\[
\Pi^*_r(p_m, w, s_m) = (w - c)[a_r(1 - \beta p)] - a\lambda(p_r - p_m) + \beta_1 s_r
\]
In what follows, we first consider the models in consistent pricing decision and the models in inconsistent. Then, we analyze the equilibrium prices under two types of competition models.

3 Main results

In this section, the manufacturer holds more bargaining power than the retailer and thus as the Stackelberg leader. The manufacturer first announces his wholesale price and the direct channel service level, the retailer determines the retail price and his service level. The manufacturer and the retailer make their decisions in order to achieve their own maximal profits, respectively. We follow a game-theoretical approach to analyze our model, and solve the problem backwards. Then, we can obtain the corresponding analytical equilibrium solutions.

3.1 The models in consistent pricing decision

Firstly, we consider consistent pricing decision scenario, where $p = p_r = p_m$. We only focus on the case, where the retailer determines the channel price $p$. We can obtain the proposition as follows.

**Proposition 1** Under the consistent pricing decision, given the wholesale price $w$ and the direct channel service level $s_m$ made earlier by the manufacturer, the retailer’s optimal response function is

$$
\begin{align*}
p^*(w, s_m) &= A_{11}w + A_{12}s_m + A_{13} \\
s^*(w, s_m) &= B_{11}w + B_{12}s_m + B_{13}
\end{align*}
$$

where $A_{11}$, $A_{12}$, $A_{13}$, $B_{21}$, $B_{22}$ and $B_{23}$ are defined in Appendix A.

After knowing the retailer’s reaction functions, the manufacturer would use them to maximize his own profit by choosing the wholesale price and the direct channel service level. The following proposition gives the closed form solution of manufacturer’s optimal wholesale price and the optimal direct channel service level.

**Proposition 2** Under the consistent pricing decision, the manufacturer’s optimal wholesale price $w_c^*$, optimal direct channel service level $s_{cm}^*$ are given as follows

$$
\begin{align*}
w_c^* &= \frac{G_{12}G_{23} - G_{22}G_{13}}{G_{11}G_{22} - G_{12}G_{21}} \\
s_{cm}^* &= \frac{G_{13}G_{21} - G_{11}G_{23}}{G_{11}G_{22} - G_{12}G_{21}}
\end{align*}
$$

Where $G_{ij}$, $G_{1i}$, $G_{12}$, $G_{21}$, $G_{22}$ and $G_{23}$ are defined in Appendix A.

Together by using Eqs. (9), (10), (11) and (12), we can easily obtain the optimal retailer price $p_c^*$ and the optimal channel service level $s_{cr}^*$.

$$
\begin{align*}
p_c^{**} &= A_{11}w_c^{**} + A_{12}s_{cm}^{**} + A_{13} \\
s_{cr}^{**} &= B_{11}w_c^{**} + B_{12}s_{cm}^{**} + B_{13}
\end{align*}
$$

We can also obtain their maximal profits $\Pi_c^*$ and $\Pi_{cm}^*$, and maximal demands $D_c^{**}$ and $D_{cm}^{**}$, by using Eqs. (1), (2), (5), (6), (11), (12), (13) and (14).

Moreover, if we let the parameters $\beta_1$, $\beta_2$, $\eta_r$ and $\eta_m$ be zero, which means both the manufacturer and retailer don’t provide channel service to customers, we can easily obtain their optimal decisions (denote $w_c^*$, $p_c^*$) and maximal profits (denote $D_c^*$, $D_{cm}^*$). The specific analysis would be shown in numerical examples in section 4.

3.2 The models in inconsistent pricing decision

Secondly, we consider inconsistent pricing decision scenario, where $p_r \neq p_m$ (the direct channel price is different from the retail channel price). We can obtain the proposition as follows.

**Proposition 3** Under the inconsistent pricing decision, given the wholesale price $w$ and the direct channel service level $s_m$ made earlier by the manufacturer, the retailer’s optimal response function is

$$
\begin{align*}
p_r^*(w, p_m, s_m) &= A_{31}w + A_{32}p_m + A_{33}s_m + A_{34} \\
s_r^*(w, p_m, s_m) &= B_{31}w + B_{32}p_m + B_{33}s_m + B_{34}
\end{align*}
$$

where $A_{31}$, $A_{32}$, $A_{33}$, $A_{34}$, $B_{31}$, $B_{32}$, $B_{33}$ and $B_{34}$ are defined in Appendix A.

We can see that the equilibrium quantities for the product in two channels are linear functions of the wholesale price, the direct channel price, the direct channel service level and the market base. Then, knowing the retailer’s reaction functions, the manufacturer sets the optimal wholesale price and the optimal direct channel service level to maximize his expected profit. We can obtain the proposition as follows.

**Proposition 4** Under the inconsistent pricing decision, the manufacturer’s optimal wholesale price $w_c^{**}$, the direct channel price $p_c^{**}$, and the optimal direct
channel service level \( s_{ir}^* \) are given as follows

\[
\begin{pmatrix}
w_{ir}^* \\
p_{ir}^* \\
s_{ir}^*
\end{pmatrix} = J^{-1}
\begin{pmatrix}
-J_{14} \\
-J_{24} \\
-J_{34}
\end{pmatrix}
\]  

(17)

Where \( J_{14}, J_{24}, J_{34} \) and \( J \) are defined in Appendix A.

Together by using Eqs. (15), (16) and (17), we can easily obtain the optimal retailer price \( p_{ir}^* \) and the optimal channel service level \( s_{ir}^* \).

\[
p_{ir}^* = A_{31}w_{ir}^* + A_{32}s_{im}^* + A_{33}s_{im}^* + A_{34}(18)
\]

\[
s_{ir}^* = B_{31}w_{ir}^* + B_{32}s_{im}^* + B_{33}s_{im}^* + B_{34}(19)
\]

We can also obtain their maximal profits \( \Pi_{ir}^* \) and \( \Pi_{im}^* \), and maximal demands \( D_{ir}^* \) and \( D_{im}^* \), by using Eqs. (3), (4), (7), (8), (17), (18) and (19).

Moreover, if we let the parameters \( \beta_1, \beta_2, \eta_r \) and \( \eta_m \) be zero, which means both the manufacturer and retailer don’t provide channel service to customers, we can easily obtain their optimal decisions (denote \( w_{ir}^*, p_{ir}^*, s_{ir}^* \) and maximal profits (denote \( \Pi_{ir}, \Pi_{im} \)). The specific analysis would be shown in numerical examples in section 4.

4 numerical examples

In this section, to better understand the equilibrium solutions, numerical examples are provided to analyze the effects of the degree of customer loyalty to the retail channel and the price-sensitive parameter on the optimal prices, channel service levels, maximal demands and maximal profits. Some managerial insights are also derived through the numerical analysis.

4.1 The discussion of parameter \( \delta \)

In this section, we explore the effects of the degree of customer loyalty to the retail channel on the optimal prices, channel service levels, maximal demands and maximal profits. The following values of parameters are assumed: \( a = 100, c = 10, \beta = 0.02, \beta_1 = 0.05, \beta_2 = 0.03, \lambda = 0.04, \eta_r = 0.03, \eta_m = 0.04. \)

The results can be illustrated in Figs. 2-8. And regardless the service levels of the manufacturer’s direct channel and the traditional retailer channel (that’s to say \( \beta_1 = \beta_2 = \eta_r = \eta_m = 0 \)), the corresponding results are shown in Figs. 2-8.

From Figs. 2 and 3, the following results are obtained:

(1-1) The manufacturer’s maximal demand will decrease with increasing \( \delta \), while the retailer’s maximal demand will increase with increasing \( \delta \), no matter in consistent or inconsistent pricing decision scenarios. It’s because that the larger the degree of customer loyalty to the retail channel, the more people choose the traditional retail channel to purchase the product, and then, the manufacturer’s maximal demand should be smaller and the retailer’s maximal demand should be larger.

(1-2) When the manufacturer and the retailer provide services to consumers, the maximal demand of manufacturer can increase, but the maximal demand of the retailer will decrease under consistent pricing decision scenario. But under inconsistent pricing decision scenario, there is little influence to the demand with/without the channel service.
optimal channel price and the optimal retail channel price increase as parameter \( \delta \) increases, but the optimal manufacturer channel price has little influence to the parameter \( \delta \).

From Fig. 6, the following results are obtained:

(3-1) Fig. 6 shows that the manufacturer’s optimal direct channel service under consistent pricing decision scenario is larger than that under inconsistent pricing decision scenario. When the parameter \( \delta \) satisfies the constraint condition \( \delta < 0.36 \), the retailer’s optimal channel service under consistent pricing decision scenario is larger than that under inconsistent pricing decision scenario. Otherwise, when \( \delta > 0.36 \), the retailer’s optimal channel service under consistent pricing decision scenario is smaller than that under inconsistent pricing decision scenario.

(3-2) Under consistent pricing decision scenario, the optimal service level will decrease with increasing \( \delta \). Under inconsistent pricing decision scenario, the optimal service level will increase with increasing \( \delta \).

Fig. 7: The change of \( \Pi_{cr}, \Pi_{tr}, \Pi_{crm} \), and \( \Pi_{crm}^* \) with \( \delta \).
From Figs. 7 and 8, the following results are obtained:

(4-1) In the manufacturer Stackelberg game model, the manufacturer’s maximal profit is always larger than the retailer’s maximal profit. That’s because the manufacturer is leader of the market. The manufacturer’s maximal profit decrease while the retailer’s maximal profit increase as parameter $\delta$ increases.

(4-2) With the dual-channel service, the manufacturer’s maximal profit will increase, while the retailer’s maximal profit will decrease. So, the manufacturer can enhance his competitiveness with retailer by offering costly direct channel service through direct channel. That’s to say, the retailer has advantage to get more profits with the dual-channel service.

(4-3) From Figs 7 and 8, we can find that the manufacturer’s maximal profit under inconsistent pricing decision scenario is higher than that under consistent pricing decision scenario, while the retailer’s maximal profit under inconsistent pricing decision scenario is lower than that under consistent pricing decision scenario.

4.2 The discussion of parameter $\lambda$

In this section, we explore the effects of the price-sensitive parameter $\lambda$ on the optimal wholesale price and the optimal retailer price, and maximal profits. The following values of parameters are assumed: $a = 100$, $c = 10$, $\beta = 0.02$, $\beta_1 = 0.05$, $\beta_2 = 0.03$, $\delta = 0.07$, $\eta_r = 0.03$, $\eta_m = 0.04$.

The results can be illustrated in Figs. 9-11. And regardless the service levels of the manufacturer’s direct channel and the traditional retailer channel (that’s to say $\beta_1 = \beta_2 = \eta_r = \eta_m = 0$), the corresponding results are shown in Figs. 9-11.

From Figs. 9-11, the following results are obtained:

(5-1) From Fig. 9, we can find that under consistent pricing decision scenario, the optimal retail channel price is larger than the optimal direct channel price and the optimal wholesale price. It is interesting to see that the optimal wholesale price is equal to the optimal direct channel price, and both of them have little influence to the parameter $\lambda$. In addition, the optimal retail...
channel price will increase with increasing $\lambda$.

(5.2) From Figs. 10 and 11, we can find that under inconsistent pricing decision scenario, the optimal retail channel price, the optimal direct channel price and the optimal wholesale price have little influence to the parameter $\lambda$. The manufacturer’s maximal profit and the retailer’s maximal profit have little influence to the parameter $\lambda$, too.

5 Conclusions

In this paper, we consider the pricing and channel service decisions of one product for one manufacturer and one retailer in a dual-channel supply chain. Unlike others studies, our work makes contributions to two points. Firstly, we consider both the manufacturer’s direct channel service and the traditional retail channel service. Secondly, we analyze two types of channel pricing decisions: the consistent pricing decision and inconsistent pricing decision.

Then, through numerical analysis, we compare analytical equilibrium solutions for the optimal pricing and the channel service level decisions, the maximal demands and the maximal profits under consistent pricing decision scenario and inconsistent pricing decision scenario. We also discuss the effect of the degree of customer loyalty to the retail channel and the price-sensitive parameter on optimal prices, optimal channel service levels and optimal expected profits.

Otherwise, there are possible extensions to improve our models. Such as, we suppose that the demand information is linear, and we just consider the supply chain under the manufacturer-leader Stackelberg scenario. So these problems may be pursued and studied in the future.

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Appendix A. Summary of Notations

$p^*_c(w, p_m, s_m) = A_{31}w + A_{32}p_m$

$A_{11} = \frac{2\beta_a + \beta_r}{2\beta_a - \beta_m^2}, A_{12} = \frac{-2}{2\beta_a - \beta_m^2}$,

$A_{13} = \frac{-2a_r}{2\beta_a - \beta_m^2}$,

$B_{11} = \frac{2\beta_m A_{11} - \beta_m}{\eta}, B_{12} = A_{12}, B_{13} = A_{13}$,

$F_1 = -2\beta_a + A_{11} - \beta_m B_{11} + \beta_r, F_{12} = \beta_1 A_{11} - \beta_2, F_{13} = \beta_1 A_{11} - \beta_2 A_{11}, F_{14} = \beta_1 A_{11} - \beta_2,$

$F_2 = \beta_1 - \beta_2 A_{11} - \beta_1 A_{12}, F_{22} = -\beta_2 A_{12}, F_{23} = \beta_1 A_{12} - \beta_3 A_{12}, F_{24} = \beta_1 A_{12} - \eta_m$,

$F_{25} = a_m A_{12} + (\beta a_m A_{12} + \beta a_r A_{12} + \beta_2 B_{12} - \beta_1 A_{11} - \beta_2 A_{11})c$.

$F_{21} = \beta_1 - \beta_2 B_{12} - \beta_3 a_m A_{12}, F_{22} = -\beta_2 A_{12}$,

$F_{23} = \beta_1 A_{12} - \beta_3 A_{12}, F_{24} = \beta_1 A_{12} - \eta_m$,

$F_{25} = a_m A_{12} + (\beta a_m A_{12} + \beta a_r A_{12} + \beta_2 B_{12} - \beta_1 A_{11} - \beta_2 A_{11})c$.

$G_{11} = F_{11} A_{11} + F_{12} B_{12} + F_{13}$,

$G_{12} = F_{11} A_{12} + F_{12} B_{12} + F_{14}$,

$G_{13} = F_{11} A_{13} + F_{12} B_{13} + F_{15}$,

$G_{21} = F_{21} A_{11} + F_{22} B_{11} + F_{23}$,

$G_{22} = F_{21} A_{12} + F_{22} B_{12} + F_{24}$,

$G_{23} = F_{21} A_{13} + F_{22} B_{13} + F_{25}$,

$A_{31} = \frac{(\beta a_r + \alpha \lambda - \beta_2)}{2\beta a_r + 2\alpha \lambda - \frac{{\beta_r^2}}{\eta}}$, $A_{32} = \frac{\alpha \lambda}{2\beta a_r + 2\alpha \lambda - \frac{{\beta_r^2}}{\eta}}$,

$A_{33} = \frac{-\beta_r}{2\beta a_r + 2\alpha \lambda - \frac{{\beta_r^2}}{\eta}}$, $A_{34} = \frac{\beta_r}{2\beta a_r + 2\alpha \lambda - \frac{{\beta_r^2}}{\eta}}$,

$B_{31} = \frac{\beta_2}{\eta} A_{31} - \frac{\beta_1}{\eta}, B_{32} = \frac{\beta_2}{\eta} A_{32}, B_{33} = \frac{\beta_2}{\eta} A_{33}$,

$B_{34} = \frac{\beta_2}{\eta} A_{34}$,

$B_{35} = \beta_1 B_{31} - \beta_2 A_{31} - \alpha A_{31}, B_{35} = -\beta_2$, $B_{36} = \alpha + (\beta_2 B_{31} - \beta_1 B_{31} + \beta a_r A_{31})c$,

$F_{11} = F_{31} A_{31} + F_{32} B_{31} + F_{33}$,

$F_{12} = F_{31} A_{32} + F_{32} B_{32} + F_{33}$,

$F_{13} = F_{31} A_{33} + F_{32} B_{33} + F_{33}$,

$F_{14} = F_{31} A_{34} + F_{32} B_{34} + F_{33}$,

$F_{15} = F_{31} A_{35} + F_{32} B_{35} + F_{33}$,

$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$

References:


