On Completeness of Inference Rules for Vague Functional and Vague Multivalued Dependencies in two-element Vague Relation Instances

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Abstract: In this paper we pay attention to completeness of the inference rules for vague functional and vague multivalued dependencies in two-element, vague relation instances. Motivated by the fact that the set of the inference rules is a complete set, that is, these exists a vague relation instance on given relation scheme which satisfies all vague functional and vague multivalued dependencies in two-element, vague relation instances. Motivated by the fact that the set of the inference rules VF1-VF4 and VM1-VM6, where VF belongs to VF5-VF7 and VM belongs to VM7-VM10, is complete set (note that the inference rules VF1-VF4 and VM1-VM6 are the main inference rules since they imply the inference rules VF5-VF7 and VM7-VM10).

This means that there exists a vague relation instance \( r^* \) on \( R(A_1, A_2, ..., A_n) \) (\( r^* \) is denoted by \( r \) in [12]), which satisfies \( A \overset{\theta}{\rightarrow} V B \) resp. \( A \overset{\theta}{\rightarrow} V B \) if \( A \overset{\theta}{\rightarrow} V B \) resp. \( A \overset{\theta}{\rightarrow} V B \) belongs to \( (\mathcal{V}, \mathcal{M})^+ \), and violates \( X \overset{\theta}{\rightarrow} V Y \) resp. \( X \overset{\theta}{\rightarrow} V Y \), where \( X \overset{\theta}{\rightarrow} V Y \) resp. \( X \overset{\theta}{\rightarrow} V Y \) is some vague functional resp. vague multivalued dependency on \( \{A_1, A_2, ..., A_n\} \) which is not a member of the closure \( (\mathcal{V}, \mathcal{M})^+ \) of \( \mathcal{V} \cup \mathcal{M} \).

The closure \( (\mathcal{V}, \mathcal{M})^+ \) of \( \mathcal{V} \cup \mathcal{M} \) is the set of all vague functional and vague multivalued dependencies on \( \{A_1, A_2, ..., A_n\} \) that can be derived from \( \mathcal{V} \cup \mathcal{M} \) by repeated applications of the inference rules VF1-VF4 and VM1-VM6, where \( \mathcal{V} \) resp. \( \mathcal{M} \) is some set of vague functional resp. vague multivalued dependencies on \( \{A_1, A_2, ..., A_n\} \).

\( X^+ (\theta, \mathcal{V}, \mathcal{M}) \) is the closure of \( X \) with respect to \( \mathcal{V} \) and \( \mathcal{M} \), i.e., \( X^+ (\theta, \mathcal{V}, \mathcal{M}) \) is the set of attributes \( A \in \{A_1, A_2, ..., A_n\} \), such that \( X \overset{\theta}{\rightarrow} V A \) belongs to \( (\mathcal{V}, \mathcal{M})^+ \).

\( W_1, W_2, ..., W_m \) are the sets in the dependency basis \( \text{dep} (X, \theta) \) of \( X \) with respect to \( \theta \), that cover

\[ \{A_1, A_2, ..., A_n\} \setminus X^+ (\theta, \mathcal{V}, \mathcal{M}) \]

Thus,

\[ \{A_1, A_2, ..., A_n\} \setminus X^+ (\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1}^{m} W_i. \]
Note that the dependency basis \( \text{dep}(X, \theta) \) of \( X \) with respect to \( \theta \) is the set \( \{Y_1, Y_2, ..., Y_k\} \) of the sets \( Y_1, Y_2, ..., Y_k \), such that \( Y_1, Y_2, ..., Y_k \) is a partition of \( \{A_1, A_2, ..., A_n\} \), and \( X \xrightarrow{\theta} V \) if and only if \( Z \) is the union of some of the sets \( Y_1, Y_2, ..., Y_k \).

For the sake of simplicity, it is assumed that \( U_1 = U_2 = ... = U_n = \{u\} = U \).

The vague sets \( V_1 \) and \( V_2 \) in \( U \) are given by

\[
V_1 = \{ \langle u, [t_{V_1}(u), 1 - f_{V_1}(u)] \rangle : u \in U \} \\
= \{ \langle u, [t_{V_1}(u), 1 - f_{V_1}(u)] \rangle \} = \{u, a\}
\]

and

\[
V_2 = \{ \langle u, [t_{V_2}(u), 1 - f_{V_2}(u)] \rangle : u \in U \} \\
= \{ \langle u, [t_{V_2}(u), 1 - f_{V_2}(u)] \rangle \} = \{u, b\}.
\]

It is assumed that \( SE_U(a, b) = \theta' \), where \( SE_U : Vag(U) \times Vag(U) \rightarrow [0, 1] \) is a similarity measure on \( Vag(U) \).

Thus, \( SE(V_1, V_2) = \theta' \).

\( \theta' \) is selected in the following way.

If \( 1 \Delta_{1}(V, M) \neq \emptyset \), then \( \theta' \in (\theta'', \theta) \) is fixed, where

\[
\theta'' = \max_{1 \Delta_{1}(V, M)} \{ 1 \theta_1(V, M) \},
\]

and

\[
1 \Delta_{1}(V, M) \\
= \left\{ A \right. \left. \xrightarrow{1 \theta} V B \in (V, M)^+ : 1 \theta_1(V, M) < \theta \right\} \\
\cup \\
\left\{ A \right. \left. \xrightarrow{1 \theta} V B \in (V, M)^+ : 1 \theta_1(V, M) < \theta \right\}.
\]

If \( 1 \Delta_{1}(V, M) = \emptyset \), then it is assumed that \( \theta' = 0 \).

Here, \( 1 \theta_1(V, M) \) denotes the limit strength of the dependency \( A \xrightarrow{1 \theta} V \) resp. \( A \xrightarrow{1 \theta} V \) with respect to \( V \) and \( M \), i.e., \( 1 \theta_1(V, M) \) belongs to \([0, 1]\), \( A \xrightarrow{1 \theta_1(V, M)} V \) B resp. \( A \xrightarrow{1 \theta_1(V, M)} V \) B belongs to \((V, M)^+ \), and \( \theta' \leq 1 \theta_1(V, M) \) for each \( A \xrightarrow{1 \theta_1(V, M)} V \) B resp. \( A \xrightarrow{1 \theta_1(V, M)} V \) B that belongs to \((V, M)^+ \).

The main purpose of this paper is to prove that the vague relation instance \( r^* \) on \( R(A_1, A_2, ..., A_n) \) may be selected to contain only two elements.

Note that the notation applied in this section will be explained in detail in the following sections.

## 2 Notation

Let \( R(A_1, A_2, ..., A_n) \) be a relation scheme on domains \( U_1, U_2, ..., U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i, i \in \{1, 2, ..., n\} = I \).

Suppose that \( V(U_i) \) is the family of all vague sets in \( U_i, i \in I \).

Here, we say that \( V_i \) is a vague set in \( U_i \), if

\[
V_i = \{ \langle u, [t_{V_i}(u), 1 - f_{V_i}(u)] \rangle : u \in U_i \},
\]

where \( t_{V_i} : U_i \rightarrow [0, 1], f_{V_i} : U_i \rightarrow [0, 1] \) are functions such that \( t_{V_i}(u) + f_{V_i}(u) \leq 1 \) for all \( u \in U_i \).

We also say that \( \{t_{V_i}(u), 1 - f_{V_i}(u)\} \subseteq [0, 1] \) is the vague value joined to \( u \in U_i \).

A vague relation instance \( r \) on \( R(A_1, A_2, ..., A_n) \) is a subset of the cross product \( V(U_1) \times V(U_2) \times ... \times V(U_n) \).

A tuple \( t \) of \( r \) is denoted by

\[
(t[A_1], t[A_2], ..., t[A_n]).
\]

Here, we consider the vague set \( t[A_i] \) as the value of the attribute \( A_i \) on \( t \).

Let \( Vag(U_i) \) be the set of all vague values associated to the elements \( u_i \in U_i, i \in I \).

A similarity measure on \( Vag(U_i) \) is a mapping \( SE_i : Vag(U_i) \times Vag(U_i) \rightarrow [0, 1] \), such that \( SE_i(x, x) = 1 \), \( SE_i(x, y) = SE_i(y, x) \), and \( SE_i(x, z) \geq \max_{y \in Vag(U_i)} (\min (SE_i(x, y), SE_i(y, z))) \) for all \( x, y, z \in Vag(U_i) \).

Suppose that \( SE_i \) is a similarity measure on \( Vag(U_i) \), \( i \in I \).

Let

\[
A_i = \{ \langle u, [t_{A_i}(u), 1 - f_{A_i}(u)] \rangle : u \in U_i \} = \{ a_{u}^{i} : u \in U_i \},
\]

\[
B_i = \{ \langle u, [t_{B_i}(u), 1 - f_{B_i}(u)] \rangle : u \in U_i \} = \{ b_{u}^{i} : u \in U_i \}
\]

to be two vague sets in \( U_i \).

The similarity measure \( SE(A_i, B_i) \) between the vague sets \( A_i \) and \( B_i \) is given by

\[
SE(A_i, B_i) = \min \left\{ \min_{a_{u}^{i} \in A_i} \max_{b_{u}^{i} \in B_i} \left\{ SE_i\left( [t_{A_i}(u), 1 - f_{A_i}(u)] \right) \right\} \right\}.
\]
[\{t_{B_i}(u), 1 - f_{B_i}(u)\}]\} \right\} \right\}.

\[
\min_{b_i \in B_i} \left\{ \max_{a_i \in A_i} \left\{ SE_i \left( [t_{B_i}(u), 1 - f_{B_i}(u)] \right) \right\} \left\{ \right. \right\}.
\]

Now, if \( r \) is a vague relation instance on \( R(A_1, A_2, ..., A_n) \), \( t_1 \) and \( t_2 \) are any two tuples in \( r \), and \( X \) is a subset of \( \{A_1, A_2, ..., A_n\} \), then, the similarity measure \( SE_X(t_1, t_2) \) between tuples \( t_1 \) and \( t_2 \) on the attribute set \( X \) is defined by

\[
SE_X(t_1, t_2) = \min_{A \subseteq X} \left\{ SE \left( t_1 [A], t_2 [A] \right) \right\}.
\]

For various definitions of similarity measures, see, [16], [5], [4], [14] and [15].

Recently, in [10] and [11], we introduced new definitions of vague functional and vague multivalued dependencies.

If \( X \) and \( Y \) are subsets of \( \{A_1, A_2, ..., A_n\} \), and \( \theta \in [0, 1] \) is a number, then, the vague relation instance \( r \) on \( R(A_1, A_2, ..., A_n) \) is said to satisfy the vague functional dependency \( X \rightarrow_V^\theta Y \), if for every pair of tuples \( t_1 \) and \( t_2 \) in \( r \),

\[
SE_Y(t_1, t_2) \geq \min \{\theta, SE_X(t_1, t_2)\}.
\]

Vague relation instance \( r \) is said to satisfy the vague multivalued dependency \( X \rightarrow_V^\theta Y \), if for every pair of tuples \( t_1 \) and \( t_2 \) in \( r \), there exists a tuple \( t_3 \) in \( r \), such that

\[
SE_X(t_3, t_1) \geq \min \{\theta, SE_X(t_1, t_2)\},
\]

\[
SE_Y(t_3, t_1) \geq \min \{\theta, SE_X(t_1, t_2)\},
\]

\[
\min \{\theta, SE_X(t_1, t_2)\}.
\]

We write \( X \rightarrow_V Y \) resp. \( X \rightarrow_V Y \) instead of \( X \rightarrow_V^\theta Y \) resp. \( X \rightarrow_V^\theta Y \) if \( \theta = 1 \).

As in [13], \( \theta \) is called the linguistic strength of the vague functional (vague multivalued) dependency \( X \rightarrow_V^\theta Y \).

Note that the authors in [24] first introduced the formal definitions of fuzzy functional and fuzzy multivalued dependencies which are given on the basis of conformance values.

For various definitions of vague functional and vague multivalued dependencies, see, [16], [19], [26] and [20].

3 Implications and interpretations of fuzzy logic

A mapping \( I : [0, 1]^2 \rightarrow [0, 1] \) is a fuzzy implication if \( I(0, 0) = I(0, 1) = I(1, 1) = 1 \) and \( I(1, 0) = 0 \).

The most important classes of fuzzy implications are: \( S \)-implications, \( R \)-implications and \( QL \)-implications (strong, residual, quantum logic implications, respectively).

For precise definitions and description of \( S \)-, \( R \)-, \( QL \)-implications, as well as for the definitions of various additional fuzzy implications, see, [23] and [3].

In this paper (as in [13]), we use the following operators:

\[
T_M(x, y) = \min \{x, y\},
\]

\[
S_M(x, y) = \max \{x, y\},
\]

\[
I_L(x, y) = \min \{1 - x + y, 1\},
\]

where \( T_M \) is the minimum \( t \)-norm (\( t \)-norms are usually applied to model fuzzy conjunctions), \( S_M \) is the maximum \( t \)-co-norm (fuzzy disjunctions are often modeled by \( t \)-co-norms), and \( I_L \) is the Lukasiewicz fuzzy implication.

The Lukasiewics fuzzy implication is an \( S \)-, an \( R \)- and a \( QL \)-fuzzy implication at the same time (see, [23], [3]).

Some of the works that deal with \( S \)-, \( R \)- and \( QL \)-implications are the following: [1], [2], [17], [25], [22], [18], [21].

Now, we extend some of the corresponding definitions in [13].

Let \( R(A_1, A_2, ..., A_n) \) be a relation scheme on domains \( U_1, U_2, ..., U_n \), where \( I \) is an attribute on the universe of discourse \( U_i, i \in I \).

Let \( r = \{t_1, t_2\} \) be a two-element vague relation instance on \( R(A_1, A_2, ..., A_n) \), and \( \beta \in [0, 1] \) be a number.

Suppose that the similarity measures \( SE_i \), \( SE \) and \( SE_X \) are given as above.

Let \( A_k \in \{A_1, A_2, ..., A_n\} \).

We calculate the similarity measure \( SE(t_1[A_k], t_2[A_k]) \) between the vague sets \( t_1[A_k] \) and \( t_2[A_k] \),

We check whether or not \( SE(t_1[A_k], t_2[A_k]) \geq \beta \).

If \( SE(t_1[A_k], t_2[A_k]) \geq \beta \), we put \( i_{r, \beta}(A_k) \) to be some value in the interval \( [\frac{1}{2}, 1] \).

Otherwise, if \( SE(t_1[A_k], t_2[A_k]) < \beta \), we put \( i_{r, \beta}(A_k) \) to be some value in the interval \( [0, \frac{1}{2}] \).

We say that \( i_{r, \beta} \) is a valuation joined to \( r \) and \( \beta \).

Thus, \( i_{r, \beta} \) is a function defined on \( \{A_1, A_2, ..., A_n\} \) with values in \([0, 1] \).

More precisely, \( i_{r, \beta} : \{A_1, A_2, ..., A_n\} \rightarrow [0, 1] \).
\[ i_{r,\beta} (A_k) > \frac{1}{2} \] if \( SE(t_1 [A_k], t_2 [A_k]) \geq \beta, \]
\[ i_{r,\beta} (A_k) \leq \frac{1}{2} \] if \( SE(t_1 [A_k], t_2 [A_k]) < \beta, \]

\( k \in \{1, 2, ..., n\}. \)

Note that the fact that \( i_{r,\beta} (A_k) \in [0, 1] \) for \( k \in \{1, 2, ..., n\} \) yields that the attributes \( A_k, k \in \{1, 2, ..., n\} \) are actually fuzzy formulas now (with respect to \( i_{r,\beta} \)).

Having in mind (1), we define

\[ i_{r,\beta} (A \wedge B) = \min \{ i_{r,\beta} (A), i_{r,\beta} (B) \}, \]
\[ i_{r,\beta} (A \vee B) = \max \{ i_{r,\beta} (A), i_{r,\beta} (B) \}, \]
\[ i_{r,\beta} (A \Rightarrow B) = \min \{ 1 - i_{r,\beta} (A) + i_{r,\beta} (B), 1 \} \]

for \( A, B \in \{A_1, A_2, ..., A_n\}. \)

Since \( T_M, S_M \) and \( I_t \) are functions defined on \([0, 1]^2\) with values in \([0, 1]\), it follows that \( A \wedge B, A \vee B \) and \( A \Rightarrow B \) are functions defined on \([0, 1]^2\) with values in \([0, 1]\), and are also fuzzy formulas with respect to \( i_{r,\beta} \).

Consequently, \( ((A \wedge B) \Rightarrow C) \vee D \), for all \( A, B, C, D \in \{A_1, A_2, ..., A_n\} \), is also a fuzzy formula with respect to \( i_{r,\beta} \).

Namely, this follows from now on the fact that

\[ i_{r,\beta} (((A \wedge B) \Rightarrow C) \vee D) \]
\[ = \max \{ i_{r,\beta} ((A \wedge B) \Rightarrow C), i_{r,\beta} (D) \} \]
\[ = \max \left\{ \min \{ 1 - i_{r,\beta} (A \wedge B) + i_{r,\beta} (C), 1 \}, i_{r,\beta} (D) \right\} \]
\[ = \max \left\{ \min \{ 1 - \min \{ i_{r,\beta} (A), i_{r,\beta} (B) \} + i_{r,\beta} (C), 1 \}, i_{r,\beta} (D) \right\}. \]

4 Auxiliary results

Let \( R (A_1, A_2, ..., A_n) \) be a relation scheme on domains \( U_1, U_2, ..., U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i, i \in I \).

Let \( r \) be a vague relation instance on \( R (A_1, A_2, ..., A_n) \), and \( X \rightarrow_{\theta} V Y \) a vague multivalued dependency on \( \{A_1, A_2, ..., A_n\} \).

Vague relation instance \( r \) is said to satisfy the vague multivalued dependency \( X \rightarrow_{\theta} V Y \), \( \theta \)-actively, if \( r \) satisfies \( X \rightarrow_{\theta} V Y \), and \( SE(t_1 [A], t_2 [A]) \geq \theta \) for all \( A \in X \) and all \( t_1, t_2 \in r \).

Suppose that \( r \) satisfies \( X \rightarrow_{\theta} V Y \), \( \theta \)-actively. It follows that

\[ SE_X (t_1, t_2) = \min_{A \in X} \{ SE(t_1 [A], t_2 [A]) \} \geq \theta \]

for all \( t_1, t_2 \in r \).

Hence, \( r \) satisfies \( X \rightarrow_{\theta} V Y \), and \( SE_X (t_1, t_2) \geq \theta \) for all \( t_1, t_2 \in r \).

Suppose that \( r \) satisfies \( X \rightarrow_{\theta} V Y \), and that \( SE_X (t_1, t_2) \geq \theta \) for all \( t_1, t_2 \in r \).

Since

\[ SE_X (t_1, t_2) = \min_{A \in X} \{ SE(t_1 [A], t_2 [A]) \}, \]

we obtain that \( SE(t_1 [A], t_2 [A]) \geq \theta \) for all \( A \in X \) and all \( t_1, t_2 \in r \).
Hence, \( r \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively.

Thus, \( r \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively if and only if \( r \) satisfies \( X \xrightarrow{\theta} Y \), and \( SE_X(t_1,t_2) \geq \theta \)
for all \( t_1,t_2 \in r \).

The following results follow immediately.

\textbf{Theorem 1.} Let \( R(A_1,A_2,...,A_n) \) be a relation scheme on domains \( U_1,U_2,...,U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i \), \( i \in I \). Let \( r = \{t_1,t_2\} \) be a vague relation instance on \( R(A_1,A_2,...,A_n) \), and \( X \xrightarrow{\theta} V \) a vague multivalued dependency on \( \{A_1,A_2,...,A_n\} \). Then, \( r \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively if and only if \( SE_X(t_1,t_2) \geq \theta \), \( SE_Y(t_1,t_2) \geq \theta \) or \( SE_X(t_1,t_2) \geq \theta \), where \( Z = \{A_1,A_2,...,A_n\} \setminus (X \cup Y) \).

\textbf{Theorem 2.} Let \( R(A_1,A_2,...,A_n) \) be a relation scheme on domains \( U_1,U_2,...,U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i \), \( i \in I \). Let \( r = \{t_1,t_2\} \) be a vague relation instance on \( R(A_1,A_2,...,A_n) \), and \( X \xrightarrow{\theta} V \) a vague multivalued dependency on \( \{A_1,A_2,...,A_n\} \). Then, \( r \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively if and only if \( SE_X(t_1,t_2) \geq \theta \), and

\[ i_{r,\theta}((\land_{A \in X}A) \Rightarrow ((\land_{B \in Y}B) \lor (\land_{C \in Z}Z))) \geq \frac{1}{2}, \]

where \( Z = \{A_1,A_2,...,A_n\} \setminus (X \cup Y) \).

\textbf{Theorem 3.} Let \( R(A_1,A_2,...,A_n) \) be a relation scheme on domains \( U_1,U_2,...,U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i \), \( i \in I \). Let \( r = \{t_1,t_2\} \) and \( q = \{u_1,u_2\} \) be any two vague relation instances on \( R(A_1,A_2,...,A_n) \), and \( X \xrightarrow{\theta} V \) a vague multivalued dependency on \( \{A_1,A_2,...,A_n\} \).

Suppose that \( r \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively, and that \( SE(u_1[A],u_2[A]) \geq \theta \) for each attribute \( A \in \{A_1,A_2,...,A_n\} \) such that \( SE(t_1[A],t_2[A]) \geq \theta \). Then, \( q \) satisfies \( X \xrightarrow{\theta} V \), \( \theta \)-actively.

\textbf{Theorem 4.} Let \( R(A_1,A_2,...,A_n) \) be a relation scheme on domains \( U_1,U_2,...,U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i \), \( i \in I \). Let \( r \) be a vague relation instance on \( R(A_1,A_2,...,A_n) \), and \( X \xrightarrow{\theta} V \) a vague functional dependency on \( \{A_1,A_2,...,A_n\} \). Then, \( r \) satisfies \( X \xrightarrow{\theta} V \) if and only if \( r \) satisfies \( X \xrightarrow{\theta} V \) for all \( B \in Y \).

\textbf{Proof.} \((\Rightarrow)\) Suppose that \( r \) satisfies \( X \xrightarrow{\theta} V \). Hence,

\[ SE_Y(t_1,t_2) \geq \min \{\theta,SE_X(t_1,t_2)\} \]

for \( t_1,t_2 \in r \).

Let \( B \in Y \). We have,

\[ SE_B(t_1,t_2) = SE(t_1[B],t_2[B]) \geq \min \{SE(t_1[B],t_2[B])\} = SE_Y(t_1,t_2) \geq \min \{\theta,SE_X(t_1,t_2)\} \]

for \( t_1,t_2 \in r \).

Therefore, \( r \) satisfies \( X \xrightarrow{\theta} B \).

\((\Leftarrow)\) Suppose that \( r \) satisfies \( X \xrightarrow{\theta} B \) for all \( B \in Y \).

Suppose that \( r \) does not satisfy \( X \xrightarrow{\theta} V \).

Now, there are tuples \( t_1,t_2 \in r \), such that

\[ SE_Y(t_1,t_2) < \min \{\theta,SE_X(t_1,t_2)\}. \]

Since \( r \) satisfies \( X \xrightarrow{\theta} V \) for all \( B \in Y \), it follows that

\[ SE_B(t_1,t_2) \geq \min \{\theta,SE_X(t_1,t_2)\} \]

for all \( B \in Y \).

Therefore,

\[ SE_B(t_1,t_2) \geq SE_Y(t_1,t_2) \]

for all \( B \in Y \).

Since

\[ SE_Y(t_1,t_2) = \min \{SE(t_1[B],t_2[B])\} \]

we know that there exists some \( B_0 \in Y \) such that

\[ SE_Y(t_1,t_2) < \min \{\theta,SE_X(t_1,t_2)\}. \]
\[ SE_Y (t_1, t_2) = SE_{B_0} (t_1, t_2). \]

Therefore,
\[ SE_{B_0} (t_1, t_2) > SE_Y (t_1, t_2) = SE_{B_0} (t_1, t_2). \]

This is a contradiction.

We conclude, \( r \) satisfies \( X \xrightarrow{\theta} Y \).

This completes the proof. \( \square \)

**Proof.** \( \text{II (⇒)} \) Suppose that \( r \) satisfies \( X \xrightarrow{\theta} Y \).

By VF2, \( r \) satisfies \( X \rightarrow Y \) for all \( B \in Y \).

By VF1, \( r \) satisfies \( X \xrightarrow{\theta} B \) for all \( B \in Y \).

(\( \Leftarrow \)) Suppose that \( r \) satisfies \( X \xrightarrow{\theta} B \) for all \( B \in Y \).

By VF5, \( r \) satisfies \( X \xrightarrow{\theta} Y \).

This completes the proof. \( \square \)

### 5 Main result

**Theorem 5.** Let \( R (A_1, A_2, ..., A_n) \) be a relation scheme on domains \( U_1, U_2, ..., U_n \), where \( A_i \) is an attribute on the universe of discourse \( U_i \), \( i \in I \). Let \((\mathcal{V}, \mathcal{M})^+\) be the closure of \( \mathcal{V} \cup \mathcal{M} \), where \( \mathcal{V} \) resp. \( \mathcal{M} \) is some set of vague functional resp. vague multivalued dependencies on \( \{A_1, A_2, ..., A_n\} \). Suppose that \( X \xrightarrow{\theta} \mathcal{V} \) resp. \( X \xrightarrow{\theta} \mathcal{M} \) is some vague functional resp. vague multivalued dependency on \( \{A_1, A_2, ..., A_n\} \) which is not and element of \((\mathcal{V}, \mathcal{M})^+\). Let \( r^* \) be a vague relation instance on \( R (A_1, A_2, ..., A_n) \) joined to \((\mathcal{V}, \mathcal{M})^+\) and \( X \xrightarrow{\theta} \mathcal{V} \) resp. \( X \xrightarrow{\theta} \mathcal{M} \) (in the way described above). Then, there exists a two-element vague relation instance \( s \subseteq r^* \) on \( R (A_1, A_2, ..., A_n) \), such that \( s \) satisfies \( A \xrightarrow{\theta} \mathcal{V} \) resp. \( A \xrightarrow{\theta} \mathcal{M} \). \( A \xrightarrow{\theta} \mathcal{V} \) if \( A \xrightarrow{\theta} B \) resp. \( A \xrightarrow{\theta} \mathcal{M} \) if \( A \xrightarrow{\theta} B \) belongs to \((\mathcal{V}, \mathcal{M})^+\), and violates \( X \xrightarrow{\theta} \mathcal{V} \) resp. \( X \xrightarrow{\theta} \mathcal{M} \).

**Proof.** Follows from Theorem 1 and Theorems 3 and 4. \( \square \)

### 6 Remarks

Motivated by the extensions of the corresponding results in the case of fuzzy functional and fuzzy multivalued dependencies through the resolution principle (see, e.g., [6], [7], [8], [9]), we may assume that the results derived in this paper will also be extended and applied accordingly.

**References:**


