Coupled transform for approximate-analytical solutions of a time-fractional one-factor Markovian bond pricing model

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Abstract: - In the theory of option pricing as regards financial mathematics, one-factor model represents a view that there exists one Wiener process in the definition of the short rate process indicating one source of randomness. In this paper, approximate-analytical solution of a time-fractional one-factor Markovian model for bond pricing is considered using a coupled technique referred to as Fractional Complex Transform (FCT) with the aid of modified differential transform method. The derivatives are defined in terms of Jumarie's sense. Illustrations are considered with a view to clarifying the effectiveness of the proposed solution method, and the solutions are presented graphically based on some financial parameters at different values of the time-fractional order. It is noted that the method requires little knowledge of fractional calculus while obtaining the approximate-analytical solutions of fractional equations without neglecting or compromising the associated accuracy. In terms of extension, the approach can be extended to multi-factor models formulated in terms of stochastic dynamics.

Key-Words: - Option pricing; Black Scholes model; RDTM; Fractional derivative; Analytical solutions; Markov process

1 Introduction

Interest rate simply denotes a certain amount mainly in monetary form charged for the use of financial property (such as money in most cases) [1]. Interest rates in financial market are non-tradable assets, instead, they are derived from the price values of tradable underlying assets like swaps as well as bonds. An interest rate model (IRM) expresses the evolution of the interest rates in connection with their dependency at maturity known as Term Structure of Interest Rates (TSIR) [2, 3]. Bond prices are mainly modeled via the application of short rate models [1, 4]. The short rate, ξ is claimed to be governed by the general Stochastic Differential Equation (SDE) of the form:

$$d\xi = \mu(\xi,t)\,dt + \sigma(\xi,t)\,dW \tag{1}$$

where $\mu(\xi,t)$ is a trend process, $\sigma(\xi,t)$ signifies random fluctuation determinant function, and $W = W(t)$ denotes a Wiener process. If $\mu(\xi,t) = \mu(\xi)$ and $\sigma(\xi,t) = \sigma(\xi)$; then (1) will give Markovian models; meaning that μ and σ are strictly functions of variable ξ (not depending on t at all). We speak of one-factor models or multi-factor models if $\xi = X$ in (1) is a scalar or a vector respectively [5]. This work will consider one-factor models (denoting the existence of one Wiener process-for one source of randomness).

The choice of the volatility function with respect to (w.r.t.) correctness has been a topic of concern. Chen, Karolyi, Longstaff, and Sanders proposed a general short rate model (SRM) defined in terms of single stochastic dynamics of the form:

$$d\xi = (\beta_1 + \beta_2\xi)\,dt + \sigma\xi^{\phi}\,dW \tag{2}$$

where β_1, β_2, ϕ are constants, and $W = W(t)$ is a Wiener process [6]. The model is referred to as CKLS model. In this regard, related references include those of [7-16]. Thus, by considering only corresponding Markov models, and taking $\tau = T - t$ (for convenient) to denote the remaining time to maturity, then the bond price, $p(\xi,\tau)$ solves the parabolic partial differential model evolving from (2) of the form:

$$\begin{cases} \dfrac{\partial p}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^{2\phi}\dfrac{\partial^2 p}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\dfrac{\partial p}{\partial \xi} - \xi p \\ p(\xi,0) = h(\xi) \end{cases} \tag{3}$$

for $p(\xi,\tau) = p,\ \xi > 0,\ \tau \in (0,T]$.

Similar PDEs such as (3) can be considered by semi-analytical, numerical, and approximate methods in terms of solutions [17-20]. In [21], Goard considered bond-pricing model w.r.t. group invariant solutions using classical Lie method approach. In computing the zero-coupon bonds, Sinkala *et al.* [22] applied symmetry analysis for the Vasicek and Cox-Ingersoll-Roll (CIR) models. Pooe *et al.* [23] transformed the one-factor bond pricing model to one-dimensional heat equation in order to obtain fundamental solutions of zero-coupon bond models. Recently, Khalique and Motsepa [24] analysed the one-factor term structure model; thereafter, they constructed new group invariant solutions to the corresponding equation.

In this paper, the extension of (3) to time-fractional order is considered. This takes the form:

$$\begin{cases} \dfrac{\partial^\alpha p}{\partial \tau^\alpha} = \dfrac{1}{2}\sigma^2 \xi^{2\phi} \dfrac{\partial^2 p}{\partial \xi^2} + \left(\beta_1 + \beta_2 \xi\right)\dfrac{\partial p}{\partial \xi} - \xi p, \\ p(\xi,0) = h(\xi), \; \alpha \in (0,1]. \end{cases} \quad (4)$$

The fractional derivative in (4) is defined in the sense of Jumarie. For related researches on fractional models and solution methods, we make reference to [25-32].

The remaining parts of the paper are structured as follows: we have in section 2: a brief note on the basic notions Fractional Derivative and its properties, section 3 is on the proposed solution method (FCT). In section 4, the proposed method is applied, thereafter, concluding remark is presented in section 5.

2 Fractional Derivative in the Sense of Jumarie

Reference is made to Jumarie's Fractional Derivative (JFD) as a modified version Riemann-Liouville derivatives [33-34]. As a result, the definition of JFD and its basic properties are presented as follows:

Let $\Psi(z)$ be considered as a continuous real valued function of z, and $D_z^\alpha \Psi = \dfrac{\partial^\alpha \Psi}{\partial z^\alpha}$ denoting JFD of h, of order α w.r.t. z. Then,

$$D_z^\alpha \Psi = \begin{cases} \left(\begin{array}{l} \dfrac{1}{\Gamma(-\alpha)} \dfrac{d}{dz} \displaystyle\int_0^z \left(z-\zeta\right)^{-\alpha-1} \Psi_*(\zeta) d\zeta, \\ \text{for } \alpha \in (-\infty, 0), \end{array} \right) \\ \left(\begin{array}{l} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dz} \displaystyle\int_0^z \left(z-\zeta\right)^{-\alpha} \Psi_{**}(\zeta) d\zeta, \\ \text{for } \alpha \in (0,1) \end{array} \right) \\ \left(\left(\Psi^{(\alpha-\phi)}(z) \right)^{(\phi)}, \; \alpha \in [\phi, \phi+1), \; \phi \ge 1 \right) \\ \Psi_*(\zeta) = \left(\Psi(\zeta) - \Psi(0) \right) \end{cases} \quad (5)$$

where $\Gamma(\cdot)$ represents a gamma function. The main features of JFD [34] as follows:

(i) $D_z^\alpha c = 0, \; \alpha > 0$, for a constant c

(ii) $D_z^\alpha \left(c\Psi(z) \right) = cD_z^\alpha \Psi(z), \; \alpha > 0$,

(iii) $D_z^\alpha z^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} z^{\beta-\alpha}, \; \beta \ge \alpha > 0,$

(iv) $D_z^\alpha \left(\Psi_1(z) \Psi_2(z) \right) = \begin{cases} D_z^\alpha \Psi_1(z) \left(\Psi_2(z) \right) \\ + \Psi_1(z) D_z^\alpha \Psi_2(z) \end{cases}$,

(v) $D_z^\alpha \left(\Psi\left(z(g) \right) \right) = D_z^1 \Psi \cdot D_g^\alpha z$.

The features (i)-(v) are fractional derivative of: constant function, constant multiple function, power function, product function, and function of function respectively. Though, (v) can be associated to Jumarie's chain rule in terms of fractional derivative.

3 The Reduced Differential Transform

Let $\omega(x,t)$ is an analytic and continuously differentiable function, defined on D a given domain, then the differential transformation form of $\omega(x,t)$ is defined and expressed as:

$$\Omega_k(x) = \frac{1}{k!} \left[\frac{\partial^k \omega(x,t)}{\partial t^k} \right]_{t=0} \quad (6)$$

where $\Omega_k(x)$ and $\omega(x,t)$ are referred to as the transformed and the original functions respectively. We make reference to [31, 35-38]. Thus, the differential inverse transform (DIT) of $\Omega_k(x)$ is defined and denoted as:

$$\omega(x,t) = \sum_{k=0}^{\infty} \Omega_k(x) t^k. \quad (7)$$

3.1 The fundamentals properties of the DTM

These properties are stated below in *D1-D5*:

D1: $$\begin{cases}\omega(x,t)=\alpha q(x,t)\pm\beta p(x,t)\\ \Rightarrow\Omega_k(x)=\alpha Q_k(x)\pm\beta P_k(x)\end{cases}.$$

D2: $$\begin{cases}\omega(x,t)=\dfrac{\alpha\partial^\eta h(x,t)}{\partial t^\eta},\ \eta\in\mathbb{N}\\ \Rightarrow\Omega_k(x)=\dfrac{\alpha(k+\eta)!}{k!}H_{k+\eta}(x).\end{cases}$$

D3: $$\begin{cases}\omega(x,t)=\dfrac{g(x)\partial^\eta h(x,t)}{\partial x^\eta},\ \eta\in\mathbb{N}\\ \Rightarrow\Omega_k(x)=\dfrac{g(x)\partial^\eta H_k(x)}{\partial x^\eta},\ \eta\in\mathbb{N}.\end{cases}$$

D4: $$\begin{cases}\omega(x,t)=p(x,t)q(x,t)\\ \Rightarrow\Omega_k(x)=\sum\limits_{\eta=0}^{k}P_\eta(x)Q_{k-\eta}(x).\end{cases}$$

D5: $$\begin{cases}\omega(x,t)=x^{n_1}t^{n_2}\\ \Rightarrow\Omega_c=x^{n_1}\delta(c-n_2),\delta(c)=\begin{cases}0,\ c\neq 0,\\ 1,\ c=0\end{cases}.\end{cases}$$

3.2 The Fractional Complex Transform [23, 28]

Suppose we consider a general fractional differential equation of the form:

$$h\left(\upsilon,D_t^\alpha\upsilon,D_x^\beta\upsilon,D_y^\lambda\upsilon,D_z^\gamma\upsilon\right)=0,\ \upsilon=\upsilon(t,x,y,z)\tag{8}$$

and define the Fractional Complex Transform (FCT) as follows:

$$T=\frac{at^\alpha}{\Gamma(1+\alpha)},\ \alpha\in(0,1],\tag{9}$$

where a is an unknown constant, then from (iii), we have:

$$D_v^\alpha z^\beta=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}z^{\beta-\alpha},\ \beta\geq\alpha>0,$$

$$\therefore\ D_t^\alpha T==\frac{a}{\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\alpha)}\right]t^{\alpha-\alpha}=a\tag{10}$$

Hence,

$$D_t^\alpha\upsilon=D_t^\alpha\upsilon\left(T(t)\right)=D_T^1\upsilon\cdot D_t^\alpha T=a\frac{\partial\upsilon}{\partial T}.\tag{11}$$

4 Applications and Illustrative Examples

In this section, the proposed FCT is applied to a time-fractional one-factor Markovian model (TF1FMM) for bond pricing as follows [3]:

$$\begin{cases}\dfrac{\partial^\alpha p}{\partial\tau^\alpha}=\dfrac{1}{2}\sigma^2\xi^{2\phi}\dfrac{\partial^2 p}{\partial\xi^2}+(\beta_1+\beta_2\xi)\dfrac{\partial p}{\partial\xi}-\xi p,\\ p(\xi,0)=1,\ \alpha\in(0,1].\end{cases}\tag{12}$$

Solution Steps:

By FCT, in section 3,

$$T=\frac{a\tau^\alpha}{\Gamma(1+\alpha)}\Rightarrow\frac{\partial^\alpha p}{\partial\tau^\alpha}=\frac{\partial p}{\partial T}$$

for $a=1$. Hence, (12) becomes:

$$\begin{cases}\dfrac{\partial p}{\partial T}=\dfrac{1}{2}\sigma^2\xi^{2\phi}\dfrac{\partial^2 p}{\partial\xi^2}+(\beta_1+\beta_2\xi)\dfrac{\partial p}{\partial\xi}-\xi p,\\ p(\xi,0)=1,\ \alpha\in(0,1].\end{cases}\tag{13}$$

By the RDTM in section 3, we have the recurrence relation from (4.3) as:

$$P_{k+1}=\frac{1}{(1+k)}\begin{pmatrix}\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,k}\\ +(\beta_1+\beta_2\xi)P'_{\xi,k}-\xi P_k\end{pmatrix},\ k\geq 0.\tag{14}$$

As a result, the recursive relation in (14) yields:

$$\begin{cases}P_0=P(\xi,0)\\ P_1=\left\{\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,0}+(\beta_1+\beta_2\xi)P'_{\xi,0}-\xi P_0\right\}\\ P_2=\dfrac{1}{2}\left\{\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,1}+(\beta_1+\beta_2\xi)P'_{\xi,1}-\xi P_1\right\}\\ P_3=\dfrac{1}{3}\left\{\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,2}+(\beta_1+\beta_2\xi)P'_{\xi,2}-\xi P_2\right\}\\ P_4=\dfrac{1}{4}\left\{\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,3}+(\beta_1+\beta_2\xi)P'_{\xi,3}-\xi P_3\right\}\\ \vdots\\ P_m=\dfrac{1}{m}\left\{\begin{matrix}\dfrac{1}{2}\sigma^2\xi^{2\phi}P''_{\xi,m-1}\\ +(\beta_1+\beta_2\xi)P'_{\xi,m-1}-\xi P_{m-1}\end{matrix}\right\},\ m\in\mathbb{N}.\end{cases}\tag{15}$$

Therefore, for $P(\xi,0)=1$, we obtained the following:

$$\begin{cases} P_0 = 1, \\ P_1 = -\xi, \\ P_2 = \left(-\frac{1}{2}\xi\beta_2 - \frac{1}{2}\beta_1 + \frac{1}{2}\xi^2 \right), \end{cases}$$

$$P_3 = \begin{pmatrix} \frac{1}{6}\sigma^2\xi - \frac{1}{6}\xi\beta_2^{\ 2} - \frac{1}{6}\beta_1\beta_2 \\ +\frac{1}{2}\xi^2\beta_2 + \frac{1}{2}\xi\beta_1 - \frac{1}{6}\xi^3 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} -\frac{1}{6}\sigma^2\xi^2 + \frac{1}{6}\sigma^2\xi\beta_2 + \frac{1}{24}\sigma^2\beta_1 - \frac{1}{24}\xi\beta_2^{\ 3} \\ -\frac{1}{24}\beta_1\beta_2^{\ 2} + \frac{7\xi^2\beta_2^{\ 2}}{24} + \frac{5\xi\beta_1\beta_2}{12} + \frac{1}{8}\beta_1^2 \\ -\frac{1}{4}\xi^3\beta_2 - \frac{1}{4}\xi^2\beta_1 + \frac{1}{24}\xi^4 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} -\frac{1}{30}\sigma^4\xi + \frac{1}{12}\sigma^2\xi^3 - \frac{\xi^5}{120} - \frac{1}{3}\xi^2\beta_1\beta_2 \\ -\frac{1}{8}\sigma^2\xi\beta_1 + \frac{1}{30}\sigma^2\beta_1\beta_2 + \frac{11\sigma^2\xi\beta_2^{\ 2}}{120} \\ +\frac{5\xi\beta_1\beta_2^{\ 2}}{24} - \frac{1}{4}\sigma^2\xi^2\beta_2 - \frac{\xi\beta_2^{\ 4}}{120} - \frac{\beta_1\beta_2^{\ 3}}{120} \\ +\frac{1}{8}\xi^2\beta_2^{\ 3} + \frac{1}{12}\beta_1^2\beta_2 - \frac{5\xi^3\beta_2^{\ 2}}{24} - \frac{1}{8}\xi\beta_1^2 \\ +\frac{1}{12}\xi^4\beta_2 + \frac{1}{12}\xi^3\beta_1 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} -\frac{119\sigma^2\xi\beta_1\beta_2}{720} + \frac{\xi^6}{720} - \frac{17\sigma^4\xi\beta_2}{360} + \frac{5\sigma^2\xi^2\beta_1}{48} \\ +\frac{13\sigma^2\xi\beta_2^{\ 3}}{360} + \frac{11\sigma^2\beta_1\beta_2^{\ 2}}{720} + \frac{7\xi\beta_1\beta_2^{\ 3}}{90} \\ +\frac{11\xi^3\beta_1\beta_2}{72} - \frac{1}{4}\xi^2\beta_1\beta_2^{\ 2} + \frac{1}{6}\sigma^2\xi^3\beta_2 - \frac{73\sigma^2\xi^2\beta_2^{\ 2}}{360} \\ -\frac{7\xi\beta_1^{\ 2}\beta_2}{48} - \frac{1}{48}\beta_1^{\ 3} - \frac{\sigma^4\beta_1}{180} - \frac{1}{48}\xi^5\beta_2 - \frac{1}{48}\xi^4\beta_1 \\ -\frac{1}{48}\sigma^2\beta_1^{\ 2} + \frac{5\beta_1^{\ 2}\beta_2^{\ 2}}{144} - \frac{\xi\beta_2^{\ 5}}{720} - \frac{\beta_1\beta_2^{\ 4}}{720} + \frac{31\xi^2\beta_2^{\ 4}}{720} \\ -\frac{1}{8}\xi^3\beta_2^{\ 3} + \frac{13\xi^4\beta_2^{\ 2}}{144} + \frac{1}{16}\xi^2\beta_1^{\ 2} + \frac{17\sigma^4\xi^2}{360} - \frac{1}{36}\sigma^2\xi^4 \end{pmatrix},$$

$$P_7 = \begin{pmatrix} -\frac{\xi^7}{5040} + \frac{137\sigma^2\xi^2\beta_1\beta_2}{720} - \frac{43\sigma^2\xi\beta_1\beta_2^{\ 2}}{360} \\ -\frac{5\sigma^2\xi^4\beta_2}{72} + \frac{17\xi\beta_1\beta_2^{\ 4}}{720} + \frac{11\xi^3\beta_1\beta_2^{\ 2}}{72} \\ +\frac{5\xi^2\beta_1^{\ 2}\beta_2}{48} - \frac{49\xi^2\beta_1\beta_2^{\ 3}}{360} - \frac{7\xi\beta_1^{\ 2}\beta_2^{\ 2}}{72} \\ -\frac{7\xi^4\beta_1\beta_2}{144} - \frac{17\sigma^4\beta_1\beta_2}{2520} - \frac{7\sigma^2\xi^3\beta_1}{144} \\ +\frac{1}{28}\sigma^2\xi\beta_1^{\ 2} - \frac{1}{28}\sigma^4\xi\beta_2^{\ 2} + \frac{8\sigma^2\xi^3\beta_2^{\ 2}}{45} \\ +\frac{7\sigma^4\xi\beta_1}{240} + \frac{11\sigma^4\xi^2\beta_2}{120} - \frac{17\sigma^2\beta_1^{\ 2}\beta_2}{720} \\ +\frac{19\sigma^2\xi\beta_2^{\ 4}}{1680} + \frac{13\sigma^2\beta_1\beta_2^{\ 3}}{2520} - \frac{7\sigma^2\xi^2\beta_2^{\ 3}}{60} \\ +\frac{\xi^6\beta_2}{240} + \frac{\xi^5\beta_1}{240} + \frac{\beta_1^{\ 2}\beta_2^{\ 3}}{90} - \frac{1}{48}\beta_1^{\ 3}\beta_2 \\ -\frac{1}{36}\xi^5\beta_2^{\ 2} - \frac{1}{48}\xi^3\beta_1^{\ 2} - \frac{\xi\beta_2^{\ 6}}{5040} - \frac{\beta_1\beta_2^{\ 5}}{5040} \\ +\frac{\xi^2\beta_2^{\ 5}}{80} - \frac{43\xi^3\beta_2^{\ 4}}{720} + \frac{5\xi^4\beta_2^{\ 3}}{72} + \frac{1}{48}\xi\beta_1^{\ 3} \\ +\frac{\sigma^2\xi^5}{144} + \frac{17\sigma^6\xi}{2520} - \frac{11\sigma^4\xi^3}{360} \end{pmatrix},$$

$$P_8=\left(\begin{array}{l}
\frac{43\sigma^4\xi\beta_1\beta_2}{840}-\frac{83\sigma^2\xi\beta_1\beta_2^3}{1344}-\frac{163\sigma^2\xi^3\beta_1\beta_2}{1440}\\
+\frac{179\sigma^2\xi^2\beta_1\beta_2^2}{960}+\frac{11\sigma^2\xi\beta_1^2\beta_2}{160}+\frac{\beta_1^4}{384}+\frac{\sigma^2\xi^4\beta_1}{64}\\
+\frac{\sigma^2\xi\beta_2^5}{336}-\frac{3\sigma^4\xi^3\beta_2}{40}-\frac{1}{32}\sigma^2\xi^2\beta_1^2+\frac{31\sigma^6\xi\beta_2}{2520}\\
-\frac{2\sigma^4\xi\beta_2^3}{105}+\frac{79\sigma^4\xi^2\beta_2^2}{840}+\frac{157\xi^3\beta_1\beta_2^3}{1440}\\
+\frac{55\xi^2\beta_1^2\beta_2^2}{576}+\frac{1}{32}\xi\beta_1^3\beta_2+\frac{17\xi^5\beta_1\beta_2}{1440}\\
+\frac{41\xi\beta_1\beta_2^5}{6720}-\frac{137\xi\beta_1^2\beta_2^3}{2880}-\frac{1}{30}\sigma^4\xi^2\beta_1\\
-\frac{11\sigma^2\xi^4\beta_2^2}{120}-\frac{89\sigma^2\xi^2\beta_2^4}{1680}-\frac{13\xi^3\beta_1^2\beta_2}{288}\\
+\frac{2}{15}\sigma^2\xi^3\beta_2^3+\frac{1}{48}\sigma^2\xi^5\beta_2-\frac{43\sigma^2\beta_1^2\beta_2^2}{2880}\\
-\frac{\sigma^4\beta_1\beta_2^2}{224}-\frac{35\xi^4\beta_1\beta_2^2}{576}-\frac{19\xi^2\beta_1\beta_2^4}{320}\\
+\frac{19\sigma^2\beta_1\beta_2^4}{13440}-\frac{\xi^7\beta_2}{1440}-\frac{\xi^6\beta_1}{1440}-\frac{\xi\beta_2^7}{40320}\\
-\frac{\beta_1\beta_2^6}{40320}+\frac{7\sigma^4\beta_1^2}{1920}+\frac{19\xi^6\beta_2^2}{2880}+\frac{\xi^4\beta_1^2}{192}\\
+\frac{\sigma^2\beta_1^3}{192}-\frac{7\beta_1^3\beta_2^2}{576}+\frac{127\xi^2\beta_2^6}{40320}+\frac{17\beta_1^2\beta_2^4}{5760}\\
-\frac{23\xi^3\beta_2^5}{960}+\frac{27\xi^4\beta_2^4}{640}-\frac{5\xi^5\beta_2^3}{192}-\frac{\xi^2\beta_1^3}{96}\\
+\frac{17\sigma^6\beta_1}{20160}+\frac{\xi^8}{40320}-\frac{\sigma^2\xi^6}{720}-\frac{31\sigma^6\xi^2}{2520}+\frac{\sigma^4\xi^4}{80}
\end{array}\right),$$

$\vdots$

Hence,

$$p(\xi,T)=\sum_{j=0}^{\infty}P_jT^j. \quad (16)$$

Therefore, (16) is the approximate-analytical solution of (13). In return, we have:

$$p(\xi,\tau)=\sum_{j=0}^{\infty}P_j\left(\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right)^j. \quad (17)$$

Equation (17) is an approximate-analytical solution of (12) corresponding to the time-fractional one-factor Markovian model for bond pricing. We consider the financial data according to [1] w.r.t CIR model with $\sigma=0.0894$, $\beta_1=0.00315$, and $\beta_2=-0.0555$. The plots of the resulting solutions for $\alpha=1$, $\alpha=0.5$, and $\alpha=0.75$ are displayed in Fig.1, Fig. 2, and Fig. 3 respectively.

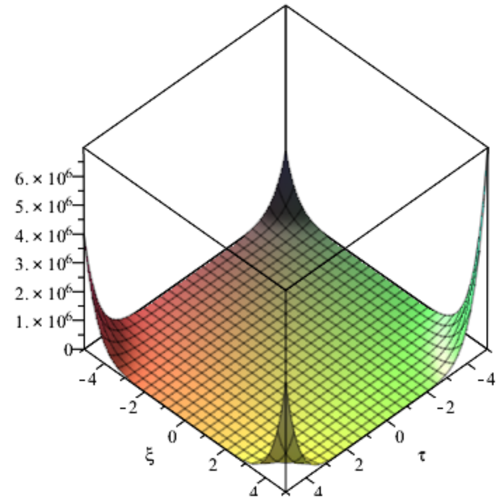

Fig 1: Solution graph for $\alpha=1$

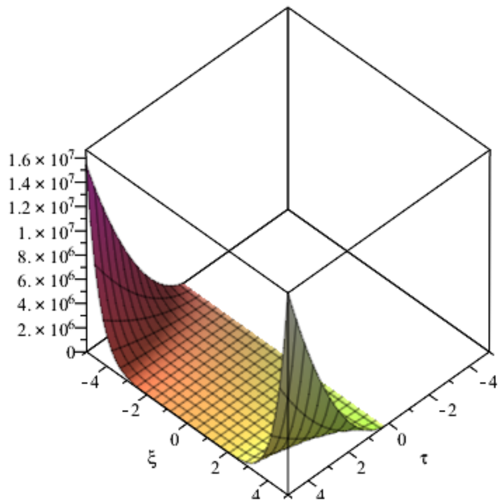

Fig 2: Solution graph for $\alpha=0.25$

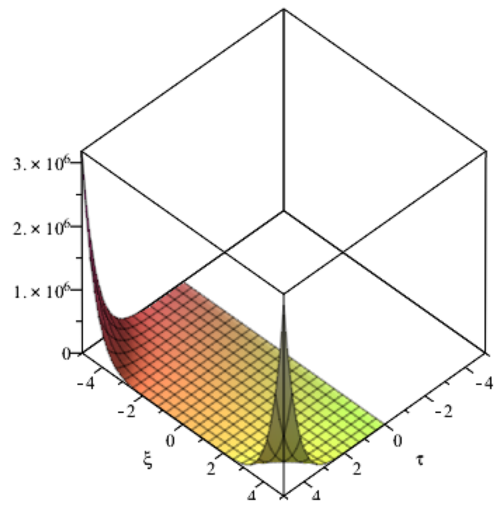

Fig 3: Solution graph for $\alpha=0.75$

5.0 Concluding Remarks

In this paper, a solution method where the Fractional Complex Transform is coupled with reduced differential transform method is proposed and applied for an approximate-analytical solution of a time-fractional one-factor Markovian model (TF1FMM) for bond pricing. The associated derivatives are defined in the sense of Jumarie. Preference is given to the CIR model version of TF1FMM with the market parameters according to Stehlikova [1] in order to clarify the effectiveness of the proposed solution method. Moreso, the solutions at different values of time-fractional order are graphically displayed. This proposed method does not necessarily require a complete knowledge of fractional calculus while computing the solutions of fractional models yet the level of accuracy is not being compromised. It is remarked that the solution obtained, and the proposed method can effectively serve as a benchmark to further researches in related areas via other semi-analytical methods.

Conflict of Interests

The authors declare no conflict of interest regarding this paper.

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