On Completeness of Inference Rules for Vague Functional and Vague Multivalued Dependencies in two-element Vague Relation Instances

DŽENAN GUŠIĆ
University of Sarajevo
Faculty of Sciences and Mathematics
Department of Mathematics
Zmaja od Bosne 33-35, 71000 Sarajevo
BOSNIA AND HERZEGOVINA
dzenang@pmf.unsa.ba

Abstract: In this paper we pay attention to completeness of the inference rules for vague functional and vague multivalued dependencies in two-element, vague relation instances. Motivated by the fact that the set of the inference rules is a complete set, that is, these exists a vague relation instance on given relation scheme which satisfies all vague functional and vague multivalued dependencies in the closure of the union of some set of vague functional and some set of vague multivalued dependencies, and violates a vague functional, respectively, a vague multivalued dependency outside of the closure, we prove that the vague relation instance may be chosen to contain only two elements.

Key–Words: Vague functional dependencies, vague multivalued dependencies, interpretations, inference rules, completeness

1 Introduction

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

By Theorem 3 in [12], the set $\{VF1 - VF4, VM1 - VM6\}$ is complete set (note that the inference rules VF1-VF4 and VM1-VM6 are the main inference rules since they imply the inference rules VF5-VF7 and VM7-VM10).

This means that there exists a vague relation instance r^* on $R(A_1, A_2, ..., A_n)$ (r^* is denoted by r in [12]), which satisfies $A \stackrel{1\theta}{\rightarrow}_V B$ resp. $A \stackrel{1\theta}{\rightarrow\rightarrow}_V B$ if $A \stackrel{1\theta}{\rightarrow}_V B$ resp. $A \stackrel{1\theta}{\rightarrow\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and violates $X \stackrel{\theta}{\rightarrow}_V Y$ resp. $X \stackrel{\theta}{\rightarrow\rightarrow}_V Y$, where $X \stackrel{\theta}{\rightarrow}_V Y$ resp. $X \stackrel{\theta}{\rightarrow\rightarrow}_V Y$ is some vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$ which is not a member of the closure $(\mathcal{V}, \mathcal{M})^+$ of $\mathcal{V} \cup \mathcal{M}$.

The closure $(\mathcal{V}, \mathcal{M})^+$ of $\mathcal{V} \cup \mathcal{M}$ is the set of all vague functional and vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$ that can be derived from $\mathcal{V} \cup \mathcal{M}$ by repeated applications of the inference rules VF1-VF4 and VM1-VM6, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$.

In [12], r^* is given by Table I.

Table 1:

$X^+(\theta, \mathcal{V}, \mathcal{M})$	W_1	...	W_m
V_1,..., V_1	V_1,..., V_1	...	V_1,..., V_1
V_1,..., V_1	V_1,..., V_1	...	V_2,..., V_2
$\vdots$	$\vdots$	...	$\vdots$
V_1,..., V_1	V_2,..., V_2	...	V_1,..., V_1
V_1,..., V_1	V_2,..., V_2	...	V_2,..., V_2

$X^+(\theta, \mathcal{V}, \mathcal{M})$ is the closure of X with respect to $\mathcal{V}$ and $\mathcal{M}$, i.e., $X^+(\theta, \mathcal{V}, \mathcal{M})$ is the set of attributes $A \in \{A_1, A_2, ..., A_n\}$, such that $X \stackrel{\theta}{\rightarrow}_V A$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

W_1, W_2,..., W_m are the sets in the dependency basis $dep(X, \theta)$ of X with respect to θ, that cover

$$\{A_1, A_2, ..., A_n\} \setminus X^+(\theta, \mathcal{V}, \mathcal{M}).$$

Thus,

$$\{A_1, A_2, ..., A_n\} \setminus X^+(\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1}^{m} W_i.$$

Note that the dependency basis $dep(X, \theta)$ of X with respect to θ is the set $\{Y_1, Y_2, ..., Y_k\}$ of the sets Y_1, Y_2,..., Y_k, such that Y_1, Y_2,..., Y_k is a partition of $\{A_1, A_2, ..., A_n\}$, and $X \overset{\theta}{\twoheadrightarrow}_V Z$ if and only if Z is the union of some of the sets Y_1, Y_2,..., Y_k.

For the sake of simplicity, it is assumed that $U_1 = U_2 = ... = U_n = \{u\} = U$.

The vague sets V_1 and V_2 in U are given by

$$\begin{aligned} V_1 &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle \,:\, u \in U\} \\ &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle\} = \{\langle u, a\rangle\} \end{aligned}$$

and

$$\begin{aligned} V_2 &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle \,:\, u \in U\} \\ &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle\} = \{\langle u, b\rangle\}. \end{aligned}$$

It is assumed that $SE_U(a, b) = \theta'$, where $SE_U : Vag(U) \times Vag(U) \to [0, 1]$ is a similarity measure on $Vag(U)$.

Thus, $SE(V_1, V_2) = \theta'$.

θ' is selected in the following way.

If ${}_1\Delta_l(\mathcal{V}, \mathcal{M}) \neq \varnothing$, then $\theta' \in \left(\theta'', \theta\right)$ is fixed, where

$$\theta'' = \max_{{}_1\Delta_l(\mathcal{V}, \mathcal{M})} \{{}_1\theta_l(\mathcal{V}, \mathcal{M})\},$$

and

$$\begin{aligned} &{}_1\Delta_l(\mathcal{V}, \mathcal{M}) \\ &= \left\{A \overset{1\theta}{\to}_V B \in (\mathcal{V}, \mathcal{M})^+ \,:\, {}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta\right\} \cup \\ &\quad \left\{A \overset{1\theta}{\twoheadrightarrow}_V B \in (\mathcal{V}, \mathcal{M})^+ \,:\, {}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta\right\}. \end{aligned}$$

If ${}_1\Delta_l(\mathcal{V}, \mathcal{M}) = \varnothing$, then it is assumed that $\theta' = 0$.

Here, ${}_1\theta_l(\mathcal{V}, \mathcal{M})$ denotes the limit strength of the dependency $A \overset{1\theta}{\to}_V B$ resp. $A \overset{1\theta}{\twoheadrightarrow}_V B$ with respect to $\mathcal{V}$ and $\mathcal{M}$, i.e., ${}_1\theta_l(\mathcal{V}, \mathcal{M})$ belongs to $[0, 1]$, $A \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\to}_V B$ resp. $A \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\twoheadrightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and $\theta_2 \leq {}_1\theta_l(\mathcal{V}, \mathcal{M})$ for each $A \overset{\theta_2}{\to}_V B$ resp. $A \overset{\theta_2}{\twoheadrightarrow}_V B$ that belongs to $(\mathcal{V}, \mathcal{M})^+$.

The main purpose of this paper is to prove that the vague relation instance r^* on $R(A_1, A_2, ..., A_n)$ may be selected to contain only two elements.

Note that the notation applied in this section will be explained in detail in the following sections.

2 Notation

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$.

Suppose that $V(U_i)$ is the family of all vagues sets in U_i, $i \in I$.

Here, we say that V_i is a vague set in U_i, if

$$V_i = \{\langle u, [t_{V_i}(u), 1 - f_{V_i}(u)]\rangle \;:\; u \in U_i\},$$

where $t_{V_i} : U_i \to [0, 1]$, $f_{V_i} : U_i \to [0, 1]$ are functions such that $t_{V_i}(u) + f_{V_i}(u) \leq 1$ for all $u \in U_i$.

We also say that $[t_{V_i}(u), 1 - f_{V_i}(u)] \subseteq [0, 1]$ is the vague value joined to $u \in U_i$.

A vague relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $V(U_1) \times V(U_2) \times ... \times V(U_n)$.

A tuple t of r is denoted by

$$(t[A_1], t[A_2], ..., t[A_n]).$$

Here, we consider the vague set $t[A_i]$ as the value of the attribute A_i on t.

Let $Vag(U_i)$ be the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping $SE_i : Vag(U_i) \times Vag(U_i) \to [0, 1]$, such that $SE_i(x, x) = 1$, $SE_i(x, y) = SE_i(y, x)$, and $SE_i(x, z) \geq \max_{y \in Vag(U_i)} (\min(SE_i(x, y), SE_i(y, z)))$ for all $x, y, z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let

$$\begin{aligned} A_i &= \{\langle u, [t_{A_i}(u), 1 - f_{A_i}(u)]\rangle \;:\; u \in U_i\} \\ &= \left\{a_u^i \;:\; u \in U_i\right\}, \\ B_i &= \{\langle u, [t_{B_i}(u), 1 - f_{B_i}(u)]\rangle \;:\; u \in U_i\} \\ &= \left\{b_u^i \;:\; u \in U_i\right\} \end{aligned}$$

be two vague sets in U_i.

The similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$\begin{aligned} &SE(A_i, B_i) \\ &= \min\Big\{ \min_{a_u^i \in A_i} \Big\{ \max_{b_u^i \in B_i} \Big\{ SE_i\Big([t_{A_i}(u), 1 - f_{A_i}(u)], \end{aligned}$$

$$[t_{B_i}(u), 1 - f_{B_i}(u)]\Big)\Big\}\Big\},$$
$$\min_{b_u^i \in B_i}\Big\{\max_{a_u^i \in A_i}\Big\{SE_i\Big([t_{B_i}(u), 1 - f_{B_i}(u)],$$
$$[t_{A_i}(u), 1 - f_{A_i}(u)]\Big)\Big\}\Big\}\Big\}.$$

Now, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between tuples t_1 and t_2 on the attribute set X is defined by

$$SE_X(t_1, t_2) = \min_{A \in X}\{SE(t_1[A], t_2[A])\}.$$

For various definitions of similarity measures, see, [16], [5], [4], [14] and [15].

Recently, in [10] and [11], we introduced new definitions of vague functional and vague multivalued dependencies.

If X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta \in [0, 1]$ is a number, then, the vague relation instance r on $R(A_1, A_2, ..., A_n)$ is said to satisfy the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r,

$$SE_Y(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}.$$

Vague relation instance r is said to satisfy the vague multivalued dependency $X \overset{\theta}{\to\to}_V Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min\{\theta, SE_X(t_1, t_2)\}, \\ SE_Y(t_3, t_1) &\geq \min\{\theta, SE_X(t_1, t_2)\}, \\ &SE_{\{A_1, A_2, ..., A_n\} \setminus (X \cup Y)}(t_3, t_2) \\ &\geq \min\{\theta, SE_X(t_1, t_2)\}. \end{aligned}$$

We write $X \to_V Y$ resp. $X \to\to_V Y$ instead of $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$ if $\theta = 1$.

As in [13], θ is called the linguistic strength of the vague functional (vague multivalued) dependency $X \overset{\theta}{\to}_V Y$ ($X \overset{\theta}{\to\to}_V Y$).

Note that the authors in [24] first introduced the formal definitions of fuzzy functional and fuzzy multivalued dependencies which are given on the basis of conformance values.

For various definitions of vague functional and vague multivalued dependencies, see, [16], [19], [26] and [20].

3 Implications and interpretations of fuzzy logic

A mapping $I : [0, 1]^2 \to [0, 1]$ is a fuzzy implication if $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$.

The most important classes of fuzzy implications are: S-implications, R-implications and QL-implications (strong, residual, quantum logic implications, respectively).

For precise definitions and description of S- , R- , QL-implications, as well as for the definitions of various additional fuzzy implications, see, [23] and [3].

In this paper (as in [13]), we use the following operators:

$$\begin{aligned} T_M(x, y) &= \min\{x, y\}, \\ S_M(x, y) &= \max\{x, y\}, \\ I_L(x, y) &= \min\{1 - x + y, 1\}, \end{aligned} \tag{1}$$

where T_M is the minimum t-norm (t-norms are usually applied to model fuzzy conjunctions), S_M is the maximum t-co-norm (fuzzy disjunctions are often modeled by t-co-norms), and I_L is the Lukasiewicz fuzzy implication.

The Lukasiewics fuzzy implication is an S- , an R- and a QL-fuzzy implication at the same time (see, [23], [3]).

Some of the works that deal with S- ,R- and QL-implications are the following: [1], [2], [17], [25], [22], [18], [21].

Now, we extend some of the corresponding definitions in [13].

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Let $r = \{t_1, t_2\}$ be a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$ be a number.

Suppose that the similarity measures SE_i, SE and SE_X are given as above.

Let $A_k \in \{A_1, A_2, ..., A_n\}$.

We calculate the similarity measure $SE(t_1[A_k], t_2[A_k])$ between the vague sets $t_1[A_k]$ and $t_2[A_k]$.

We check whether or not $SE(t_1[A_k], t_2[A_k]) \geq \beta$.

If $SE(t_1[A_k], t_2[A_k]) \geq \beta$, we put $i_{r,\beta}(A_k)$ to be some value in the interval $\left(\frac{1}{2}, 1\right]$.

Otherwise, if $SE(t_1[A_k], t_2[A_k]) < \beta$, we put $i_{r,\beta}(A_k)$ to be some value in the interval $\left[0, \frac{1}{2}\right]$.

We say that $i_{r,\beta}$ is a valuation joined to r and β.

Thus, $i_{r,\beta}$ is a function defined on $\{A_1, A_2, ..., A_n\}$ with values in $[0, 1]$.

More precisely, $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0, 1]$,

$$i_{r,\beta}(A_k) > \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) \geq \beta,$$
$$i_{r,\beta}(A_k) \leq \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) < \beta,$$

$k \in \{1, 2, ..., n\}$.

Note that the fact that $i_{r,\beta}(A_k) \in [0,1]$ for $k \in \{1, 2, ..., n\}$ yields that the attributes A_k, $k \in \{1, 2, ..., n\}$ are actually fuzzy formulas now (with respect to $i_{r,\beta}$).

Having in mind (1), we define

$$i_{r,\beta}(A \wedge B) = \min\{i_{r,\beta}(A), i_{r,\beta}(B)\},$$
$$i_{r,\beta}(A \vee B) = \max\{i_{r,\beta}(A), i_{r,\beta}(B)\},$$
$$i_{r,\beta}(A \Rightarrow B) = \min\{1 - i_{r,\beta}(A) + i_{r,\beta}(B), 1\}$$

for A, $B \in \{A_1, A_2, ..., A_n\}$.

Since T_M, S_M and I_L are functions defined on $[0,1]^2$ with values in $[0,1]$, it follows that $A \wedge B$, $A \vee B$ and $A \Rightarrow B$, A, $B \in \{A_1, A_2, ..., A_n\}$, are also fuzzy formulas with respect to $i_{r,\beta}$.

Consequently, $((A \wedge B) \Rightarrow C) \vee D$, where A, B, C, $D \in \{A_1, A_2, ..., A_n\}$, for example, is a fuzzy formula with respect to $i_{r,\beta}$.

Namely, this follows from now from the fact that

$$\begin{aligned} & i_{r,\beta}(((A \wedge B) \Rightarrow C) \vee D) \\ &= \max\{i_{r,\beta}((A \wedge B) \Rightarrow C), i_{r,\beta}(D)\} \\ &= \max\Big\{\min\{1 - i_{r,\beta}(A \wedge B) + i_{r,\beta}(C), 1\}, \\ & \quad i_{r,\beta}(D)\Big\} \\ &= \max\Big\{\min\Big\{1 - \min\{i_{r,\beta}(A), i_{r,\beta}(B)\} + \\ & \quad i_{r,\beta}(C), 1\Big\}, i_{r,\beta}(D)\Big\}. \end{aligned}$$

In this paper we are interested in the following fuzzy formulas with respect to $i_{r,\beta}$:

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B),$$
$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C)),$$

where X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $Z \subseteq \{A_1, A_2, ..., A_n\}$ is given by $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$, where X and Y are given.

Through the rest of the paper we shall assume that each time some $r = \{t_1, t_2\}$ and some $\beta \in [0,1]$ are given, the fuzzy formula

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$$

resp.

$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$$

with respect to $i_{r,\beta}$ is joined to $X \xrightarrow{\theta}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$, where $X \xrightarrow{\theta}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$ is a vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$, and $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

4 Auxiliary results

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$, and $X \overset{\theta}{\to\to}_V Y$ a vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$.

Vague relation instance r is said to satisfy the vague multivalued dependency $X \overset{\theta}{\to\to}_V Y$, θ-actively, if r satisfies $X \overset{\theta}{\to\to}_V Y$, and $SE(t_1[A], t_2[A]) \geq \theta$ for all $A \in X$ and all t_1, $t_2 \in r$.

Suppose that r satisfies $X \overset{\theta}{\to\to}_V Y$, θ-actively.

It follows that

$$\begin{aligned} & SE_X(t_1, t_2) \\ &= \min_{A \in X}\{SE(t_1[A], t_2[A])\} \geq \theta \end{aligned}$$

for all t_1, $t_2 \in r$.

Hence, r satisfies $X \overset{\theta}{\to\to}_V Y$, and $SE_X(t_1, t_2) \geq \theta$ for all t_1, $t_2 \in r$.

Suppose that r satisfies $X \overset{\theta}{\to\to}_V Y$, and that $SE_X(t_1, t_2) \geq \theta$ for all t_1, $t_2 \in r$.

Since

$$SE_X(t_1, t_2) = \min_{A \in X}\{SE(t_1[A], t_2[A])\},$$

we obtain that $SE(t_1[A], t_2[A]) \geq \theta$ for all $A \in X$ and all t_1, $t_2 \in r$.

Hence, r satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively.

Thus, r satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively if and only if r satisfies $X \overset{\theta}{\to}\to_V Y$, and $SE_X(t_1, t_2) \geq \theta$ for all $t_1, t_2 \in r$.

The following results follow immediately.

Theorem 1. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $r = \{t_1, t_2\}$ be a vague relation instance on $R(A_1, A_2, ..., A_n)$, and $X \overset{\theta}{\to}\to_V Y$ a vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$. Then, r satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively if and only if $SE_X(t_1, t_2) \geq \theta$, $SE_Y(t_1, t_2) \geq \theta$ or $SE_X(t_1, t_2) \geq \theta$, $SE_Z(t_1, t_2) \geq \theta$, where $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.*

Theorem 2. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $r = \{t_1, t_2\}$ be a vague relation instance on $R(A_1, A_2, ..., A_n)$, and $X \overset{\theta}{\to}\to_V Y$ a vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$. Then, r satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively if and only if $SE_X(t_1, t_2) \geq \theta$, and*

$$i_{r,\theta}\left((\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}Z))\right) > \frac{1}{2},$$

where $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

Theorem 3. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $r = \{t_1, t_2\}$ and $q = \{u_1, u_2\}$ be any two vague relation instances on $R(A_1, A_2, ..., A_n)$, and $X \overset{\theta}{\to}\to_V Y$ a vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$. Suppose that r satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively, and that $SE(u_1[A], u_2[A]) \geq \theta$ for each attribute $A \in \{A_1, A_2, ..., A_n\}$ such that $SE(t_1[A], t_2[A]) \geq \theta$. Then, q satisfies $X \overset{\theta}{\to}\to_V Y$, θ-actively.*

Theorem 4. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$, and $X \overset{\theta}{\to}_V Y$ a vague functional dependency on $\{A_1, A_2, ..., A_n\}$. Then, r satisfies $X \overset{\theta}{\to}_V Y$ if and only if r satisfies $X \overset{\theta}{\to}_V B$ for all $B \in Y$.*

Proof. I $(\Rightarrow)$ Suppose that r satisfies $X \overset{\theta}{\to}_V Y$.

Hence,

$$SE_Y(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}$$

for $t_1, t_2 \in r$.

Let $B \in Y$. We have,

$$\begin{aligned} SE_B(t_1, t_2) =& SE(t_1[B], t_2[B]) \\ &\geq \min_{B\in Y}\{SE(t_1[B], t_2[B])\} \\ &= SE_Y(t_1, t_2) \\ &\geq \min\{\theta, SE_X(t_1, t_2)\} \end{aligned}$$

for $t_1, t_2 \in r$..

Therefore, r satisfies $X \overset{\theta}{\to}_V B$.

$(\Leftarrow)$ Suppose that r satisfies $X \overset{\theta}{\to}_V B$ for all $B \in Y$.

Suppose that r does not satisfy $X \overset{\theta}{\to}_V Y$.

Now, there are tuples $t_1, t_2 \in r$, such that

$$SE_Y(t_1, t_2) < \min\{\theta, SE_X(t_1, t_2)\}.$$

Since r satisfies $X \overset{\theta}{\to}_V B$ for all $B \in Y$, it follows that

$$SE_B(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}$$

for all $B \in Y$.

Therefore,

$$SE_B(t_1, t_2) \geq SE_Y(t_1, t_2)$$

for all $B \in Y$.

Since

$$\begin{aligned} SE_Y(t_1, t_2) &= \min_{B\in Y}\{SE(t_1[B], t_2[B])\} \\ &= \min_{B\in Y}\{SE_B(t_1, t_2)\}, \end{aligned}$$

we know that there exists some $B_0 \in Y$ such that

$$SE_Y(t_1,t_2) = SE_{B_0}(t_1,t_2).$$

Therefore,

$$SE_{B_0}(t_1,t_2) > SE_Y(t_1,t_2) = SE_{B_0}(t_1,t_2).$$

This is a contradiction.
We conclude, r satisfies $X \overset{\theta}{\rightarrow}_V Y$.
This completes the proof. $\square$

Proof. II $(\Rightarrow)$ Suppose that r satisfies $X \overset{\theta}{\rightarrow}_V Y$.
By VF2, r satisfies $X \rightarrow_V B$ for all $B \in Y$.
By VF1, r satisfies $X \overset{\theta}{\rightarrow}_V B$ for all $B \in Y$.
$(\Leftarrow)$ Suppose that r satisfies $X \overset{\theta}{\rightarrow}_V B$ for all B $\in Y$.
By VF5, r satisfies $X \overset{\theta}{\rightarrow}_V Y$.
This completes the proof. $\square$

5 Main result

Theorem 5. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $(\mathcal{V}, \mathcal{M})^+$ be the closure of $\mathcal{V} \cup \mathcal{M}$, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$ is some vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$ which is not and element of $(\mathcal{V}, \mathcal{M})^+$. Let r^* be a vague relation instance on $R(A_1, A_2, ..., A_n)$ joined to $(\mathcal{V}, \mathcal{M})^+$ and $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$ (in the way described above). Then, there exists a two-element vague relation instance $s \subseteq r^*$ on $R(A_1, A_2, ..., A_n)$, such that s satisfies $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \overset{1\theta}{\rightarrow\rightarrow}_V B$ if $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \overset{1\theta}{\rightarrow\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.*

Proof. Follows from Theorem 1 and Theorems 3 and 4. $\square$

6 Remarks

Motivated by the extensions of the corresponding results in the case of fuzzy functional and fuzzy multivalued dependencies through the resolution principle (see, e.g., [6], [7], [8], [9]), we may assume that the results derived in this paper will also be extended and applied accordingly.

References:

[1] M. Baczyński and B. Jayaram, On the characterizations of (S,N)-implications generated from continuous negations, in: Proceedings of the 11th Conf. Information Processing and Management of Uncertainity in Knowledge-based Systems, Paris 2006, pp. 436–443.

[2] M. Baczyński and B. Jayaram, On the characterizations of (S,N)-implications, *Fuzzy Sets and Systems* 158, 2007, pp. 1713–1727.

[3] M. Baczyński and B. Jayaram, *Fuzzy Implications,* Springer–Verlag, Berlin–Heidelberg 2008

[4] S.–M. Chen, Measures of Similarity Between Vague Sets, *Fuzzy Sets and Systems* 74, 1995, pp. 217–223.

[5] S.–M. Chen, Similarity Measures Between Vague Sets and Between Elements, *IEEE Transactions on Systems, Man and Cybernetics* 27, 1997, pp. 153–159.

[6] N. Dukić, Dž. Gušić, A. Muratović-Ribić, A. Alihodžić, E. Tabak and H. Dukić, From fuzzy dependences to fuzzy formulas and vice versa, for Kleene-Dienes fuzzy implication operator, *WSEAS Trans. on Systems and Control* 13, 2018, pp. 285–297.

[7] N. Dukić, Dž. Gušić and N. Kajmović, Reichenbach and f-generated implications in fuzzy database relations, *Int. J. of Circuits, Systems and Signal Processing* 12, 2018, pp. 285–297.

[8] Dž. Gušić, Continuous Maps in Fuzzy Relations, *WSEAS Trans. on Systems and Control* 13, 2018, pp. 324–344.

[9] Dž. Gušić, On Fuzzy Dependencies and g-generated Fuzzy Implications in Fuzzy Relations, *WSEAS Trans. on Systems and Control* 14, 2019, pp. 71–89.

[10] Dž. Gušić, Soundness and Completeness of Inference Rules for New Vague Functional Dependencies, *MATEC Web of Conferences*, to appear.

[11] Dž. Gušić, Soundness of Inference Rules for New Vague Multivalued Dependencies, *MATEC Web of Conferences*, to appear.

[12] Dž. Gušić, Completeness of Inference Rules for New Vague Multivalued Dependencies, *WSEAS Trans. on Math.*, to appear.

[13] Dž. Gušić, Vague Functional Dependencies and Resolution Principle, *WSEAS Trans. on Math.*, to appear.

[14] D.–H. Hong and C. Kim, A note on Similarity Measures Between Vague Sets and Between Elements, *Information Sciences* 115, 1999, pp. 83–96.

[15] F. Li and Z. Xu, Measures of Similarity Between Vague Sets, *Journal of Software* 12, 2001, pp. 922–927.

[16] A. Lu and W. Ng, Managing Merged Data by Vague Functional Dependencies, in: ER 2004 LNCS, Berlin 2004, pp. 259–272.

[17] M. Mas, M. Monserrat and J. Torrens, QL-implications versus d-implications, *Kybernetika* 42, 2006, pp. 956–966.

[18] R. Mesiar and A. Mesiarova, Residual implications and left-continuous t-norms which are ordinal sums of semigroups, *Fuzzy Sets and Systems* 143, 2004, pp. 47–57.

[19] J. Mishra and S. Ghosh, A New Functional Dependency in a Vague Relational Database Model, *International Journal of Computer Applications* 39, 2012, pp. 29–36.

[20] J. Mishra and S. Ghosh, A Vague Multivalued Data Dependency, *Fuzzy Inf. Eng.* 4, 2013, pp. 459–473.

[21] D. Pei, R_0 implication: characteristics and applications, *Fuzzy Sets and Systems* 131, 2002, pp. 297–302.

[22] Y. Shi, B. Van Gasse, D. Ruan and E. Kerre, On the first place antitonicity in ql-implications, *Fuzzy Sets and Systems* 159, 2008, pp. 2988–3013.

[23] Y. Shi, A Deep Study of Fuzzy Implications, Ph.D. dissertation, Faculty of Science, Ghent University, Ghent, 2009.

[24] M. Sozat and A. Yazici, A complete axiomatization for fuzzy functional and multivalued dependencies in fuzzy database relations, *Fuzzy Sets and System* 117, 2001, pp. 161–181.

[25] E. Trillas, C. Del Campo and S. Cubillo, When qm-operators are implication functions and conditional fuzzy relations, *Int. J. Intelligent Systems* 15, 2000, pp. 647–655.

[26] F. Zhao and Z.–M. Ma, Functional Dependencies in Vague Relational Databases, in: IEEE International Conference on Systems, Man, and Cybernetics, Taipei 2006, pp. 4006–4010.