Mixed $\mathrm{H}_2/\mathrm{H}_\infty$ Control Synthesis for Discrete-time Linear Positive Systems using Enhanced Set of Linear Matrix Inequalities

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Abstract: - This paper provides an idea, resulting from linear matrix inequality representation of parameter constraints of discrete-time linear positive systems, to formulate state-feedback synthesis for this class of systems. The design conditions are imposed to obtain control respecting existing strictly positivity or non-negativity in the system matrix description. Formulated as a linear matrix inequality feasibility problem it is reiterated that approach leads iteratively to estimation of norms of closed-loop system.

Key-Words: - Convex optimization, diagonal stabilisation, discrete-time linear positive systems, linear matrix inequalities,positivityconstraints.

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1 Introduction

Many structures are alternating in design, to reflect the specific structure of Metzlerian continuous-time systems [6] or positive discrete-time linear systems [3], [20], or in solving problem of diagonal stabilisation of the closed-loop system [13], [14]. Because of the positivity constraints, the synthesis of positive systems can be limited by using linear programming [1] for some problems implying from parametric constrain effects. Defining parametric constraints by means of linear matrix inequalities [9], [10] greatly facilitates state control synthesis for positive systems. Formulating for partly unconstrained cases to derive the stability condition, H_2 state-feedback synthesis problem under structural positivity constraints may stay non-convex [5]. To regularize it, a set of additive linear matrix inequalities (LMI) [4], [19] is combined to the basic set of LMIs guaranteing the system asymptotic stability. On the other hand, the advantage of procedures based on H_2 norm is the possibility to combine into the synthesis conditions the boundaries defined also by the system environment [7], [16]. The common idea is to calculate the matrix gains of the control law by solving the optimization on extended set of LMIs.

The paper presents an approach in state control synthesis for discrete-time positive systems within mixed $\mathrm{H}_2/\mathrm{H}_\infty$ norm formulation. Motivated by the ideas presented in [10]-[12] design is covered via extended set of LMIs with fixing of H_2 norm of closed-loop system transform matrix combined with bounded real lemma LMI structure. LMI-based approach, characterizing synthesis of controllers is computationally simple, efficient, applicable to square multiple input and multiple output (MIMO) systems for strictly positive discrete-time linear systems as well as for non-negative class of these systems. Authors try to apply enhanced LMI based ideas and to show that the above conjecture is true for standard classes of positive discrete-time linear systems.

The outline of the paper is as follows. Section 2 characterises a way of accounting system parametric constraints into control design for the considered system class and, subsequently, Sections 3 - 5 present adaptations of H_2 and H_∞ norm principles to formulate associate LMI-based design conditions for discrete-time linear strictly positive systems. An enhanced approach, reflecting norm-based principles to formulate control design conditions, is given in Section 6, gradually focusing on differences in design tasks formulation for strictly and non-strictly positive systems. Section 7 gives numerical examples, illustrating obtained results and Sec. 8 presents some conclusions.

Throughout the paper, the notations are narrowly standard in such way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, for a square matrix $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is a symmetric negative definite matrix, diag$[\,\cdot\,]$ enters up a diagonal matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $\mathbb{R}_n^n$ ($\mathbb{R}_+^n$) points to the set of all n-dimensional real (non-negative as well as positive) vectors, $\mathbb{R}_n^{n \times n}$ ($\mathbb{R}_+^{n \times r}$) refers to the set of all $n \times r$ real (non-negative as well as positive) matrices and $\mathbb{R}_+$ denotes the set of positive real numbers.

2 Control of Strictly Positive Systems

Considered systems class admits the description

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{2}$$

where $\boldsymbol{q}(i) \in \mathbb{R}^n_+$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, $\boldsymbol{y}(t) \in \mathbb{R}^m_+$ are state, input and output vectors, $\boldsymbol{G} \in \mathbb{R}^{n\times r}_+$, $\boldsymbol{C} \in \mathbb{R}^n_+$ are nonnegative matrices and $\boldsymbol{F} \in \mathbb{M}^{n\times n}_{-+}$ is strictly positive natrix (all its elements are greater then zero). Such systems are noted as the strictly positive discrete-time systems [2]. A strictly positive matrix structure of $\boldsymbol{F}$ implies n^2 structural constraints

$$f_{lh} > 0 \,\forall\, l,h = 1,\dots n \tag{3}$$

which, in consequence mean that strictly positive systems (1), (2) are diagonally stabilizable [9]. To do with diagonal constraints, the operations based on permutation matrices are exploited.

Definition 1. *[8] A square matrix $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is a permutation matrix if exactly one entry in each row and column is equal to 1 and all other entries are 0.*

Remark 1. *Defining $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ in the circulant form*

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix} \tag{4}$$

and a square diagonal matrix $\boldsymbol{Y}$ of dimension $n\times n$

$$\boldsymbol{Y} = diag\,[\, y_1 \;\; y_2 \;\; \cdots \;\; y_n \,] \tag{5}$$

then

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{L} = diag\,[\, y_2 \;\; \cdots \;\; y_n \;\; y_1 \,] \tag{6}$$

To substitute (3) as a set of n linear matrix inequalities, the following lemma determines solutions.

Lemma 1. *(adapted from [9]) Let system (1), (2) is strictly positive then it is asymptotically stable if and only if there exists a positive definite diagonal matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}_+$ such that for $j = 1,\dots,n$, $h = 0,1,\dots n-1$, $\boldsymbol{L} \in \mathbb{R}^{n\times n}_+$, $(\Delta) = (1 \leftrightarrow n)/n$, the following set of LMIs is feasible for*

$$\boldsymbol{F}(j,j+h)_{(\Delta)} = diag\,[a_{1,1+h}\cdots a_{n-j,n}\cdots a_{nh}] \tag{7}$$

$$\boldsymbol{P} \succ 0 \tag{8}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} \succ 0 \tag{9}$$

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F} - \boldsymbol{P} \prec 0 \tag{10}$$

Note, the set of LMIs (9) reflects the structural constraints (3) and the Lyapunov matrix inequality (10) guaranties that $\boldsymbol{F}$ is Schur.

Considering state feedback with a strictly positive matrix $\boldsymbol{K} \in \mathbb{R}^{r\times n}_+$ and strictly positive $\boldsymbol{F} \in \mathbb{R}^{n\times n}_+$

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) \tag{11}$$

$$\mathrm{K} = \begin{bmatrix} \mathrm{k}_1^{\mathrm{T}} \\ \vdots \\ \mathrm{k}_r^{\mathrm{T}} \end{bmatrix}, \quad \mathrm{G} = [\, \mathrm{g}_1 \;\; \cdots \;\; \mathrm{g}_r \,] \tag{12}$$

then the closed-loop system description is

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) = \boldsymbol{F}_c\boldsymbol{q}(i) \tag{13}$$

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} = \boldsymbol{F} - \sum_{k=1}^{r} \boldsymbol{g}_k\boldsymbol{k}_k^{\mathrm{T}} \tag{14}$$

has to be also a strictly positive matrix. Then $\boldsymbol{G} \in \mathbb{R}^{n\times r}_+$, $\boldsymbol{F}_c \in \mathbb{R}^{n\times n}_+$ prescribe algebraic constraints

$$f_{clj} = f_{lj} - \sum_{k=1}^{r} g_{lk}k_{kl} > 0 \;\;\forall\;\; l,j = 1,\dots,n \tag{15}$$

while, in details,

$$\boldsymbol{F} = \begin{bmatrix} f_{11}\cdots f_{1n} \\ \vdots \\ f_{n1}\cdots f_{nn} \end{bmatrix}, \boldsymbol{g}_k = \begin{bmatrix} g_{1k} \\ \vdots \\ g_{nk} \end{bmatrix}, \boldsymbol{k}_k = \begin{bmatrix} k_{k1} \\ \vdots \\ g_{kn} \end{bmatrix} \tag{16}$$

that is, the positiveness constraints for the solvability of a strictly positive gain matrix $\boldsymbol{K}$ are given by set of n^2 scalar inequalities (15).

To explain the diagonal principle of positiveness constraints definition by linear matrix inequalities, the full structure of $\boldsymbol{F}_c$ is considered as follows

$$\boldsymbol{F}_c = \begin{bmatrix} f_{11}\cdots f_{1n} \\ \vdots \\ f_{n1}\cdots f_{nn} \end{bmatrix} - \sum_{k=1}^{r} \begin{bmatrix} g_{1k} \\ \vdots \\ g_{nk} \end{bmatrix} [k_{k1}\cdots k_{kn}] \tag{17}$$

and the positiveness constraints on the diagonal elements of (17), if $\mathrm{K} \in \mathbb{R}^{r\times n}_+$ is strictly positive, can be

$$\boldsymbol{F}(j,j)_{(\Delta)} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dk} \succ 0 \tag{18}$$

$$\mathrm{F}(j,j)_{(\Delta)} = \mathrm{diag}\,[f_{11}\cdots f_{nn}] \succ 0 \tag{19}$$

$$\boldsymbol{K}_{dk} = \mathrm{diag}\,[k_{k1}\cdots k_{kn}] \succ 0 \tag{20}$$

$$\boldsymbol{G}_{dk} = \mathrm{diag}\,[g_{1k}\cdots g_{nk}] \tag{21}$$

Multiplying the right side of (14) by $\boldsymbol{L}$ results in

$$\begin{aligned} &\boldsymbol{F}_c\boldsymbol{L} \\ &= \begin{bmatrix} f_{12}\cdots f_{1n}\, f_{11} \\ \ddots \\ f_{n2}\cdots f_{nn}\, f_{n1} \end{bmatrix} - \sum_{k=1}^{r} \begin{bmatrix} g_{1k} \\ \vdots \\ g_{nk} \end{bmatrix} [k_{k2}\cdots k_{kn}\, k_{k1}] \end{aligned} \tag{22}$$

and it has to yield for diagonal elements of (22)

$$\boldsymbol{F}(j,j+1)_{(\Delta)} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dkc1} \succ 0 \tag{23}$$

$$\mathrm{F}(j,j+1)_{(\Delta)} = \mathrm{diag}\,[f_{12}\, f_{23} \cdots f_{n1}] \succ 0 \tag{24}$$

$$\boldsymbol{K}_{dkc1} = \mathrm{diag}\,[k_{k2} \cdots k_{kn}\, k_{k1}] \succ 0 \tag{25}$$

and $\boldsymbol{K}_{dkc1}$ denotes the diagonal matrix $\boldsymbol{K}_{dk}$ with one circular shift of its diagonal elements.

Applying this procedure $(n-1)$-times with circular shift of diagonal elements of $\boldsymbol{K}_{dk}$ results in

$$\begin{aligned} \mathrm{F}_c = \boldsymbol{F}(j,j)_{(\Delta)} &- \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dk} + \\ + \mathrm{F}(j,j+1)_{(\Delta)}\boldsymbol{L}^{T} &- \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dkc1}\boldsymbol{L}^{\mathrm{T}} + \\ &\vdots \\ + \mathrm{F}(j,j+n-1)_{(\Delta)}&\boldsymbol{L}^{(n-1)\mathrm{T}} - \\ - \sum_{k=1}^{r} \boldsymbol{G}_{dk}&\boldsymbol{K}_{dkc(n-1)}\boldsymbol{L}^{(n-1)\mathrm{T}} \end{aligned} \tag{26}$$

for all

$$\boldsymbol{F}(j,j+h)_{(\Delta)} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dkch} \succ 0 \tag{27}$$

$$\mathrm{F}(j,j+h)_{(\Delta)} = \mathrm{diag}\,[f_{1,h+1}\, f_{2,h+2} \cdots f_{nh}] \tag{28}$$

$$\mathrm{K}_{dkch} = \mathrm{diag}\,[k_{k,1+h} \quad k_{k,2+h} \cdots \quad k_{kh}] \succ 0 \tag{29}$$

while $\boldsymbol{K}_{dkch}$ represents a diagonal matrix structure of $\boldsymbol{K}_{dk}$ with h circular shifts of its diagonal elements and used index summation $(\Delta) = (1 \leftrightarrow n)/n$ denotes generalized sum $modulo\ n$ [9].

Set of LMIs (26) for $h = 0, 1, \dots n-1$ defines conditions to design strictly positive $\boldsymbol{K}$ guaranteing, in general not Shur, strictly positive matrix $\boldsymbol{F}_c$. The following lemma formulates this result.

Lemma 2. *Using state feedback control law (11) in control of (1), (2), then matrix $\boldsymbol{F}_c$ is strictly positive if for given strictly positive matrix $\boldsymbol{F} \in \mathbb{R}_{+}^{n\times n}$ and non-negative $\boldsymbol{G} \in \mathbb{R}_{+}^{n\times r}$ there exist strictly positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}^{n\times n}$ such that for $j = 1, \dots, n$, $h = 0, 1, \dots, n-1$, $k = 1, \dots r$*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0 \tag{30}$$

$$\boldsymbol{L}^{h}\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{L}^{h}\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{31}$$

where design parameters are (19), (21) and (4).

If the above conditions are feasible

$$\boldsymbol{K}_{dk} = \boldsymbol{R}_k\boldsymbol{P}^{-1}, \boldsymbol{k}_k^{\mathrm{T}} = \boldsymbol{l}^{\mathrm{T}}\boldsymbol{K}_{dk},\ \boldsymbol{l}^{\mathrm{T}} = [1 \cdots 1] \tag{32}$$

Proof. Using the permutation matrix (4) it yields for $h = 0, 1, 2, \dots n-1$ that

$$\boldsymbol{K}_{dk} = \boldsymbol{L}^{h}\boldsymbol{K}_{dkch}\boldsymbol{L}^{h\mathrm{T}} \tag{33}$$

Thus, multiplying the right side by $\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P}$, where $\boldsymbol{P}$ is a positive definite diagonal matrix, then (33) implies

$$\begin{aligned} &\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{K}_{dkch}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} \\ &= \boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{K}_{dk}\boldsymbol{P} \\ &\succ 0 \end{aligned} \tag{34}$$

and multiplying the left side of (34) by $\boldsymbol{L}^{h}$ to preserve diagonal structure, then with the notation

$$\boldsymbol{R}_k = \boldsymbol{K}_{dk}\boldsymbol{P} \tag{35}$$

(34) implies (31). This concludes the proof. □

Remark 2. *Formulation of the diagonal stabilization principle can be simple illustrated by replacing square form od (17) by the following rhombic notation*

$$\boldsymbol{F}_{c(\Delta)} = \begin{bmatrix} f_{c11}\, f_{c12}\, f_{c13} \cdots & f_{c1n} & & & \\ f_{c22}\, f_{c23} \cdots & f_{c2n} & f_{p21} & & \\ f_{c33} \cdots & f_{c3n} & f_{c31} & f_{c32} & \\ \ddots & \vdots & \vdots & \vdots & \ddots \\ & f_{cnn} & f_{cn1} & f_{cn2} \cdots & f_{cn,n-1} \end{bmatrix} \tag{36}$$

where $f_{clj}\ \forall\, l, j = 1, \dots, n$ are given in (15).

It can see that positive elements of diagonal matrix $\boldsymbol{F}_c(j,j)_{(\Delta)}$ are on the first diagonal of $\boldsymbol{F}_{c(\Delta)}$, positive elements of diagonal matrix $\boldsymbol{F}_c(j,j+1)_{(\Delta)}$ are on the second diagonal of $\boldsymbol{F}_{c(\Delta)}$, etc. Thus, since elements of $\boldsymbol{F}_c(j,j+h)_{(\Delta)}$ are on the $(h+1)$ diagonal of $\boldsymbol{F}_{c(\Delta)}$, their positivity can be expressed by diagonal matrix inequality (31).

In addition, the fundamental ordering if elements of the associated rows in $\boldsymbol{F}_{c(\Delta)}$ agree with (multiple) circular shift of the reference rows of (17).

Note, the conditions given by Lemma 2 (if are feasible for given system matrix parameters) result in strictly positive $\boldsymbol{K} \in \mathbb{R}_{+}^{r\times n}$ and strictly positive $\boldsymbol{F}_c \in \mathbb{R}_{+}^{n\times n}$ but do not guarantee that $\boldsymbol{F}_c$ is Schur. Therefore, these inequalities has to be supplemented by another linear matrix inequality (eventually by a set of linear matrix inequalities) to ensure that the asymptotic stability of the closed-loop system is achieved (if it is possible) [10].

Such a solution is also used in the following, when the stabilizing condition takes into account the Lyapunov inequality.

Theorem 1. *Using state feedback control law (1) in control of (1), (2), then matrix $\boldsymbol{F}_c$ is strictly positive and Schur if for given strictly positive matrix $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}^{n\times n}$ such that for $j = 1,\dots,n$, $h = 0,1,\dots,n-1$ and $k = 1,\dots r$,*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0 \tag{37}$$

$$\begin{bmatrix} -\boldsymbol{P} & * \\ \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{m} \boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_{dk} & -\boldsymbol{P} \end{bmatrix} \prec 0 \tag{38}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r}\boldsymbol{L}^h\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{39}$$

where matrix design parameters are given in (4), (7) and (21).

If the above conditions are feasible it yields (32) and the strictly positive $\boldsymbol{K}$ is constructed using (12).

Hereafter, $*$ is the symmetric item in a symmetric matrix.

Proof. Considering a positive definite diagonal matrix $\boldsymbol{P} \in \mathbb{R}_+^{n\times n}$ and defining the Lyapunov function candidate as

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^{\mathrm{T}}(i)\boldsymbol{P}\boldsymbol{q}(i) > 0 \tag{40}$$

then, substituting (3), it yields

$$\Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^{\mathrm{T}}(i)(\boldsymbol{F}_c^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}_c - \boldsymbol{P})\boldsymbol{q}(i) < 0 \tag{41}$$

which implies

$$\boldsymbol{F}_c^{\mathrm{T}}\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{F}_c - \boldsymbol{P} \prec 0 \tag{42}$$

and it in what follows, using the Schur complement property,

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{F}_c^{\mathrm{T}}\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{F}_c & -\boldsymbol{P} \end{bmatrix} \prec 0 \tag{43}$$

Since

$$\begin{aligned} \boldsymbol{F}_c\boldsymbol{P} &= \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{m}\boldsymbol{g}_k\boldsymbol{k}_k^{\mathrm{T}}\boldsymbol{P} \\ &= \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{m}\boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{K}_{dk}\boldsymbol{P} \end{aligned} \tag{44}$$

where $\boldsymbol{K}_{dk}$, $\boldsymbol{G}_{dk}$ are defined in (20), (21) and $\boldsymbol{l}^{\mathrm{T}}$ in (32). Thus, with (35) then (43) implies (38). This concludes the proof. □

It can be in this way concluded, generalizing the idea of the design condition structure to state-feedback control design for strictly positive linear systems means that the LMIs guaranteing closed-loop system matrix parameter positivity stay unchanged and only set of LMIs, guaranteing stability and transfer function performances has to be redefined.

3 H_∞ Control Synthesis

To demonstrate the applicability of H_∞ norm principle in control design for strictly positive discrete-time linear systems, the following lemma is required in proofs of the proposed theorem.

Lemma 3. *[7] (Bounded real lemma (BRL)) If the closed-loop discrete-time linear system is given as*

$$\boldsymbol{q}(i+1) = \boldsymbol{F}_c\boldsymbol{q}(i) + \boldsymbol{D}\boldsymbol{d}(i) \tag{45}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{46}$$

where $\boldsymbol{D} \in \mathbb{R}^{n\times p}$, $\boldsymbol{d} \in \mathbb{R}^p$, $\boldsymbol{w} \in \mathbb{R}^r$, then $\boldsymbol{F}_c$ is Schur if there exist a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and a positive scalar $\gamma \in \mathbb{R}_+$ such that

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \succ 0, \quad \gamma > 0 \tag{47}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}_c\boldsymbol{P} & -\boldsymbol{P} & * & * \\ \boldsymbol{C}\boldsymbol{P} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0 \tag{48}$$

If a strictly positive discrete-time linear systems is characterised by (45), (46) but with nonnegative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ and strictly positive $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and (48) is taken as a basis to redefine LMI guaranteing stability and disturbance transfer function matrix H_∞ performances, the following theorem results.

Theorem 2. *Using state feedback control law (1) in control of (1), (2) with unknown disturbance, then matrix $\boldsymbol{F}_c$ is strictly positive and Schur if for given strictly positive matrix $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative matrices $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}^{n\times n}$ such that for $j = 1,\dots,n$, $h = 0,1,\dots,n-1$ and $k = 1,\dots r$,*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0, \quad \gamma > 0 \tag{49}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{m}\boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_{dk} & -\boldsymbol{P} & * & * \\ \boldsymbol{C}\boldsymbol{P} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0 \tag{50}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r}\boldsymbol{L}^h\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{51}$$

where matrix design parameters are given in (4), (7) and (21).

If the above conditions are feasible it yields (32) and the strictly positive $\boldsymbol{K}$ is constructed using (12).

Proof. Inserting (44) into (48) then (48) implies (50). This concludes the proof. □

4 H_2 Control Synthesis

Before giving the results for H_2 control synthesis the following preliminaries are presented.

Lemma 4. *[17] Given Schur $\boldsymbol{F}_c$ and $\boldsymbol{G}$, $\boldsymbol{C}$ from (45), (46), then*

$$\delta^2 = tr\,(\boldsymbol{CSC}^{\mathrm{T}}) \tag{52}$$

$$\boldsymbol{F}_c\boldsymbol{S}\boldsymbol{F}_c^{\mathrm{T}} - \boldsymbol{S} + \boldsymbol{G}\boldsymbol{G}^{\mathrm{T}} = \boldsymbol{0} \tag{53}$$

where $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ is symmetric positive definite controllability Gramian of $(\boldsymbol{F}_c, \boldsymbol{G})$ and $\delta \in \mathbb{R}_+$ is H_2 norm of the reference input system transfer function.

Corollary 1. *As a consequence, there exists symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that*

$$\boldsymbol{F}_c\boldsymbol{P}\boldsymbol{F}_c^{\mathrm{T}} - \boldsymbol{P} + \boldsymbol{G}\boldsymbol{G}^{\mathrm{T}} \prec 0 \tag{54}$$

while the above Lyapunov inequality implies that (54) is negative definite if and only if $\boldsymbol{F}_c$ is Schur.

The above gives the base to formulate the design condition for discrete-time strictly positive systems.

Theorem 3. *Using state feedback control law (1) in control of (1), (2), then matrix $\boldsymbol{F}_c$ is strictly positive and Schur if for given strictly positive $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ there exist strictly positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}_+^{n\times n}$, $\boldsymbol{V} \in \mathbb{R}_+^{m\times m}$ and a positive scalar $\eta \in \mathbb{R}_+$ such that for $j = 1, \ldots, n$, $h = 0, 1, \ldots, n-1$ and $k = 1, \ldots r$*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{V} \succ 0, \quad \boldsymbol{R}_k \succ 0, \quad \eta > 0 \tag{55}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j, j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r}\boldsymbol{L}^h\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{56}$$

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{FP} - \sum\limits_{k=1}^{r}\boldsymbol{G}_{dk}\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k & \boldsymbol{G} \\ * & -\boldsymbol{P} & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{57}$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{PC}^{\mathrm{T}} \\ * & \boldsymbol{V} \end{bmatrix} \succ 0 \tag{58}$$

$$\min_{\eta} \ \textit{subject to}\ (\boldsymbol{V} - \eta\boldsymbol{I}_m) \prec 0 \tag{59}$$

When the above conditions are satisfied then, using in (2), the strictly positive $\mathrm{k}_k^{\mathrm{T}}$ are given by (32) and guaranties H_2 norm performances such that $\eta \geq \delta$.

Proof. Using (54) the equivalent matrix inequality is

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{F}_c\boldsymbol{P} & \boldsymbol{G} \\ \boldsymbol{P}\boldsymbol{F}_c^{\mathrm{T}} & -\boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{G}^{\mathrm{T}} & \boldsymbol{0} & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{60}$$

and inserting (44) then (60) results in (57).

Moreover, $\boldsymbol{P} \succ \boldsymbol{S}$ leads to the fact relating (52)

$$\mathrm{tr}\,(\boldsymbol{CPC}^{\mathrm{T}}) > \mathrm{tr}\,(\boldsymbol{CSC}^{\mathrm{T}}) = \delta^2 \tag{61}$$

and adjusting that

$$\mathrm{V} \succ \mathrm{CPC}^{\mathrm{T}} = \mathrm{CPP}^{-1}\mathrm{PC}^{\mathrm{T}} \tag{62}$$

with $\boldsymbol{V} \in \mathbb{R}^{m\times m}$ being diagonal positive definite matrix, then (62) implies (58) and it yields [15]

$$\|\boldsymbol{V}\|_2 = \sqrt{\zeta_{max}(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V})} = \lambda_{max}(\mathrm{V}) \tag{63}$$

where λ_{max} is maximal eigenvalue of $\boldsymbol{V}$. This gives

$$\boldsymbol{V} - \eta\boldsymbol{I}_m \prec 0, \quad \eta > \lambda_{max} \tag{64}$$

which leads to minimization in the sense of (59). This concludes the proof. □

5 Mixed $\mathrm{H}_2/\mathrm{H}_\infty$ Control Synthesis

The following theorem can be derived directly from Theorem 2 and Theorem 3 and its proof is omitted.

Theorem 4. *Using state feedback control law (1) in control of (1), (2), then matrix $\boldsymbol{F}_c$ is strictly positive and Schur if for given strictly positive $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times p}$ there exist strictly positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}_+^{n\times n}$, $\boldsymbol{V} \in \mathbb{R}_+^{m\times m}$ and positive scalars $\gamma, \eta \in \mathbb{R}_+$ such that for $j = 1, \ldots, n$, $h = 0, 1, \ldots, n-1$ and $k = 1, \ldots r$*

$$\boldsymbol{P} \succ 0,\ \boldsymbol{V} \succ 0,\ \boldsymbol{R}_k \succ 0,\ \gamma > 0,\ \eta > 0 \tag{65}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j, j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{r}\boldsymbol{L}^h\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{66}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{FP} - \sum\limits_{k=1}^{m}\boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_{dk} & -\boldsymbol{P} & * & * \\ \boldsymbol{CP} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0 \tag{67}$$

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{FP} - \sum\limits_{k=1}^{r}\boldsymbol{G}_{dk}\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k & \boldsymbol{G} \\ * & -\boldsymbol{P} & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{68}$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{PC}^{\mathrm{T}} \\ * & \boldsymbol{V} \end{bmatrix} \succ 0 \tag{69}$$

$$\min_{\eta} \ \textit{subject to}\ (\boldsymbol{V} - \eta\boldsymbol{I}_m) \prec 0 \tag{70}$$

When the above conditions are satisfied then, using in (2), the strictly positive $\mathrm{k}_k^{\mathrm{T}}$ are given by (32) and guaranties H_2 norm in such a way that $\eta > \delta$.

Remark 3. *In the case of more than one criterion it can deduce that by this way multiple closed-loop performances can be ensured concurrently, since in the constraints represented by given set of LMIs (the LMIs related to parametric constraint, an Lyapunov LMI guaranteing asymptotic stability, the set of LMI extensions representing H_2 and H_∞ norm constraints given on the associated transfer function matrices) the controller considered in each inequality is the same. Concretely, it can be summarized that such kind of the multi-objective synthesis is definable for strictly positive discrete-time linear systems as an LMI optimization problem, which may be efficiently solved.*

Moreover, because the H_∞ synthesis is essentially based on the worst-case performance analysis and the H_2 norm reflects an average performance, the idea of combining these two types of closed-loop performances means that mixed H_2/H_∞ approach indirectly forces the formalism based on the H_2 norm. Such construction of this formulation for synthesis is crucial since if H_∞ norm would not be tied to the disturbance transfer function matrix but to the control input transfer function, the condition $\delta < \gamma$ will cause that the H_∞ constraint stays redundant.

6 Enhanced Mixed H$_2$/H$_\infty$ Control

Introducing a slack matrix $\boldsymbol{H}$, matrices $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{C}$, $\boldsymbol{D}$ can be decoupled from the Lyapunov matrix which admits more freedom and reduces the conservativeness.

Theorem 5. *Using state feedback control law (1) in control of (1), (2), then matrix $\boldsymbol{F}_c$ is strictly positive and Schur if for given strictly positive $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times p}$ there exist strictly positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{H}, \boldsymbol{R}_k \in \mathbb{R}_+^{n\times n}$, $\boldsymbol{V} \in \mathbb{R}_+^{m\times m}$ and positive scalars $\gamma, \eta \in \mathbb{R}_+$ such that for $j = 1, \ldots, n$, $h = 0, 1, \ldots, n-1$, $k = 1, \ldots r$*

$$\boldsymbol{P} \succ 0, \boldsymbol{H} \succ 0, \boldsymbol{V} \succ 0, \boldsymbol{R}_k \succ 0, \gamma > 0, \eta > 0 \quad (71)$$

$$\boldsymbol{L}^h \boldsymbol{F}(j, j+h)_{(\Delta)} \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{H} - \sum_{k=1}^{r} \boldsymbol{L}^h \boldsymbol{G}_{dk} \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{R}_k \succ 0 \quad (72)$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}\boldsymbol{H} - \sum_{k=1}^{m} \boldsymbol{G}_{dk} \boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} \boldsymbol{R}_{dk} & \boldsymbol{P} - 2\boldsymbol{H} & * & * \\ \boldsymbol{C}\boldsymbol{H} & \boldsymbol{0} & -\gamma \boldsymbol{I}_m & * \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma \boldsymbol{I}_p \end{bmatrix} \prec 0 \quad (73)$$

$$\begin{bmatrix} \boldsymbol{P} - 2\boldsymbol{H} & \boldsymbol{F}\boldsymbol{H} - \sum_{k=1}^{r} \boldsymbol{G}_{dk} \boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} \boldsymbol{R}_k & \boldsymbol{G} \\ * & -\boldsymbol{P} & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \quad (74)$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{H}\boldsymbol{C}^{\mathrm{T}} \\ * & \boldsymbol{V} \end{bmatrix} \succ 0 \quad (75)$$

$$\min_{\eta} \ \textit{subject to} \ (\boldsymbol{V} - \eta \boldsymbol{I}_m) \prec 0 \quad (76)$$

When the above conditions are satisfied then the strictly positive $\mathrm{k}_k^{\mathrm{T}}$ are given as

$$\boldsymbol{K}_{dk} = \boldsymbol{R}_k \boldsymbol{H}^{-1}, \boldsymbol{k}_k^{\mathrm{T}} = \boldsymbol{l}^{\mathrm{T}} \boldsymbol{K}_{dk}, \ \boldsymbol{l}^{\mathrm{T}} = [1 \cdots 1] \quad (77)$$

and guaranties H_2 norm system performances in such a way that $\eta > \delta$ while γ is minimized interactively.

Proof. Defining the diagonal positive definite matrices $\boldsymbol{Q}, \boldsymbol{H} \in \mathbb{R}_+^{n\times n}$ such that it holds

$$(\boldsymbol{Q}^{-1} - \boldsymbol{H}^{-1}) \boldsymbol{Q} (\boldsymbol{Q}^{-1} - \boldsymbol{H}^{-1}) \succ 0 \quad (78)$$

$$\boldsymbol{H}^{-1} \boldsymbol{Q} \boldsymbol{H}^{-1} \succ 2\boldsymbol{H}^{-1} - \boldsymbol{Q}^{-1} \quad (79)$$

$$\boldsymbol{Q} \succ 2\boldsymbol{H} - \boldsymbol{H}\boldsymbol{Q}^{-1}\boldsymbol{H} \quad (80)$$

Considering that (48) is satisfied for the above given diagonal positive definite matrix $\boldsymbol{Q}$ that is

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{Q}\boldsymbol{F}_c^{\mathrm{T}} & \boldsymbol{Q}\boldsymbol{C}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{F}_c \boldsymbol{Q} & -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{D} \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{0} & -\gamma \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma \boldsymbol{I}_p \end{bmatrix} \prec 0 \quad (81)$$

and introducing the block diagonal matrix

$$\boldsymbol{T}_\infty = \begin{bmatrix} \boldsymbol{H}\boldsymbol{Q}^{-1} & \boldsymbol{I}_n & \boldsymbol{I}_m & \boldsymbol{I}_p \end{bmatrix} \quad (82)$$

then pre-multiplying the left side by $\boldsymbol{T}_\infty$ and post-multiplying the right side by $\boldsymbol{T}_\infty^{\mathrm{T}}$ (81) implies

$$\begin{bmatrix} -\boldsymbol{H}\boldsymbol{Q}^{-1}\boldsymbol{H} & \boldsymbol{H}\boldsymbol{F}_c^{\mathrm{T}} & \boldsymbol{H}\boldsymbol{C}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{F}_c \boldsymbol{H} & -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{D} \\ \boldsymbol{C}\boldsymbol{H} & \boldsymbol{0} & -\gamma \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma \boldsymbol{I}_p \end{bmatrix} \prec 0 \quad (83)$$

Denoting

$$\boldsymbol{P} = \boldsymbol{H}\boldsymbol{Q}^{-1}\boldsymbol{H}, \quad \boldsymbol{Q} \succ 2\boldsymbol{H} - \boldsymbol{P} \quad (84)$$

then (83) can be approximated as

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{H}\boldsymbol{F}_c^{\mathrm{T}} & \boldsymbol{H}\boldsymbol{C}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{F}_c \boldsymbol{H} & -2\boldsymbol{H} + \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{D} \\ \boldsymbol{C}\boldsymbol{H} & \boldsymbol{0} & -\gamma \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma \boldsymbol{I}_p \end{bmatrix} \prec 0 \quad (85)$$

and it is evident that the Lyapunov matrix $\boldsymbol{P}$ is decoupled from the matrix system parameters.

Writing analogously to (44)

$$\boldsymbol{F}_c\boldsymbol{H} = \boldsymbol{F}\boldsymbol{H} - \sum_{k=1}^{m} \boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{K}_{dk}\boldsymbol{H} \tag{86}$$

then with the notation

$$\boldsymbol{R}_k = \boldsymbol{K}_{dk}\boldsymbol{H} \tag{87}$$

(85) implies (73) and (34) is modified as

$$\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{H} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_{dk} \succ 0 \tag{88}$$

to obtain (72) multiplying the left side of (88) by $\boldsymbol{L}^h$.

Also it can consider that (60) is satisfied for the above defined $\boldsymbol{Q}$ that is

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{F}_c\boldsymbol{Q} & \boldsymbol{G} \\ \boldsymbol{Q}\boldsymbol{F}_c^{\mathrm{T}} & -\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{G}^{\mathrm{T}} & \boldsymbol{0} & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{89}$$

Introducing the block diagonal matrix

$$\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{H}\boldsymbol{Q}^{-1} & \boldsymbol{I}_r \end{bmatrix} \tag{90}$$

then pre-multiplying the left side by $\boldsymbol{T}_2$ and post-multiplying the right side by $\boldsymbol{T}_2^{\mathrm{T}}$ (89) implies

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{F}_c\boldsymbol{H} & \boldsymbol{G} \\ \boldsymbol{H}\boldsymbol{F}_c^{\mathrm{T}} & -\boldsymbol{H}\boldsymbol{Q}^{-1}\boldsymbol{H} & \boldsymbol{0} \\ \boldsymbol{G}^{\mathrm{T}} & \boldsymbol{0} & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{91}$$

and with the notation (84) then LMI (91) can be approximated as

$$\begin{bmatrix} -2\boldsymbol{H}+\boldsymbol{P} & \boldsymbol{F}_c\boldsymbol{H} & \boldsymbol{G} \\ \boldsymbol{H}\boldsymbol{F}_c^{\mathrm{T}} & -\boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{G}^{\mathrm{T}} & \boldsymbol{0} & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{92}$$

Thus, using (86) then (92) implies (74).

Since for the given $\boldsymbol{Q}$ also it has to be satisfied the matrix inequality

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{Q}\boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{V} \end{bmatrix} \succ 0 \tag{93}$$

then introducing the positive definite block diagonal matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{H}\boldsymbol{Q}^{-1} & \boldsymbol{I}_m \end{bmatrix} \tag{94}$$

and pre-multiplying the left side by $\boldsymbol{T}$ and post-multiplying the right side by $\boldsymbol{T}^{\mathrm{T}}$ (93) implies the equivalent inequality

$$\begin{bmatrix} \boldsymbol{H}\boldsymbol{Q}^{-1}\boldsymbol{H} & \boldsymbol{H}\boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{C}\boldsymbol{H} & \boldsymbol{V} \end{bmatrix} \succ 0 \tag{95}$$

Thus, with notation (84) then (95) implies (75). Because the inequality (59) stays unchanged, this concludes the proof. □

Zero elements of non strictly positive $\boldsymbol{F}$ a priori generate new boundaries to establish a non-negative gain matrix $\boldsymbol{K}$. Reflecting such boundaries, it is possible to mimic the theory developed above to solve design task for only positive linear discrete-time systems. Since in design conditions only structured variables are defined, the proof of the theorem is omitted.

Theorem 6. *If system (1), (2) is only positive and an element $f_{\alpha\beta}$, $\alpha,\beta \in \langle 1,n\rangle$ of the matrix $\boldsymbol{F}$ is zero, then $\boldsymbol{F}_c$ is positive and Schur and its element $f_{c\alpha\beta}$ is zero if for given non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times p}$ there exist positive definite diagonal matrices $\boldsymbol{P},\boldsymbol{H} \in \mathbb{R}_+^{n\times n}$, $\boldsymbol{V} \in \mathbb{R}_+^{m\times m}$, positive scalars $\gamma,\eta \in \mathbb{R}_+$ and alternatively*

i. positive definite diagonal matrices $\mathrm{R}_k \in \mathbb{R}_+^{n\times n}$ (if all elements $g_{\alpha k}$, are zero),

ii. positive semi-definite diagonal matrices $\mathrm{R}_k \in \mathbb{R}_+^{n\times n}$ (if minimally one elements of $g_{\alpha k}$ is positive)

such that for $j = 1,\ldots,n$, $h = 0,1,\ldots,n-1$, $k = 1,\ldots,m$,

$$\boldsymbol{P} \succ 0,\ \boldsymbol{H} \succ 0,\ \boldsymbol{V} \succ 0,\ \gamma > 0,\ \eta > 0 \tag{96}$$

$$\boldsymbol{L}^h\boldsymbol{F}(j,j+h)_{(\Delta)}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{H} - \sum_{k=1}^{r} \boldsymbol{L}^h\boldsymbol{G}_{dk}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0 \tag{97}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}\boldsymbol{H} - \sum_{k=1}^{m} \boldsymbol{G}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_{dk} & \boldsymbol{P}-2\boldsymbol{H} & * & * \\ \boldsymbol{C}\boldsymbol{H} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{0} & \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0 \tag{98}$$

$$\begin{bmatrix} \boldsymbol{P}-2\boldsymbol{H} & \boldsymbol{F}\boldsymbol{H} - \sum_{k=1}^{r} \boldsymbol{G}_{dk}\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k & \boldsymbol{G} \\ * & -\boldsymbol{P} & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{99}$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{H}\boldsymbol{C}^{\mathrm{T}} \\ * & \boldsymbol{V} \end{bmatrix} \succ 0 \tag{100}$$

$$\min_{\eta}\ \textit{subject to}\ (\boldsymbol{V} - \eta\boldsymbol{I}_m) \prec 0 \tag{101}$$

where

$i.\quad \boldsymbol{R}_k = \mathit{diag}\begin{bmatrix} r_{1k} & \cdots & r_{\beta,k} & \cdots & r_{nk} \end{bmatrix}$
$ii.\quad \boldsymbol{R}_k = \mathit{diag}\begin{bmatrix} r_{1k} & \cdots & r_{\beta-1,k} & 0 & r_{\beta+1,k} & \cdots & r_{nk} \end{bmatrix}$

and

$i.\ \ r_{lk} > 0\ for\ l = 1,\ldots,n,\ k = 1,\ldots,m$
$ii\ \ r_{lk} > 0\ for\ l = 1,\ldots,n,\ k = 1,\ldots,m,\ l \neq \beta$ $and\ r_{\beta k} = 0\ for\ l = \beta.$

When the above conditions hold (77) implies $\boldsymbol{k}_k^T$ and

i. $\boldsymbol{K}$ is a strictly positive matrix,

ii. $\boldsymbol{K}$ is a non-negative matrix.

The procedure can also be applied when more than one zero element occurs in $\boldsymbol{F}$. It should be noted, however that the number of columns of a matrix $\boldsymbol{F}$ containing at least one non-zero element must be less than n, while existence of a solution being bound to fulfill the above LMIs conditions for structured diagonal matrix variables $\boldsymbol{R}_k$, having the zero columns reflecting those columns of $\boldsymbol{F}$ which contains one or more zero elements. Obviously, such a design of structured matrix variables is always done ad hoc.

Note, the above given set of matrix inequalities in Theorem 5 can be simple restructured to obtain enhanced design conditions for H_2 or H_∞ control design of strictly linear positive systems. Similarly, the set of matrix inequalities in Theorem 6 can be restructured to obtain standard or enhanced design conditions for H_2 or H_∞ control design for non-strictly linear positive systems.

Proposition 1. *Considering the state space description of an autonomous linear continuous-time system in the form*

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) \tag{102}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$*, then (102) implies*

$$\boldsymbol{q}(t) = e^{\boldsymbol{A}t}\boldsymbol{q}(0) = \boldsymbol{\Phi}(t)\boldsymbol{q}(0) \tag{103}$$

where the fundamental matrix is constructed by the following application of the inverse Laplace transform $\boldsymbol{\Phi}(t) = \mathcal{L}^{-1}\{(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\}$.

The discrete-time form associated with (102) is

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) \tag{104}$$

where [18]

$$\boldsymbol{F} = e^{\boldsymbol{A}t_s} = \boldsymbol{I}_n + \boldsymbol{A}\frac{t_s}{1!} + \boldsymbol{A}^2\frac{t_s^2}{2!} + \boldsymbol{A}^3\frac{t_s^3}{3!} + \cdots \tag{105}$$

and t_s *is the optimal sampling period satisfying the Whittaker-Nyquist-Kotelnikov-Shannon theorem.*

When the derivative of $\boldsymbol{q}(t)$ *at the point* $t = it_s$ *is approximated by the difference quotient*

$$\dot{\boldsymbol{q}}(t)|_{t=it_s} \doteq \frac{\boldsymbol{q}(i+1) - \boldsymbol{q}(i)}{t_s} \tag{106}$$

(102) implies the relation

$$\boldsymbol{q}(i+1) \doteq \left(\boldsymbol{I}_n + \boldsymbol{A}\frac{t_s}{1!}\right)\boldsymbol{q}(i) = \boldsymbol{F}_1\boldsymbol{q}(i) \tag{107}$$

If the considered class of positive linear continuous-time systems is characterized by Metzler structure of the system matrix $\boldsymbol{A}$ *in which all the off-diagonal elements are nonnegative (equal to or greater than zero) and all the diagonal elements are negative, it is evident that* $\boldsymbol{F}_1$ *certainly takes a nonnegative structure (positive diagonal elements and nonnegative off-diagonal elements) but* $\boldsymbol{F}$ *takes, in general, a strictly positive structure.*

If all the off-diagonal elements of a Metzler matrix are positive (it contains no zero entry), this matrix structure is noted as strictly Metzler matrix. In this case $\boldsymbol{F}_1$ *is strictly positive and* $\boldsymbol{F}$*, in general, too.*

Thus, linear discrete-time not strictly positive systems are rather an exception, obtained only in the case when the sampling period is so small that $\boldsymbol{F}$ *is equal to* $\boldsymbol{F}_1$ *and the off-diagonal element of a Metzler matrix are nonnegative.*

7 Illustrative Examples

To illustrate proposed concepts, the results are presented for a strictly positive discrete-time linear system and for a positive discrete-time linear system.

Retaining the nomenclature of strictly positive systems, system (1), (2) is defined by the parameters

$$\mathrm{F} = \begin{bmatrix} 0.9361 & 0.0116 & 0.1219 & 0.1149 \\ 0.0112 & 0.9197 & 0.0375 & 0.0156 \\ 0.0198 & 0.0792 & 0.8784 & 0.1098 \\ 0.0022 & 0.0428 & 0.0035 & 0.9593 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0081 & 0.0043 \\ 0.0110 & 0.0041 \\ 0.0028 & 0.0063 \\ 0.0025 & 0.0034 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0.0140 \\ 0.0150 \\ 0.0223 \\ 0.0061 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which in the considered notation means that $\boldsymbol{G}$, $\boldsymbol{D}$ are strictly positive, $\boldsymbol{C}$ is non-negative and $\boldsymbol{F}$ is strictly positive but not Schur, since

$$\rho(\boldsymbol{F}) = \{\ 1.0273,\quad 0.8393,\quad 0.9134 \pm 0.0217\,\mathrm{i}\ \}$$

The auxiliary parameters implying from (4), (7) and (21)are

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^T & 1 \\ \boldsymbol{I}_3 & \boldsymbol{0} \end{bmatrix}, \quad 1^T = [1\,1\,1\,1]\,, \ (\Delta) = (1 \leftrightarrow 4)/4$$

$$\boldsymbol{F}(j,j)_{(\Delta)} = \mathrm{diag}\,[0.9361\ 0.9197\ 0.8784\ 0.9593]$$

$$\boldsymbol{F}(j,j+1)_{(\Delta)} = \mathrm{diag}\,[0.0116\ 0.0375\ 0.1098\ 0.0022]$$

$$\boldsymbol{F}(j,j+2)_{(\Delta)} = \mathrm{diag}\,[0.1219\ 0.0156\ 0.0198\ 0.0428]$$

$$\boldsymbol{F}(j,j+3)_{(\Delta)} = \mathrm{diag}\,[0.1149\ 0.0112\ 0.0792\ 0.0035]$$

$$\boldsymbol{G}_{d1} = \mathrm{diag}\,[0.0081\ \ 0.0110\ \ 0.0028\ \ 0.0025]$$

$$\boldsymbol{G}_{d2} = \mathrm{diag}\,[0.0043\ \ 0.0041\ \ 0.0063\ \ 0.0034]$$

Setting iteratively $\eta = 0.056$ when solving (71)-(76) by Self-Dual-Minimization (SeDuMi) package, the LMI variables are

$$\boldsymbol{V} = \mathrm{diag}\,[0.0512\ \ 0.0462]\,, \quad \gamma = 1.7247$$

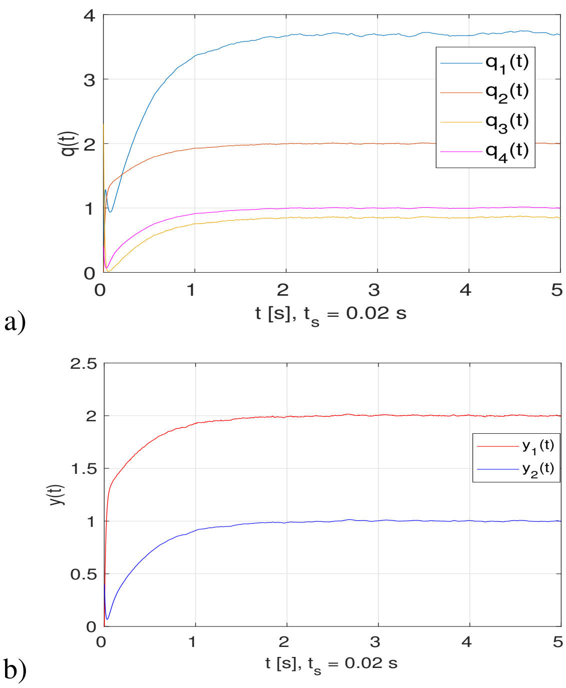

Figure 1: Positive system: a) state response b) output response

$$
\begin{aligned}
\boldsymbol{P} &= \operatorname{diag}\,[0.8122 \;\; 0.0485 \;\; 0.2631 \;\; 0.0362] \\
\boldsymbol{H} &= \operatorname{diag}\,[0.8247 \;\; 0.0474 \;\; 0.2606 \;\; 0.0355] \\
\boldsymbol{R}_1 &= \operatorname{diag}\,[0.3437 \;\; 0.0113 \;\; 0.3293 \;\; 0.0036] \\
\boldsymbol{R}_2 &= \operatorname{diag}\,[0.0240 \;\; 0.0994 \;\; 0.0157 \;\; 0.1196]
\end{aligned}
$$

Correspondingly, the strictly positive matrices

$$
\boldsymbol{K} = \begin{bmatrix} 0.4168 & 0.2376 & 1.2636 & 0.1026 \\ 0.0291 & 2.0969 & 0.0602 & 3.3648 \end{bmatrix}
$$

$$
\boldsymbol{F}_c = \begin{bmatrix} 0.9326 & 0.0007 & 0.1114 & 0.0996 \\ 0.0065 & 0.9085 & 0.0234 & 0.0007 \\ 0.0184 & 0.0653 & 0.8745 & 0.0883 \\ 0.0001 & 0.0351 & 0.0001 & 0.9476 \end{bmatrix}
$$

are obtained as the results, where $\boldsymbol{F}_c$ is Schur because

$$
\rho(\mathrm{F}_c) = \{\; 0.8422 \quad 0.9908 \quad 0.9151 \pm 0.0193\,\mathrm{i} \;\}
$$

Note, H_2 norm of the closed-loop transfer functions matrix is $\delta = 0.0557$ and is within tolerance of η.

Demonstration for a not strictly positive case with $\boldsymbol{F}(1,2) = \boldsymbol{F}(3,4) = 0$ considers the system matrix

$$
\boldsymbol{F} = \begin{bmatrix} 0.9361 & 0 & 0.1219 & 0.1149 \\ 0.0112 & 0.9197 & 0.0375 & 0.0156 \\ 0.0198 & 0.0792 & 0.8784 & 0 \\ 0.0022 & 0.0428 & 0.0035 & 0.9593 \end{bmatrix}
$$

The non-negative $\boldsymbol{F}$ defined in such a way is not Schur because

$$
\rho(\boldsymbol{F}) = \{\; 1.0122, \quad 0.8313, \quad 0.9250 \pm 0.0118\,\mathrm{i} \;\}
$$

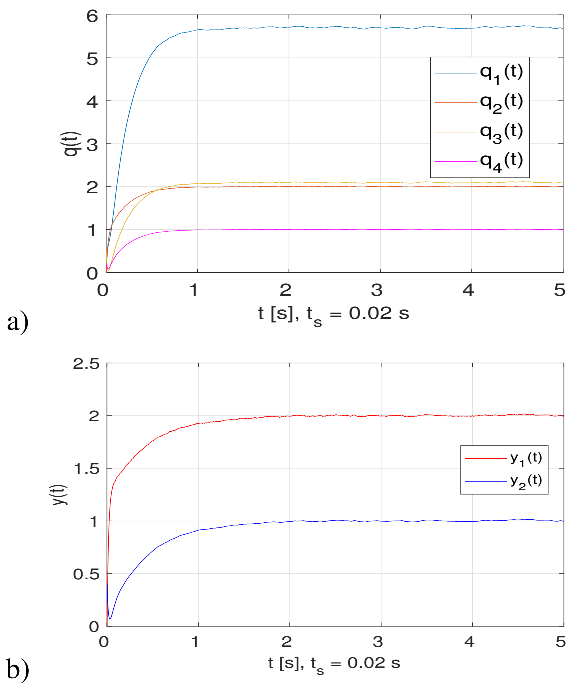

Figure 2: Strictly positive system: a) state response b) output response

This parameter structure of $\boldsymbol{F}$ means that only the axially parameter noted above as the diagonal matrix $\boldsymbol{F}(j,j{+}1)_{(\Delta)}$ is changed in such a way that

$$
\mathrm{F}(j,j{+}1)_{(\Delta)} = \operatorname{diag}\,[0 \quad 0.0375 \quad 0 \quad 0.0022]
$$

while $\boldsymbol{F}(j,j)_{(\Delta)}$, $\boldsymbol{F}(j,j{+}2)_{(\Delta)}$, $\boldsymbol{F}(j,j{+}3)_{(\Delta)}$, $\boldsymbol{G}_{d1}$, $\boldsymbol{G}_{d2}$ stay unchanged.

To solve this task using the set of LMIs defined by Theorem 6, the following structured matrix variable are defined

$$
\boldsymbol{R}_1 = \operatorname{diag}\,[r_{11} \quad 0 \quad r_{31} \quad 0]
$$

$$
\boldsymbol{R}_2 = \operatorname{diag}\,[r_{12} \quad 0 \quad r_{32} \quad 0]
$$

Thus, setting interactively $\eta = 0.092$ to solve (96)–(101), the feasible solution means

$$
\mathrm{V} = \operatorname{diag}\,[0.0756 \;\; 0.0769]\,, \qquad \gamma = 4.6988
$$

$$
\begin{aligned}
\boldsymbol{P} &= \operatorname{diag}\,[1.2379 \;\; 0.0548 \;\; 0.2250 \;\; 0.0582] \\
\boldsymbol{H} &= \operatorname{diag}\,[1.2393 \;\; 0.0544 \;\; 0.2234 \;\; 0.0579] \\
\boldsymbol{R}_1 &= \operatorname{diag}\,[0.5803 \;\; 0 \;\; 0.3047 \;\; 0] \\
\boldsymbol{R}_2 &= \operatorname{diag}\,[0.0069 \;\; 0 \;\; 0.0039 \;\; 0]
\end{aligned}
$$

and this set of variables predefines non-negative matrices

$$
\boldsymbol{K} = \begin{bmatrix} 0.4683 & 0 & 1.3637 & 0 \\ 0.0055 & 0 & 0.0177 & 0 \end{bmatrix}
$$

$$\boldsymbol{F}_c = \begin{bmatrix} 0.9323 & 0 & 0.1108 & 0.1149 \\ 0.0060 & 0.9197 & 0.0224 & 0.0156 \\ 0.0185 & 0.0792 & 0.8745 & 0 \\ 0.0001 & 0.0428 & 0.0001 & 0.9593 \end{bmatrix}$$

where the set of eigenvalues

$$\rho(\boldsymbol{F}_c) = \{\ 0.9955 \quad 0.8364 \quad 0.9289 \pm 0.0038\,\mathrm{i}\ \}\,.$$

conveys that $\boldsymbol{F}_c$ is Schur and H_2 norm of the closed-loop transfer functions matrix $\delta = 0.0914$ is within tolerance of $\eta = 0.092$.

It can also see that the second and forth column of $\boldsymbol{F}$ are identical with associated columns of $\boldsymbol{F}_c$. For example, if f_{24} of $\boldsymbol{F}$ would be also zero, the structure of $\boldsymbol{K}$ stays unchanged (the same structural matrix variables $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ are used) but the matrix elements of $\boldsymbol{K}$ and $\boldsymbol{F}_c$ will be different comparing with the last given, since $\boldsymbol{F}(j, j{+}2)_{(\Delta)}$ would be also changed in consequence.

The obtained solution for strictly positive system is illustrated in Fig. 2, and is indicated by $\boldsymbol{K}$ solved within conditions defined by Theorem 5 for a strictly positive system. The used schemes have to reflect facts that the state vector $\boldsymbol{q}(i)$ as well as the output vector $\boldsymbol{y}(i)$ would be positive also in the case of negative feedback related to forced mode control

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \boldsymbol{W}\boldsymbol{w}(i)$$

Since, by using the static decoupling principle, the signal gain matrix

$$\boldsymbol{W} = \begin{bmatrix} 17.1517 & -19.4574 \\ -22.9244 & 29.5269 \end{bmatrix}$$

is signum indefinite, a non-negative initial system state $\boldsymbol{q}(0)$ has to be set to obtain positive state and output time responses. This concept of positive simulations is executed in Matlab framework, with the desired system output vector $\boldsymbol{w}^{\mathrm{T}}(i) = [2\ 1]$ and $\mathrm{q}^{\mathrm{T}}(0) = [0\ 0\ 0.5\ 0.2]$, $\sigma_d^2 = 0.0064$.

Using the nonnegative matrix $\boldsymbol{K}$ which is given by feasible solution of Theorem 6 for the considered positive system parameter modification, the forced mode is supported by the same desired output vector and

$$\boldsymbol{K} = \begin{bmatrix} 0.4683 & 0 & 1.3637 & 0 \\ 0.0055 & 0 & 0.0177 & 0 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 17.1605 & -14.5538 \\ -25.2042 & 22.6488 \end{bmatrix}$$

$$\boldsymbol{q}^{\mathrm{T}}(0) = [0\ 0\ 2.2\ 0.4]\,, \quad \sigma_d^2 = 0.0064$$

while time responses are in Fig. 1. It demonstrates also for non-strictly positive linear systems that all system output variables reach their desired values at the steady state.

This kind of simulations are included to highlight the presented theoretical results and some performances of control structures of discrete-time positive systems also from system working point setting view.

The synthesis principle is based on the construction of a base set of LMIs, the feasibility of which guarantees that the control law gain matrix will be (strictly) positive and the closed-loop system matrix will also be (strictly) positive. The addition of additional LMIs introduces into synthesis the requirements regarding generally asymptotic stability, D-stability, boundaries of the transfer function matrix norms, limits of variables in terms of LQ control, etc. Analysing the above feasible example results, it is evident that prescribed algebraic parametric constraints are met and a solution guaranties strictly positiveness (non-negativeness) of the control gain matrices $\boldsymbol{K}$ and stability of the strictly positive (non-negative) closed-loop system matrix $\boldsymbol{F}_c$. At the same time, in the forced control mode it necessitates to ensure positiveness of the state and input variables by appropriate choice of non-negative system initial state vector.

One from the properties of enhanced design conditions is that $\boldsymbol{F}_c$ is more closest to nonnegative structure than $\boldsymbol{F}_c$ with $\boldsymbol{K}$ designed in a standard way. Since (84) implies that $2\boldsymbol{H} - \boldsymbol{P} \succ 0$, it gives a feasible set of LMI variables being closest to potential singularities in parameter constraints, defined by desired positive structure of closed-loop system matrix, as it is indicated also by the numerical results.

8 Concluding Remarks

An extended approach for synthesis of the state feedback controllers, destined for MIMO linear positive discrete-time systems and reflecting optimized $\mathrm{H}_2/\mathrm{H}_\infty$ norm attenuation is derived in the paper. The concept of the proposed design method for the control of positive linear systems was created on the basis of an analysis of the properties of methods that use only LMI formulation. Since control design principle for mentioned system class exploits the principle of diagonal stabilization, to accomplish that the closed-loop system matrix be (strictly) positive and Schur, design conditions are established in terms of LMIs linked to positive (semi)definite diagonal matrix variables. The illustrative examples confirm effectiveness of the design principles.

The main idea of the presented state control synthesis, which optimizes upper boundary values of the transfer function matrix norms, is reduced to a feasible problem using the technique of linear matrix inequalities. The LMIs representation of closed-loop system matrix parametric constraints is found, the design conditions are deduced with expression through a slack matrix principle and the variables positivity of the system controlled in the forced mode is proposed by a nonnegative system initial state. The analysis carried out clearly exhibits the usefulness of LMI representation to apply the state-space relations for solutions with Schur positive matrices for the multivariate case of linear discrete-time systems and the reduction the conservativeness when enhanced principle is applied.

The simulation conditions correspond to the considered class of discrete-time systems, feasibility of the synthesis conditions with respect to the defined parametric boundaries and reflect the behavior of the controlled system in a noise environment. The only exception that does not enter the synthesis conditions is the control signal gain, resulting from the principle of static untying. Since structure of this gain matrix is not compatible with the principle of diagonal stabilization of positive systems, the desired positive system behavior can be achieved only by appropriate choice of the system initial conditions.

The obtained results cover as possible complexity of the given type of algorithms. Modified approach efficiency is a scope of further study in developing research ways to formulate conditions on the in general not positive discrete-time linear systems, concerning the existence of a positive realization of the control law gain matrix (existence problem), to provide approaches in constructing such gains (optimization problem) and to find how the positive realizations can be related to continuous-time non Metzlerian systems (generalization problem). It is also planed to study the effects of additive constraints integration into the linear matrix inequalities that may help in deciding among a set of feasible solutions.

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Contribution of individual authors

Anna Filasová elaborated the principles of attenuation of the closed-loop H_2 norm in control law parameter synthesis and implemented their simulation verification, Dušan Krokavec addressed slack matrix principle into mounting an enhanced set of linear matrix inequalities for mixed H_2/H_∞ norm relations in control design for positive and strictly positive discrete-time linear systems.

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