# Comparison Between Chaos-Control Methods Efficiency for Discrete Systems

Lina A. Shalby
Moscow Institute of Physics and Technology
Department of Management and Applied Mathematics
Dolgobrudny, Moscow Region
Russia
Faculty of Women, Ain Shams University
Department of Applied Mathematics
Asmaa Fahmy st., 1, Cairo
Egypt
lina.khamis@gmail.com, lina.shalby@phystech.edu

*Abstract:* Controlling unstable behaviour of many nonlinear dynamical systems is one of the recent interesting topics for researchers. Many methods are proposed to stabilize chaotic discrete time systems. In this paper, a comparison between different control methods is performed for distinguishing their efficiency. The used control methods are Ott-Grebogi-Yorke (OGY), Predictive Feedback Control (PFC), Time Delay Auto Synchronization (TDAS) and its extended (ETDAS), control methods based on self-organizing migrating algorithm (SOMA) and differential evolution (DE). They are briefly introduced and then applied to most popular discrete nonlinear systems 100 times. The controlled orbits of period-1 characteristics are evaluated, presenting the robust of each method according to autocorrelation, the number of required iterations, number of successfully controlled orbits, and max absolute value of the control input. TDAS and PFC methods are the most convenient to stabilize the chaotic attractor of the system.

*Key–Words:* Control Methods, Nonlinear Chaotic Discrete Systems, and Autocorrelation.

## 1 Introduction

The chaotic behaviour of many dynamical systems under certain conditions is one of recent attractive studies for scientists. The most two famous properties of chaos are the highly sensitivity to the initial condition and the parameter value. In the last decades, many researchers concerned with how to control chaotic phenomena depending on these two properties. The aim is to bring a trajectory close enough to a desired location in the chaotic attractor by using a very small perturbation. Numerous researches are focusing on stabilizing special chaotic map or system describing their study case such as [7, 11, 24]. Few others, we are concerning with them, those are focusing to stabilize chaos generally. In 1990, E. Ott was the first publisher who concerned with stabilizing chaos. This method introduced to control discrete data, so continuous time systems should first be discretized by Poincaré map. This control method is called "the Ott-Grebogi-Yorke (OGY) method" [14], it suggests firstly linearizing the nonlinear chaotic system about the desired fixed point and the nominal parameter. It was followed by many other publications improving his method or introducing new methods (see for example [8] and [9], see also survey [2]). In [16, 17], they represent the delayed feedback control (DFC) to stabilize the unstable periodic orbits (UPO). The control input is set to be the multiplication of a gain by the difference between current system state and a state with the time delay as it is discussed in [5]. DFC method was firstly proposed for continuous systems and then extended to the discrete such as in [22]. Many applications, such as in [4, 12], were controlled by using DFC method. Also, many developments for DFC method were performed, see [10, 13]. One of the recent surveys on DFC is presented in [18]. Ushio [23] introduced the idea of predicting control in his paper and it was extended in [3] to stabilize continuous time systems using predictive-based control method. Predictive Feedback Control (PFC) [15] is an easily implemented method to stabilize unknown UPOs in discrete time dynamical systems. In PFC, the control input was given as the multiplication of a gain with the difference between two predicted future system states. PFC depends on applying small controls that completely change the nature of system's behaviour and devotes its attention to the most important problem of stabilizing or controlling chaos. In [21], PFC method has been extended to stabilize continuous time dynamical system. Detailed survey on existing approaches and methods for chaos control can be found in the Handbook of Chaos Control [19]. The extended time delay auto synchronization (ETDAS) method is a control law that was based on the previous states feedback method [17]. The control law is based on the the idea of time delay auto synchronization (TDAS) [16]. Senkerik method is only specified in stabilizing Hénon map by explaining the application of Analytic programming (AP)[20]. AP is a superstructure of evolutionary algorithms and is used for constructing an analytic solution according to the required behaviour. The AP dataset had to be expanded to cover a longer system output history (the states $x_n$ to $x_{n-9}$), which imitates the inspiring control method for the successful control law for stabilization of higher periodic orbits.

In this paper, we focus on the methods ability to stabilize discrete chaotic systems and compare their efficiency. These methods are briefly represented and used to stabilize UPOs. 100 random initial conditions are generated for Logistic and Hénon maps, then the proposed control inputs of each method are inserted to stabilize their attractors. The 100 resulted orbits are examined due to the controllability achievement percent, minimum number of required iterations, maximum autocorrelation and the inserted control value (which is recommended to be small enough to be negligible). Finally, TDAS and PFC are the methods those are theoretically constructed to change any unstable behaviour to stable for discrete time systems generally. So that they have the capability to stabilize the system for any initial point in its domain, and PFC method showed highest accuracy by inserting minimum control values to the system to achieve stability.

In Section 2, the chaos-control methods are introduced briefly. Their inserted control laws to the system are represented to stabilize the chaotic attractor. Section 3 presents the comparison between the methods applied to 100 unstable orbits of Logistic and Hénon maps.

# 2 Chaos-Control methods

Let us in this section introduce the idea of controlling unstable cycles. Nonlinear discrete systems can be written in state space notation

$$x_{i+1} = f(x_i) \tag{1}$$

where $f : \mathbf{R}^N \to \mathbf{R}^N$ is a differentiable map, $x$ is $N \times 1$ state of the system. Let $F$ be the controlled function or the perturbation such that the system has the form

$$F(x_i, u_i) = f(x_i) + g(u_i) = x_{i+1} \tag{2}$$

where $g(u)$ is $M \times 1$ ($M \leq N$) input which is needed to stabilize the system.

## 2.1 OGY method

Now suppose system (1) has a fixed point $x^*$, for which holds that $F(x^*, u) = 0$ or $x^*_{i+1} = x^*_i$. Then around $x^*$ system (2) is linearized to be in the form

$$x_{i+1} = Ax_i + Bu_i \tag{3}$$

where $A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$, $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial u_1} & \cdots & \frac{\partial f_N}{\partial u_N} \end{bmatrix}$. Then around this equilibrium point we can approximate system (2) by using (3). For the discrete maps, let the value $u(j)$ close to $u_0$ and in the neighborhood of the unstable fixed point $x^*$ for some state $j$, the linear map can be approximated in the form

$$x(j+1) - x^* = A(x(j) - x^*) + B(u(j) - u_0) \tag{4}$$

. To calculate the time dependent parameter perturbation $(u(j) - u_0)$, we assume that it is a linear function of $x$

$$u(i) - u_0 = -K^T(x(i) - x^*) \tag{5}$$

where the matrix $K^T$ is $1 \times N$ matrix, require to be determined so that the fixed point $x^*$ becomes stable. Substituting Eq. (6) into Eq. (5), we obtain

$$x(j+1) - x^* = (A - BK^T)[x(j) - x^*] \tag{6}$$

which shows that the fixed point will be stable if the matrix $(A - BK^T)$ is asymptotically stable; that is, all its eigenvalues have modulus smaller than unity.
The solution to the problem of determining $K^T$, such that the eigenvalues of the matrix $(A - BK^T)$ have specified values, is known from control systems theory as the ”pole placement technique” [1].
Finding an appropriate control law $u = -Kx$ is the essential consideration to get stable. For linear systems the control law will control any point $x \in \mathbf{R}$, whereas for nonlinear systems the control law will only work for $x$ sufficiently close to $x^*$, since the linearization (5) is only valid in the vicinity of the equilibrium point.

### 2.2 TDAS and ETDAS methods

The simpler TDAS control method for the $i-1$ state, in its discrete form, is given as

$$F(x_i) = K[x_{i-m} - x_i] \tag{7}$$

where $m$ is the period of the $m$-periodic orbit to be stabilized. For the purpose of stabilizing higher periodic orbits, the ETDAS method was used. The original control method, ETDAS in the discrete form suitable for the two-dimensional Hénon map, has this form

$$\begin{aligned} x_{i+1} &= a – x_i^2 + by_i + F_i, \\ F_i &= K(1-R)[S_{i-m} - x_i], \\ S_i &= x_i + RS_{i-m} \end{aligned} \tag{8}$$

where $K$ and $R$ are adjustable constants and $S$ is given by a delay equation utilizing previous states of the system.

### 2.3 PFC method

A great effort by author in [15] to introduce a generalized PFC method, which is capable of stabilizing periodic orbits with an arbitrary small perturbation. We will introduce some important notations at first about PFC. The periodicity of an orbit can be represented in the form of the composition of itself, i.e. if $f$ is periodic function of period $p$, then

$$f^p = \underbrace{fofof \dots f}_{p-times},$$

and there exist a $s$-cycle has the number $\mu = f'(x_s{}^*) \times \cdots \times f'(x_1{}^*)$ is the multiplicator of the cycle. In this paper, we concerns with $s = 1$, so $\mu = f'(x^*)$. The periodicity of the function is also used as the predictive term of the control methods due to its future indication for the function after $p$ iterations.

Let the $x^*$ under consideration be unstable and $|\ \mu\ | > 1$, and if a control term is added to stabilize the system, then the resulting closed-loop system takes the form

$$x_{i+1} = F(x_i), \quad F(x) = f(x) - \epsilon(f^{p+1}(x) - f^p(x)) \tag{9}$$

where,

$$\frac{|\epsilon-\epsilon^*|}{|\epsilon^*|} < \frac{1}{|\mu|}, \quad \epsilon^* = \frac{1}{\mu^p(\mu-1)}$$

where $p$ is a nonnegative integer. It is important to note that the quantity $\epsilon^*$ arbitrarily small for sufficiently large values of $p$, so that the control parameter will be a small value add to the original function. The choice of $p$ and $\mu$ is clearly discussed in [15].
For higher dimension function $f$, the stability condition depends on the eigenvalues $\lambda_i$ of the jacobian matrix $J(f)$ and the choice of $\mu$ will be taken to be the value of $\lambda$ greater than 1.

### 2.4 Methods based on SOMA-DE

In [20], they proosed methods to stabilize Hénon map. They used two evolutionary algorithms, self-organizing migrating algorithm (SOMA) and differential evolution (DE) (for more details, see [6]). They represented best control laws based on SOMA and DE for the $i-1$ orbit stabilization. Their coast functions $F_i$ are

$$\begin{aligned} CL1 : F(x_i) &= \frac{x_i x_{i-1}}{x_{i-1}(2x_i - x_{i-1}) - (x_i + 0.035)/(x_{i-1} - x_i)} \\ CL2 : F(x_i) &= \frac{1.111(x_{i-1} - x_i)(0.0112x_i - x_{i-1}x_i)}{x_i} \\ CL3 : F(x_i) &= \frac{(-x_{i-1} + x_i - 0.497)(x_i - x_{i-1})}{x_{i-1}^2} \end{aligned} \tag{10}$$

In the next section, the control methods are applied to the most popular chaotic systems, Logistic map and Hénon map. And some statistical tests are used to analysis the difference between the chaotic sequences before control and after control.

## 3 Comparison between control methods

The Comparison between control methods is evaluated for controlling the orbits of two of the most popular chaotic maps, Logistic and Hénon, to be of period-1. Logistic map is the simplest 1D nonlinear map, given by

$$x_{i+1} = ax_i(1 - x_i) \tag{11}$$

where $a$ is parameter. For system (11), 100 randomly initial condition $x_0$ are chosen and 1000 iterates are computed considering the case $a = 3.9$ for which the system is apparently chaotic as shown in Fig. 1. An important characteristic of a chaotic attractor is that there exist an infinite number of unstable periodic orbits embedded within it. Now suppose we want to avoid chaos at $a = 3.9$.

Hénon map is one of the known 2D discrete systems, has the form

$$x_{i+1} = a - x_i{}^2 + by_i, \quad y_{i+1} = x_i, \tag{12}$$

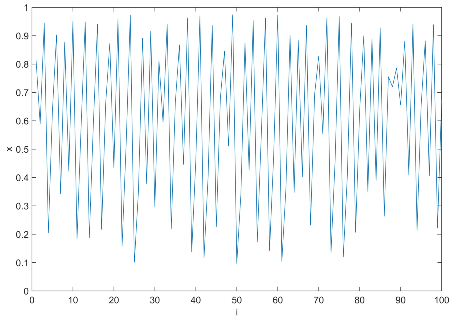


Figure 1: $x_i$ values for Logistic map

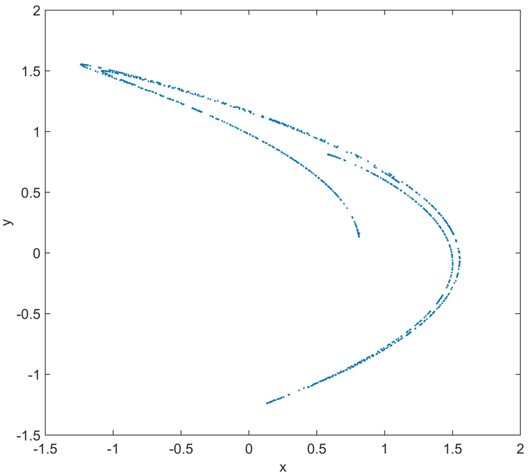


Figure 2: ($x$, $y$) values for Hénon map

where $a$ and $b$ are parameters. For $a = 1.2$ and $b = 0.3$, Eq. (12) has fixed point 0.8. Taking initial conditions $x_0 = y_0 = 0.7$, its eigenvalues for the jacobian matrix are $-1.58883$ and $0.18$, which shows instability and Fig. 2 represents its chaotic series.

Let us begin with applying OGY method to control Logistic map. Suppose that the parameter $a$ is allowed to vary in the range $[a_0 - \delta, a_0 + \delta]$, where $\delta \ll 1$. Denote the target $x^*$ (period-1 orbit), that gives the best system performance, to be controlled. Assume that at time $j$, the trajectory falls into the neighborhood of $x^*$. The linearized dynamics in the neighborhood of component $x^*$ is then

$$x_{j+1} - x^* = \frac{\partial f}{\partial x}[x_j - x^*] + \frac{\partial f}{\partial a}\Delta a_j =$$
$$a_0[1 - 2x^*][x_j - x^*] + x^*[1 - x^*]\Delta a_j,$$

where the partial derivatives are evaluated at $x = x^*$ and $a = a_0$. We require $x_{i+1}$ to stay in the neighborhood of $x^*$. Hence, we set $x_{j+1} = x^*$, which gives

$$\Delta a_j = a_0 \frac{(2x^* - 1)(x_j - x^*)}{(x^*[1 - x^*])} \tag{13}$$

Eq. (13) holds only when the trajectory $x_j$ enters a small neighborhood of $x^*$, i.e., when $|x_j - x^*| \ll 1$ and hence the required parameter perturbation $\Delta a_j$ is small. So for our example, we can adjust$|\ x_j - x^*\ | < 0.01$.

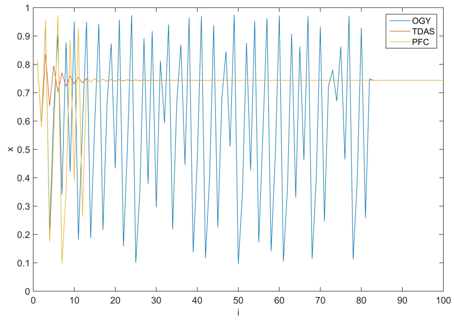


Figure 3: $x_i$ values for each of OGY, TDAS and PFC methods

The TDAS and PFC methods could be applied to stabilize Logistic map at the fixed point. We set $K = -0.48$ for TDAS method, while $\mu = -1.9$ and $p = 5$ for PFC method.

The three control methods are applied to Logistic map 100 times for the 100 random initial conditions. TDAS method succeeded to control Logistic map trajectory for only 41 out of the 100 initial conditions,but for the close ones to 0 and 1, and the series diverges. The results of the first 100 iterates for each method are plotted as in Fig. 3, starting from $x_0 = 0.8147...$ (one of the 100 random chosen initial conditions). Fig. 3 shows that OGY method is controlling Logistic map lately than TDAS and PFC methods.

The autocorrelation (AC) of a random sequence describes the correlation between its values for finding repeated ones. The maximum AC values are evaluated as shown in Table 1.

Table 1 shows that TDAS and PFC methods has high AC compared to that of Logistic uncontrolled map, while OGY method is less AC than TDAS and PFC method. Also the range of iterations (minimum and maximum iterations) to achieve $x^*$ with accuracy $10^{-5}$ for the 100 samples. PFC method gives the best method for general use to control Logistic of period one without restrictions and with minimum value of the maximum control term $F_i$ for all the 100 samples.

Control terms are applied to Hénon map, defined in Eq. (12). All the control methods are inserted 100

Table 1: Maximum autocorrelation, range of iterations (it.) and maximum absolute value of $F_i$ for 100 Logistic map samples.

| Methods | Max. AC | Range of it. | Max. $\|F_i\|$ |
|---|---|---|---|
| OGY | 0.7207 | 2 - 822 | 0.0181 |
| TDAS | 0.8955 | 37 - 73 | 0.3230 |
| PFC | 0.8102 | 22 - 118 | 0.0075 |

times for 100 random initial conditions$(x_0, y_0)$. For OGY method, $\mid x_i - x^* \mid < 0.1$ and $K = 1.6354$. TDAS control parameter is $K = -0.85$, while the used values of the control parameters for ETDAS method $R$ and $K$ are $0.326949$ and $1.03809$ respectively. The PFC method value of $\mu$ is taken equal to the eigenvalue greater than 1, $\mu = -1.58883$. Fig. 4 shows the first 50 iterations for each method as they are all stabilizing the H$\acute{e}$non map to the fixed point $(x^*, y^*) = (0.8, 0.8)$.

Table 2 compares the methods results. TDAS, ETDAS, and PFC methods are capable to control all the inserted initial conditions. The higher max. AC values are for CL1, CL2 and PFC methods as shown in Table 2. Minimum iterations to control H$\acute{e}$non map is 10 for CL1 method and 12 for ETDAS and PFC methods, while minimum numbers of maximum required iterations are 23, 32, 40 and 50 for ETDAS, CL3, TDAS and PFC methods respectively. Finally, minimum values of the coast functions are $(0.1574, 0)$, $(0.2217, 0.2687)$ and $(0.4428, 0)$ for OGY, PFC and CL3 methods.

TDAS and PFC methods are capable to control any discrete nonlinear map starting at any initial condition. PFC mehtod showed higher efficiency than TDAS method due to its small control term values and quick convergence. OGY method defect is that the control term takes place only when the iterations come close to the desired point, so it is not a good choice for short short time use of a real life application. ETDAS method is suitable to be applied for wide range of variables, to prevent divergence. The control methods based on SOMA and DE show weakness points due to its high control term values and divergence of some orbits.

## 4 Conclusions

A comparison is done to the recent control methods OGY, TDAS, ETDAS, PFC, and three control methods based on SOMA and DE. The methods are discussed briefly and then applied to control 100 orbits of

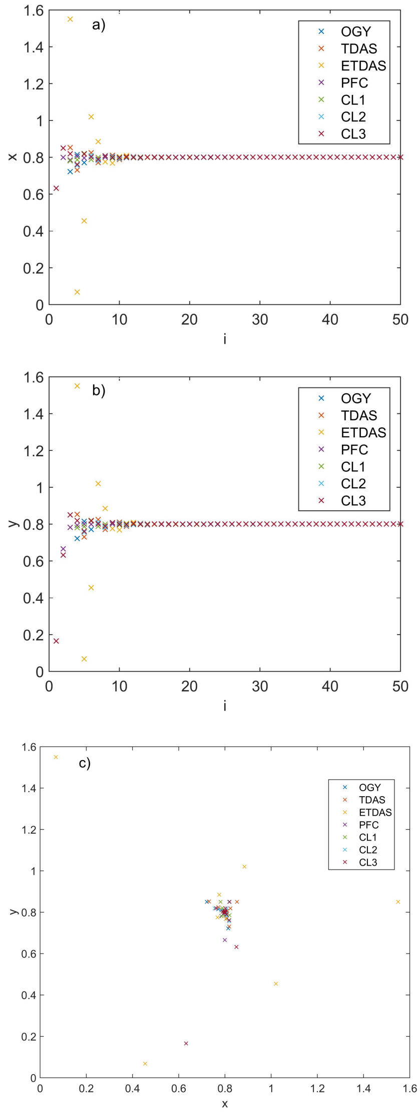


Figure 4: a) $x$ values, b) $y$ values, and c) $x - y$ plot of H$\acute{e}$non map after applying OGY, TDAS, ETDAS, PFC, CL1, CL2, and CL3.

Table 2: Successfully controlled percent(SC%) orbits, maximum autocorrelation (AC), range of iterations (it.) and maximum absolute value of $F_i$ for 100 Hénon samples.

| Methods | SC % | Max. AC | | Range of it. | | Max.$\|F_i\|$ | |
|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | Min. | Max. | $x$ | $y$ |
| OGY | 76 | 0.7897 | 0.7913 | 19 | 1000 | 0.1574 | 0 |
| TDAS | 100 | 0.6358 | 0.6923 | 19 | 40 | 1.2338 | 0 |
| ETDAS | 100 | 0.6404 | 0.7331 | 12 | 23 | 1.6233 | 0 |
| PFC | 100 | 0.8535 | 0.7874 | 12 | 50 | 0.2217 | 0.2687 |
| CL1 | 74 | 0.9317 | 0.9210 | 10 | 129 | 3.4058 | 0 |
| CL2 | 79 | 0.8642 | 0.8904 | 16 | 64 | 2.9337 | 0 |
| CL3 | 35 | 0.7522 | 0.6643 | 16 | 32 | 0.4428 | 0 |

Logistic and Hénon maps generated randomly. They are compared according to the percent of successfully controlled period-1 orbit, maximum absolute autocorrelation, minimum and maximum required number of iteration to reach accuracy less than $10^{-5}$, and maximum absolute value of additive control term. TDAS and PFC methods showed the most capability to control the discrete nonlinear systems with high efficiency, but PFC method has higher autocorrelation and less value of maximum absolute control term.

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