On the Construction of the Advanced Hybrid Methods and Application to Solving Volterra Integral Equation

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Abstract: - As is known there are some classes of the numerical methods for solving the Volterra integral equations. Each of them has the advantages and disadvantages. Therefore the scientists in often construct the methods for solving Volterra integral equations, having some advantages. Here for the construction of the methods with the best properties have used the advanced multistep and hybrid methods. Prove that, there are stable methods on the junction of these methods. And also, prove the existence of the stable methods with the degree $p > 3k+1$, and one of the constructed, here methods have applied to the solving of the model equation. And also described the way for finding the coefficients of the proposed method and also have defined the necessary condition for satisfying the coefficients of these methods.

Key-Words: - Volterra integral equation, hybrid methods, forward jumping methods

1 Introduction

The mathematical models for the many problems of natural science are described by the integral equations with the variable bounders. The first scientific work in this has been published by Abel (see [1, p.12]). And the integral equations with the variable boundaries have fundamentally investigated by V.Volterra (see [2, p.83-112]). For the finding the numerical solution of such equation, Volterra proposed to use the quadrature method, which successfully uses at present time.

Let us consider the following nonlinear Volterra integral equations:

$$y(x) = f(x) + \int_{x_0}^{x} K(x,s,y(s))ds,$$
$$x_0 \le s \le x \le X. \qquad (1)$$

Assume that the equation (1) has a unique solution, defined on the interval $[x_0, X]$. In order to find the numerical solution of the equation (1), the segment $[x_0, X]$ divided into N equal parts by using constant step size $h > 0$ and the mesh points is defined as the $x_{i+1} = x_i + h\,(i = 0,1,...,N-1)$ and is denoted approximate value of the solution of equation (1) at the point $x_i(0 \le i \le N)$ by $y_i(0 \le i \le N)$, and the exact value of the solution of equation (1) at the point $x_i(0 \le i \le N)$ by $y(x_i)(0 \le i \le N)$.

Remark that the basic disadvantages of the quadrature methods to consist in using the integral sum with the variable bounders. For the explanation this, let us consider following quadrature formula:

$$\int_{x_0}^{x} K(x_n, s, y(s))ds = \sum_{j=0}^{n} A_j K(x_n, x_i, y_i) + R_n(x).$$

Here $R_n(x)$ -is the remainder term. It follows that by the increasing the value of P also the volume of computational work is increasing. For correction these disadvantages of the quadrature method, scientists have constructed different methods for solving of equation (1) (see. for example [3]-[8]). But in this work in basically have used the similar way to quadrature method. Let us noted that in [9] have constructed the method, which is freed from this disadvantage of the quadrature method. By using the disadvantages of that method some authors have constructed the methods with the higher order of exactness which are freed of the above-mentioned disadvantages of quadrature methods and belong to the class of multistep, hybrid, advanced and others type of methods.

Here, by using some combination of these methods we have constructed the stable methods, with the higher order of exactness.

2 Construction of the forward jumping methods

The advanced multistep method was suggested by Cowell (see [10]) for solving initial-value problem for ODE. Such methods were constructed by the famous scientists as the Laplace and Steklov (see [11]). The advantage of this method was proved in [12], and in [13] the predictor-corrector method was constructed for their realization. The authors of using the advanced multistep method gave the next explanation. As is known, while calculating y_i- approximate value of the solution to equation (1) by the multistep method, information on the function $y(x)$ at the previous points are used. But for using the advanced multistep method there arises necessity to use the values of the function $y(x)$ not only at the previous but also at the next points. In this connection, advanced the multistep method is proposed for solving the equation of (1) can be written as follows:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = \sum_{i=0}^{k-m}\alpha_i f_{n+i} + + h\sum_{i=0}^{k}\sum_{j=i}^{k}\beta_i^{(j)}K(x_{n+j}, x_{n+i}, y_{n+i}), \tag{2}$$

Here the coefficients $\alpha_i, \beta_i^{(j)} (i, j = 0,1,...,k)$ are real numbers moreover $\alpha_k \neq 0$.

If in this formula put $k = 1$, then receive the one-step method. As is known the one-step and multistep methods have the advantages and disadvantages. The general one-step method is the Runge-Kutta method, which fundamentally investigated by many authors (see for example [14]-[15]).

For the meeting with the application of Runge-Kutta methods to solving Volterra integral equations, one can use some source (see for example [16], [17]). For the construction of the numerical methods with the best properties to solving initial value problem for the ODE, Butcher and Gear (see [18], [19]) have constructed the new class of methods, which have called the hybrid methods. These methods have developed by some scientists (see [22]-[26]).

By using the advantages of advanced multistep and hybrid methods, here consider to construction the methods on the joined of these methods and the application of them to solving Volterra integral equations. For this aim let us use the next methods:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} \ (m > 0), \tag{3}$$

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\gamma_i y'_{n+i+\nu_i} \quad (|\nu_i| < 1; i = 0,1,...,k). \tag{4}$$

The method (3) is the advanced multistep method and in this class of methods, there are the stable methods with the degree of $p = k + m + 1 (k + m \le 2k - m + 1)$ but in the class of methods of type (4), there are the stable methods with the degree $p = k + 2$. Let us construct the stable method by using the following linear presentation of the methods (3) and (4):

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h\sum_{i=0}^{k}\gamma_i y'_{n+i+\nu_i} \quad (|\nu_i| < 1; i = 0,1,...,k). \tag{5}$$

This method for the case $m = 0$ has investigated (see [23], [27]). Let us note that application of the method (5) is not simple. For the application of the method (5) to solving equation (1) in the case of $m > 0$, we must have some additional information about the solution of the equation (1). Let us consider the application of the method (5) to solving equation (1). Then receive the following advanced multistep method:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = \sum_{i=0}^{k-m}\alpha_i f_{n+i} + + h\sum_{i=0}^{k}\sum_{j=i}^{k}(\beta_i^{(j)}K(x_{n+j}, x_{n+i}, y_{n+i}) + + \gamma_i^{(j)}K(x_{n+j}, x_{n+i+\nu_i}, y_{n+i+\nu_i})), \quad (|\nu_i| < 1; i = 0,1,...,k), m > 0, \tag{6}$$

here $f_l = f(x_l), l \ge 0$.

Now let us consider, how one can apply the methods of type (6) to the solving the equation (1). For this aim let us consider the following equality:

$$y(x_m) = f(x_m) + \int_{x_0}^{x_m} K(x_m, s, y(s))ds. \tag{7}$$

It is clear that the exactness of calculation of the value $y(x_m)$ depends from the degree of the method, which has applied to the calculation of the integral and independent from the value of the variable x.Therefore the kernel $K(x, s, y)$ can be approached by any interpolation polynomials on the interval $[x_0, x_m]$ .It follows, that we can present the kernel as the following:

$$K(x,s,y) = \sum_{i=0}^{k} l_i(x) K(x_{m-i}, s, y) + \\ + R_m(x), x \in [x_0, x_m],$$

here x-is a fixed point, $l_i(x)(i = 0,1,...,k)$ are the Lagrange interpolation basic functions, but R_m- remainder term.

By taking this into account in the equality of (7) for the fixed value of x receive the following:

$$y(x) = f(x) + \sum_{i=0}^{k} l_i(x) \int_{x_0}^{x} K(x_{m-i}, s, y(s)) ds + \\ + R_m(x), x \geq x_m. \tag{8}$$

By differentiable this equality, receive that (see for example [28, p. 158]):

$$y' = f'(x) + \sum_{i=0}^{k} l_i'(x) K(x_{m-i}, s, y(x)) + \\ + R_m'(x), y(x_0) = f(x_0). \tag{9}$$

By this way, we replay to determine the solution of the Volterra integral equations by the determined of the solution of the initial value problem for ODE. But it does not follow from here, that the Volterra integral equations and the initial value problem for ODE are equivalent.

For the application of the method (6) to solving the equation (1) let us consider the following method:

$$\sum_{i=0}^{k-m} \alpha_i (y_{n+i} - f_{n+i}) = h \sum_{i=0}^{k} \beta_i (y_{n+i} - f_{n+i})' + \\ + h \sum_{i=0}^{k} \gamma_i (y_{n+i+\nu_i} - f_{n+i+\nu_i})' (|\nu_i| < 1; i = 0,1,...,k). \tag{10}$$

If we suppose that the order of the remainder term of $R_m(x)$ satisfies the condition $R_m(x) = O(h^{p+1})$, then after discarding the term of $R_m'(x)$ in the equality (9), receive the following:

$$y' = f'(x) + \sum_{j=0}^{k} l_j'(x) K(x_{m-j}, x, y(x)), \\ y(x_{m-k}) = y_{m-k}, x \in [x_{m-k}, x_m].$$

After applying of the method (10) to solving this problem, we have:

$$\sum_{i=0}^{k-m} \alpha_i (y_{n+i} - f_{n+i}) = \\ h \sum_{i=0}^{k} \sum_{j=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + \\ + h \sum_{i=0}^{k} \sum_{j=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+i+\nu_i}, y_{n+i+\nu_i}) \\ (|\nu_i| < 1; i = 0,1,...,k).$$

This method coincides with the method (6). As is known, one of the basic problems in the modern computational mathematics is the construction of the methods with the higher order of exactness. Above we have described two directions for the construction of the stable methods with a higher degree, and also have constructed the stable methods on the intersection of this direction. It is not difficult to understand that all properties of the numerical methods are depended on the values of its coefficients. Therefore, let us consider to define the values of the coefficients of the method (6) and for this aim to use special case, when $K(x,s,y) = \varphi(s,y)$. If applied the method (6) to the solving of the problem (11), then receive that

$$y(x) = f(x) + \int_{x_0}^{x} \varphi(s, y(s)) ds. \tag{11}$$

If applied the method (6) to solving the problem (11), then receive that

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = \sum_{i=0}^{k-m} \alpha_i f_{n+i} + \\ h \sum_{i=0}^{k} (\beta_i \varphi_{n+i} + \gamma_i \varphi_{n+i+\nu_i}), (|\nu_i| < 1; i = 0,1,...,k), \tag{12}$$

here $\varphi_m = \varphi(x_m, y_m)(m = 0,1,...)$ and $m > 0$.

The relation between the methods (6) and (12) can be determined by the following:

$$\sum_{i=0}^{k} \beta_i^{(j)} = \beta_i; \sum_{i=0}^{k} \gamma_i^{(j)} = \gamma_i \ (i = 0,1,...,k). \tag{13}$$

If the values $\beta_i, \gamma_i (i = 0,1,...,k)$ are known, then by using the solution of the system (13), one can be determined the values of the variable $\beta_i^{(j)}$ and $\gamma_i^{(j)} (i, j = 0,1,...,k)$. Remark, that the solution of this system more than one. It is not difficult to understand that the order of the exactness of the methods received from the formula (6) depends from the values of the variables $\beta_i, \gamma_i (i = 0,1,...,k)$. Therefore, to define the maximal value of the degree for the methods of type (6), let us determine any

relation between the order k and the degree p for the methods of type (6).
If using the following

$$z(x) = y(x) - f(x)$$

in the method of (6), then receive:

$$\sum_{i=0}^{k-m} \alpha_i z_{n+i} = h\sum_{i=0}^{k} \beta_i z'_{n+i} + + h\sum_{i=0}^{k} \gamma_i z'_{n+i+\nu_i}, (|\nu_i| < 1, i = 0,1,...,k). \tag{14}$$

Assume that the method (14) has the degree of p, then receive the following is hold:

$$\sum_{i=0}^{k-m} \alpha_i z(x+ih) - h\sum_{i=0}^{k} (\beta_i z'(x+ih) - \gamma_i z'(x+(i+\nu_i)h)) = O(h)^{p+1}, \quad h \to 0 (m > 0) \tag{15}$$

By using the next Teylor series

$$z(x+ih) = z(x) + ihz'(x) + ... + + \frac{(ih)^p}{p!} z^{(p)}(x) + O(h)^{p+1},$$

$$z'(x+ih) = z'(x) + ihz''(x) + ... + + \frac{(ih)^{p-1}}{(p-1)!} z^{(p)}(x) + O(h)^p.$$

in the equality of (14) receive:

$$\sum_{i=0}^{k-m} \alpha_i z(x) + h\sum_{i=0}^{k-m} i\alpha_i z'(x) - h\sum_{i=0}^{k} (\beta_i + \gamma_i) z'(x) + + h^2 \sum_{i=0}^{k-m} \frac{i^2}{2} \alpha_i z''(x) - h^2 \sum_{i=0}^{k} (i\beta_i + (i+\nu_i)\gamma_i) z''(x) + ... + + h^p \sum_{i=0}^{k-m} \frac{i^p}{p!} \alpha_i z^{(p)}(x) - - h^p \sum_{i=0}^{k} \left(\frac{i^{p-1}}{(p-1)!} \beta_i + \frac{(i+\nu_i)^{p-1}}{(p-1)!} \gamma_i \right) z^{(p)}(x) = O(h^{p+1}), \quad h \to 0. \tag{16}$$

If take into account that the method (14) has the degree of p, then by the compares the asymptotic equalities (16) and (15) obtain, that the following is holds:

$$\sum_{i=0}^{k-m} \alpha_i z(x) + h\sum_{i=0}^{k-m} i\alpha_i z'(x) - h\sum_{i=0}^{k} (\beta_i + \gamma_i) z'(x) - + h^2 \sum_{i=0}^{k-m} \frac{i^2}{2} \alpha_i - h^2 \sum_{i=0}^{k} (i\beta_i + (i+\nu_i)\gamma_i) z''(x) + ... + h^p \sum_{i=0}^{k-m} \frac{i^p}{p!} \alpha_i - h^p \sum_{i=0}^{k} (\frac{i^{p-1}}{(p-1)!} \beta_i - \frac{(i+\nu_i)^{p-1}}{(p-1)!} \gamma_i) z^{(p)}(x) = 0. \tag{17}$$

It is known that the $1, x, x^2, ..., x^p$ is a linearly independent system, therefore, equality of (17) is equivalent to the following:

$$\sum_{i=0}^{k-m} \alpha_i = 0, \ \sum_{i=0}^{k} (\beta_i + \gamma_i) = \sum_{i=0}^{k-m} i\alpha_i, ..., \sum_{i=0}^{k} \left(\frac{i^{p-1}}{(p-1)!} \beta_i - \frac{(i+\nu_i)^{p-1}}{(p-1)!} \gamma_i \right) = \sum_{i=0}^{k-m} \frac{i^p}{p!} \alpha_i. \tag{18}$$

Thus, for construction of the methods of type (6) receive the system of nonlinear algebraic equations.

3 The bounders imposed on the coefficients of the advanced hybrid method

As is known in the application of the method (6) to solving some problems there arises the necessity to impose any bounders on the coefficients of this method. For example, define the necessary condition for convergence of the method (6). Therefore, let us consider the determination of the conditions, which satisfy the coefficients of the method (6) if it is convergence.

For the investigation of the bounders which are imposed on the coefficients of the method (6), let us consider the special case and put $K(x,s,y) = \varphi(s,y)$. Taking into account that the basic properties of the method (6) are defined by the values of the variable $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,2,...,k)$, suppose that these coefficients satisfy the following conditions:

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,2,...,k)$ of the method (14) are some real numbers, moreover, $\alpha_k \neq 0$.

B: Characteristic polynomials of the method (14)

$$\rho(\lambda) \equiv \sum_{i=0}^{k-m} \alpha_i \lambda^i,$$

$$\sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i;$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}.$$

have no common factor different from the constant.
C: $\sigma(1)+\gamma(1)\neq 0$ and $p\geq 1$.
The condition A is obvious, therefore let us prove the condition B. For this aim assume the contrary and denote by $\psi(\lambda)$ the common factor of these polynomials. Then by means of the shift operator
$E^l z(x)=z(x+lh)$,
we rewrite method (14) in the form:

$$\rho(E)z_n-h\sigma(E)z'_n-h\gamma(E)z'_n=0. \quad (19)$$

It is known from the theory of finite-difference equations that for the existence of the unique solution of the equation (19) should be known the set of initial values $z_0,z_1,...,z_{k-1}$ .By using the above describe assumption, the equation of (19) can be rewritten in the following form:

$$\psi(E)(\rho_1(E)z_n-h\sigma_1(E)z'_n-h\gamma_1(E)z'_n)=0 \quad (20)$$

where

$$\rho_1(\lambda)=\rho(\lambda)/\psi(\lambda),$$
$$\sigma_1(\lambda)=\sigma(\lambda)/\psi(\lambda),\ \gamma_1(\lambda)=\gamma(\lambda)/\psi(\lambda).$$

Hence it follows that

$$\rho_1(E)z_n+h\sigma_1(E)z'_n-h\gamma_1(E)z'_n=0, \quad (21)$$

since $\psi(\lambda)\neq const$.
Obviously, the order of difference equation (21) is at most $k-1$. Therefore, taking into account the equivalence of equation (19) and (21) we get that the equation (14) has a unique solution if the $k-1$ initial conditions are known that contradict the general theory of difference equations. Consequently, the condition B holds. Let us in the equality of (19) pass to limit for the $h\to 0$. In this case, receive that $\rho(1)z(x)=0$. Here x-is a fixed point. From here receive that $\rho(1)=0$, which is the necessary condition for the convergence of method (19).
Now prove that if the method (14) converges, then the condition C holds. Using the shift operator and $\rho(1)=0$ the necessary condition of convergence of the multistep method, we can rewrite relation (19) in the form

$$\rho_1(E)(z_i-z_{i-1})-h\sigma(E)z'_i-h\gamma(E)z'_i=0, \quad (\rho_1(\lambda)=\rho(\lambda)/(\lambda-1)). \quad (22)$$

Here we change the parameter i from 1 to n and sum the obtained equalities, then get:

$$\rho_1(E)(z_n-z_0)-\sigma(E)h\sum_{j=1}^{n}z'_j-\gamma(E)h\sum_{j=1}^{n}z'_j=0. \quad (23)$$

Passing to limit as $h\to 0$, we have:

$$\rho_1(1)(z(x)-z_0)=(v(1)+\gamma(1))\int_{x_0}^{x}\varphi(s,y(s))ds,$$

here $x=x_0+nh$ is a fixed point.
From here we get

$$\rho_1(1)=v(1)+\gamma(1). \quad (24)$$

By using the following equalities
$\rho_1(\lambda)=\rho(\lambda)/(\lambda-1)$ and $\lim_{\lambda\to 1}\rho_1(\lambda)=\rho'(1)$
in the equality of (24) receive:

$$\rho'(1)=v(1)+\gamma(1).$$

Thus we obtain that if the method of (6) is convergent, then $\rho(1)=0$ and $\rho'(1)=v(1)+\gamma(1)$, coincide with the first two equations of the system (18) from where it follows that $p\geq 1$. Note that if $v(1)+\gamma(1)=0$, then we get, $\rho'(1)=0$. Hence it follows that method (14) is not stable. But it is known that the stability is a necessary and sufficient condition for the convergence of the method (14). Consequently, the condition C holds. Noted that the number of the equation in the system of (18) is equal to $p+1$, but the number of unknowns is equal to $4k+4$. It is not difficult to prove, that there exist the methods with the degree $p=4k+2$ and the stable methods with the degree $p=3k+2$. In special cases one can construct the stable methods of type (6), having the degree $p>3k+2$.
And now to consider the construction of the stable methods of type (14) having a high degree. To this end, put $m=1$ and $k=2$. Then, to determine the values of variables $\alpha_i,\beta_i,\gamma_i,v_i\ (i=0,1,...,k)$, we obtain the following system nonlinear algebraic equations:

$$\begin{gathered}\beta_2+\beta_1+\beta_0+\gamma_2+\gamma_1+\gamma_0=1,\\ 2^j\beta_2+\beta_1+l_2^j\gamma_2+l_1^j\gamma_1+l_0^j\gamma_0=1/(j+1)\\ (j=1,...,7).\end{gathered} \quad (25)$$

By using the solution of the system (25) one can be constructed stable methods with the degree $p\leq 8$. If in the system (25) we use the values $l_2=1+\alpha,\ l_1=1,\quad l_0=1-\alpha\ (|\alpha|<1)$ then receive the following solution:
$\beta_2=-1/180;\beta_1=6/90;\beta_0=29/180;\ l_0=1/2,$
$\gamma_2=1/45,\quad \gamma_1=6/90;\quad \gamma_0=31/45;\quad l_2=3/2.$
In this case the stable method can be written in the following form:

$$\begin{aligned}y_{n+1}=y_n&+h(29y'_n+24y'_{n+1}-y'_{n+2})/180+\\&+h(62y'_{n+1/2}+2y'_{n+3/2})/90.\end{aligned} \quad (26)$$

This method is stable and has the degree $p = 5$.
In [12, p. 277] is proved that if the method (14) for $m = 1$ is stable and has a maximum degree, then the coefficients β_k and β_{k-1} satisfies the following conditions $\beta_{k-1} > 0$, $\beta_{k-1} \cdot \beta_k < 0$ and $|\beta_{k-1}| > |\beta_k|$. As is known, if the method (14) for sufficiently smooth solution of the original problem has the degree of p, then it is local error can be written in the following form:

$$C_1 h^{p+1} y^{p+1} + C_2 h^{p+2} y^{n+2} + ... + O(h^{p+s}),\ h \to 0.$$

Usually, when increases the value of p, the value of the coefficient C_1 is decreases. These parameters for the method of (26), are defined as $p = 5$ and $C_1 = -1/720$. For the finding of the coefficients $\alpha_i, \beta_i, \gamma_i$ $(i = 0,1,2,...,k)$ and variables ν_i $(i = 0,1,...,k)$ one can be use the some standard program. If p is sufficiently large (in the redistribution of p = 10), then the values of the constant C_1 may be less than the machine zero. In this case values for the degree of the method (14) can be extended. Therefore, in such cases, it is desirable to compliance with caution.
Now consider the application of the method (26) to solving of the equation (1). Then we receive:

$$\begin{aligned} y_{n+1} = y_n + f_{n+1} - f_n + h(10K(x_n, x_n, y_n) + \\ +10K(x_{n+1}, x_n, y_n) + 9K(x_{n+2}, x_n, y_n) + \\ +6K(x_{n+1}, x_{n+1}, y_{n+1}) + 6K(x_{n+2}, x_{n+1}, y_{n+1}) - \\ -K(x_{n+2}, x_{n+2}, y_{n+2})/180 + h(31K(x_{n+1}, x_{n+1/2}, y_{n+1/2}) + \\ +31K(x_{n+1/2}, x_{n+1/2}, y_{n+1/2}) + 6K(x_{n+1}, x_{n+1}, y_{n+1}) + \\ +K(x_{n+2}, x_{n+3/2}, y_{n+3/2}) + K(x_{n+3/2}, x_{n+3/2}, y_{n+3/2}))/90. \end{aligned} \tag{27}$$

In the work [8] have investigated the comparison of the method (26) with the following hybrid method by using the solutions of some model equations:

$$y_{n+1} = y_n + h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha})\ (\alpha = \sqrt{3}/6).$$

For the illustration of the received here results let us consider the finding of the numerical solution of the following equations by using above-mentioned method:

1. $y(x) = 1 + x^2/2 + \int_0^x y(s)ds,\quad x \le 1,$ the exact solution is $y(x) = 2e^x - x - 1$.

Table 1. Comparison of errors for different values of step size

Variable x	Step size $h = 0.01$	$h = 0.05$	$h = 0.1$
0.10	2.59E-9	3.04E-7	2.25E-6
0.30	1.17E-8	1.38E-6	1.03E-5
0.50	2.78E-8	3.30E-6	2.48E-5
0.70	5.34E-8	6.38E-6	4.82E-5
1.00	1.17E-7	1.41E-5	1.07E-4

4 Conclusion

As is known the hybrid methods and multistep forward-jumping methods have some advantages, therefore the construction of the methods on the intersection of these methods are promising. Therefore here for solving of the nonlinear Volterra integral equation here proposed and investigated the hybrid multistep method of advanced (forward-jumping) type. To application of the received results to solving some practical problems, we have determined the conditions imposed on the coefficients of proposed methods. And also have shown the advantages of the proposed methods. Remark that the result received in the solving of the model equation agrees with the theoretical. We are considered proposed here method is promising.

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