The Interpolation Method for Calculating Eigenvalues of Matrices

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Abstract: - The problem of solving systems of linear algebraic equations (SLAEs) is connected with finding the eigenvalues of the matrix of the system. Often it is necessary to solve SLAEs with positive definite symmetric matrices. The eigenvalues of such matrices are real and positive. Here we propose an interpolation method for finding eigenvalues of such matrices. The proposed method can also be used to calculate the real eigenvalues of an arbitrary matrix with real elements. This method uses splines of Lagrangian type of fifth order and/or polynomial integro-differential splines of fifth order. To calculate the eigenvalue, it is necessary to calculate several determinants and solve the nonlinear equation. Examples of numerical experiments are given.

Key-Words: - eigenvalue problem, integro-differential splines, approximation

1 Introduction

In the process of solving various physical problems, it is necessary to solve systems of linear algebraic equations with positive definite symmetric matrices. Systems of linear algebraic equations (SLAEs) with positive definite symmetric matrices arise quite often, for example, during the solution of ordinary differential equations by the Ritz method, which leads to Gram matrices.

An important problem related to solving SLAEs is the calculation of the eigenvalues of a matrix. Using some methods one can find all the eigenvalues of the matrix (as with Krylov's method, see [1]), using other methods one can find the largest eigenvalue in its absolute value (as with the power iteration method, see [1]). For the initial localization of eigenvalues, one can use Gershgorin's theorem or use a norm of the matrix to be consistent and subordinate to the vector norm. The theory of finding eigenvalues is constantly being improved [2]-[4].

Here we propose an interpolation method for finding the real eigenvalues of a matrix with real elements, based on the use of polynomial splines of Lagrangian type and polynomial integro-differential splines [5]-[9]. This method can be used to calculate the eigenvalues of symmetric positive definite matrices, and for obtaining real eigenvalues of matrices with real elements. The main features of integro-differential splines are the following: the approximation is constructed separately for each grid interval, the approximation constructed as the sum of products of the basic splines and the values of function in nodes and/or the values of its derivatives and/or the values of integrals of this function over subintervals. Basic splines are determined by using a solving system of equations which are provided by the set of functions. It is known that when integrals of the function over the intervals are equal to the integrals of the approximation of the function over the intervals then the approximation has some physical parallel.

Here we construct polynomial integro-differential splines of fifth order. In this paper we discuss the construction of polynomial integro-differential splines which use integrals over subintervals in addition to the values of the function in the nodes. To calculate the eigenvalue, it is necessary to calculate five determinants and solve the nonlinear equation.

2 About Calculation Eigenvalues

Often it is very important to know all eigenvalues or the largest eigenvalue of a matrix. For solving this problem it is possible to use different types of splines [10]-[12]. For the calculation of the eigenvalues we can recommend a modification of the interpolation method (the idea of an interpolation method was suggested in 1960 by D.K.Faddeyev). The interpolation polynomial in the Lagrange form is well known. Here we suggest the following modification of the interpolation method for obtaining the eigenvalues of a positive defined symmetric matrix (or a near symmetrical positive defined matrix) using splines of Lagrange type.

Let matrix $\mathrm{H}_n = (h_{ij})_{i,j=1}^{n}$ of the SLAE $\mathrm{H}_n z=f$ be symmetric and positive definite (for example, the Hilbert matrix, i.e. the matrix where $h_{ij} = 1/(i+j-1)$). For example, we calculate the eigenvalues for positive definite non symmetric matrices that can be constructed in this way. Let us put $n = 20$, replace element $h_{20,1} = 1/20$ of the matrix H_{20} with the element $\tilde{h}_{20,1} = 1/(20.1)$ and replace element $h_{1,20} = 1/20$ of the matrix H_{20} with the element $\tilde{h}_{1,20} = 1/(19.9)$. Now we obtain matrix $\widetilde{H}_{20}$ in which all elements except $\tilde{h}_{20,1}$ and $\tilde{h}_{1,20}$ coincide with the elements of the matrix H_{20}. Matrix $\widetilde{H}_{20}$ is not symmetric, $\|\widetilde{H}_{20} - \mathrm{H}_{20}\| \leq 0.000251 = \delta$ (using Euclidean norm).

Let us calculate the largest eigenvalue of $\widetilde{H}_{20}$. It is not difficult to verify that all eigenvalues of $\widetilde{H}_{20}$ are positive. The calculation of the value of determinant is rather complicated, so we should do it as seldom as possible. At first we construct the function $F(t) = \widetilde{H}_{20} - tE_{20}$. Using Gershgorin's theorem we can calculate domains $D_k = [a_k, b_k]$, where there are eigenvalues μ_i. Let $\{t_j\}$ be a set of nodes, $t_j \in D_k$. On every $[t_i, t_{i+1}]$ we construct polynomial $P_4(t) = \sum_j \det\left(F(t_j)\right) w_j(t)$, where $w_j(t)$ are local basic functions, that are obtained from system of equations $P_4(t) = P(t)$, $P(t) = t^s, s = 0,1,2,3,4$. Such local interpolation splines have been studied in [5]-[9].

Further we denote by $\|g\|_{[a,b]} = \max_{[a,b]} |g(x)|$. If $\operatorname{supp} w_j$ consists of 5 intervals and $h = \max_i h_i$, $h_i = t_{i+1} - t_i$, we have

$$\|P_4 - F\| \leq Kh^5 \|F^{(5)}\|_{[a_k,b_k]}, \quad F \in C^5(D_k), \; K > 0.$$

Here $F(t)$ is the characteristic polynomial. On every interval $[t_i, t_{i+1}]$ we can construct the interpolation polynomial in one of the following relations. We denote them as $P_4^k(t)$.

2.1 The first option

Approximation on the rightmost interval may be obtained with the following approximation:

$$P_4^1(t) = \sum_{j=i-4}^{j=i} \det\left(F(t_j)\right) w_j(t), \; t \in [t_i, t_{i+1}] \quad (1)$$

where

$$w_i(t) = \frac{(t-t_{i-4})(t-t_{i-3})(t-t_{i-2})(t-t_{i-1})}{(t_i-t_{i-4})(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})},$$

$$w_{i-1}(t) = \frac{(t-t_{i-4})(t-t_{i-3})(t-t_{i-2})(t-t_i)}{(t_{i-1}-t_{i-4})(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)},$$

$$w_{i-2}(t) = \frac{(t-t_{i-4})(t-t_{i-3})(t-t_{i-1})(t-t_i)}{(t_{i-2}-t_{i-4})(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)},$$

$$w_{i-3}(t) = \frac{(t-t_{i-4})(t-t_{i-2})(t-t_{i-1})(t-t_i)}{(t_{i-3}-t_{i-4})(t_{i-3}-t_{i-2})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)},$$

$$w_{i-4}(t) = \frac{(t-t_{i-3})(t-t_{i-2})(t-t_{i-1})(t-t_i)}{(t_{i-4}-t_{i-3})(t_{i-4}-t_{i-2})(t_{i-4}-t_{i-1})(t_{i-4}-t_i)}.$$

The error of approximation we receive from remainder term of the Lagrange interpolation. For $t \in [t_i, t_{i+1}]$ using (1) we obtain

$$\|P_4^1 - F\|_{[t_i,t_{i+1}]} \leq Kh^5/5! \; \|F^{(5)}\|_{[t_{i-4},t_{i+1}]}, \; K = 120.$$

2.2 The second option

Approximation on the interval to the right of the rightmost interval may be obtained with the following approximation:

$$P_4^2(t) = \sum_{j=i-3}^{j=i+1} \det\left(F(t_j)\right) w_j(t), \; t \in [t_i, t_{i+1}], \quad (2)$$

where

$$w_{i-3}(t) = \frac{(t-t_{i-2})(t-t_{i-1})(t-t_i)(t-t_{i+1})}{(t_{i-3}-t_{i-2})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)(t_{i-3}-t_{i+1})},$$

$$w_{i-2}(t) = \frac{(t-t_{i-3})(t-t_{i-1})(t-t_i)(t-t_{i+1})}{(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})},$$

$$w_{i-1}(t) = \frac{(t-t_{i-3})(t-t_{i-2})(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})}$$

$$w_i(t) = \frac{(t-t_{i-3})(t-t_{i-2})(t-t_{i-1})(t-t_{i+1})}{(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})},$$

$$w_{i+1}(t) = \frac{(t-t_{i-3})(t-t_{i-2})(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-3})(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}.$$

The error of approximation we receive from the remainder term of the Lagrange interpolation. For $t \in [t_i, t_{i+1}]$ using (2) we obtain:

$$\|P_4^2 - F\| \leq K\, h^5/5!\ \|F^{(5)}\|_{[t_{i-3},t_{i+1}]}, K = 3.632.$$

2.3 Next option

On the interval to the right of the previous one, we can apply the approximation

$$P_4^3(t) = \sum_{j=i-2}^{j=i+2} \det\left(F(t_j)\right) w_j(t),\ t \in [t_i, t_{i+1}], \quad (3)$$

$$w_{i-2}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})(t_{i-2}-t_{i+2})},$$

$$w_{i-1}(t) = \frac{(t-t_{i-2})(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})},$$

$$w_i(t) = \frac{(t-t_{i-2})(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})}{(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})},$$

$$w_{i+1}(t) = \frac{(t-t_{i-2})(t-t_{i-1})(t-t_i)(t-t_{i+2})}{(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})},$$

$$w_{i+2}(t) = \frac{(t-t_{i-2})(t-t_{i-1})(t-t_i)(t-t_{i+1})}{(t_{i+2}-t_{i-2})(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}.$$

The error of approximation we receive from the remainder term of the Lagrange interpolation:

$$\|P_4^3 - F\| \leq K\, \|F^{(5)}\|_{[t_{i-2},t_{i+2}]} h^5/5!,\ K = 0.864.$$

2.4 Another option

On the interval to the right of the previous one, we can apply the approximation

$$P_4^4(t) = \sum_{j=i-1}^{j=i+3} \det\left(F(t_j)\right) w_j(t),\ t \in [t_i, t_{i+1}], \quad (4)$$

$$w_{i-1}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})(t_{i-1}-t_{i+3})},$$

$$w_i(t) = \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})(t_i-t_{i+3})},$$

$$w_{i+1}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})},$$

$$w_{i+2}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})},$$

$$w_{i+3}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_{i-1})(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})}.$$

The error of approximation we receive from the remainder term of the Lagrange interpolation. For $t \in [t_i, t_{i+1}]$ from (4) we obtain

$$\|P_4^4 - F\|_{[t_i,t_{i+1}]} \leq K\, \|F^{(5)}\|_{[t_{i-1},t_{i+3}]} h^5/5!, K = 1.21$$

2.5 One more option

On the interval to the right of the previous one, we can apply the approximation

$$P_4^5(t) = \sum_{j=i}^{j=i+4} \det\left(F(t_j)\right) w_j(t),\ t \in [t_i, t_{i+1}], \quad (5)$$

$$w_i(t) = \frac{(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})(t-t_{i+4})}{(t_i-t_{i+1})(t_i-t_{i+2})(t_i-t_{i+3})(t_i-t_{i+4})},$$

$$w_{i+1}(t) = \frac{(t-t_i)(t-t_{i+2})(t-t_{i+3})(t-t_{i+4})}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})(t_{i+1}-t_{i+4})},$$

$$w_{i+2}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+3})(t-t_{i+4})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})(t_{i+2}-t_{i+4})},$$

$$w_{i+3}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+4})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})(t_{i+3}-t_{i+4})},$$

$$w_{i+4}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i+4}-t_i)(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})}.$$

The error of approximation we receive from the remainder term of the Lagrange interpolation: for $t \in [t_i, t_{i+1}]$ by (5)

$\|P_4^5 - F\| \le K \,\|F^{(5)}\|_{[t_i,t_{i+4}]} h^5/5!$, $\quad K = 3.632$. Formulae (3), (4) give us the less error of approximation. If it is possible it is better to use formulae (3) or (4) to obtain the root on $[t_i, t_{i+1}]$. Now we will discuss the method of calculation of the largest eigenvalue using values of $F(t), F(t) = \det(\widetilde{H}_{20} - tE_{20})$, at no less than 5 different points t_j. Let the largest eigenvalue μ be such that $\mu \in D_m = [a_m, b_m]$. We put $h_m = (b_m - a_m)/5$, $t_i = a_m + ih_m, i = 0,1,\dots,5$. Eigenvalues of $\widetilde{H}_{20}$ are positive, so we can put $a_m = 0$. Thus, for the calculation of eigenvalues on $[t_i, t_{i+1}]$ we have to find the roots of polynomial $P_4(t)$. We begin from the right side of domain D_m. If $i = 4$ then we have $t \in [b_m - h, b_m]$ and we have to solve $P_4^1(t) = 0$, where $P_4^1(t)$ is given by (1) (we can also use $P_4^2(t)$, which is given by (2)). If there is a real root in the interval $[b_m - h, b_m]$ then this root is the largest eigenvalue. If there is no real root in this interval then we try to obtain a real root in the next interval. For $i = 3$ we have $t \in [b_m - 2h, b_m - h]$ and solve $P_4^2(t) = 0$, where $P_4^2(t)$ has the form (2). If there is a real root in the interval $[b_m - 2h, b_m - h]$ then this root is the largest eigenvalue. If there is no real root in this interval then we should try to obtain a real root in the next interval. If $i = 2$ then we have $t \in [b_m - 3h, b_m - 2h]$ and we have to solve $P_4^3(t) = 0$, where $P_4^3(t)$ is (3). If there is a real root in the interval $[b_m - 3h, b_m - 2h]$ then this root is the largest eigenvalue. If there is no real root in this interval then we should try to obtain a real root in the next interval. If $i = 1$ we have $t \in [a_m + h, a_m + 2h]$ and solve $P_4^4(t) = 0$, where $P_4^4(t)$ is given by (4). If there is a real root in the interval $[a_m + h, a_m + 2h]$ then this root is the largest eigenvalue. If there is no real root in this interval then we should try to obtain a real root in the next interval. If $i = 0$ we have $t \in [a_m, a_m + h]$ and solve $P_4^5(t) = 0$, where $P_4^5(t)$ is given by (5).

Example. Results of the calculation of the largest eigenvalue of matrix $\widetilde{H}_{20}$. First we have to calculate $D_m = [a_m, b_m]$, So, with the help of the Gershhorin's theorem, we get $b_m \approx 3.60, a_m \approx -1.59$. As all eigenvalues of $\widetilde{H}_{20}$ are positive, we can put $a_m = 0$. After that we obtain $h_m = (b_m - a_m)/5 \approx 0.72$. Thus we have $x_0 = 0$, $x_1 \approx 0.72$, $x_2 \approx 1.439, x_3 \approx 2.1588$, $x_4 \approx 2.878$, $x_5 \approx 3.598$. Using (1), (2) we see that there aren't any roots at $[x_4, x_5]$ and $[x_3, x_4]$. Using (3) we calculate root $\mu \in [x_2, x_3]$: $\mu \approx 2.1573$. Thus we know the upper and lower boundaries of the domain where the largest eigenvalue is $|t_m - t_r| \le K\,h^5 \approx 0.2\,K$, $K = const$. Here t_r is the exact value of the largest eigenvalue. Now, using (1)-(5), we can continue the process of determining the boundaries of the largest eigenvalue. Let us take $[a_m, b_m] = [x_2, x_3]$, $h_1 = (x_3 - x_2)/5$. So we have $h_1 = 0.088$ and $x_{00} = 1.439$ $\quad x_{10} \approx 1.583$, $\quad x_{20} \approx 1.727, x_{30} \approx 1.871$, $\quad x_{40} \approx 2.015$, $\quad x_{50} \approx 2.159$. The new approach to the largest eigenvalue is 1.8993. Choosing $[a_m, b_m] = [x_{30}, x_{40}] = [1.899768, 1.928536]$, we obtain 1.907127. The value of the largest eigenvalue, which may be calculated using Maple, is approximately 1.907135. So the error is less then $0.8 \cdot 10^{-5}$. We calculated 13 determinates. Suppose that $D_m \cap D_i \ne \emptyset$. If our aim is to obtain all eigenvalues then we can take into account all real roots which possibly belong to D_i and not belong to D_m. Continuing the process, we obtain the remaining eigenvalues.

3 Integro-differential Splines Application

There are other possibilities for interpolating polynomial spline construction. Here for obtaining the eigenvalues we can use integro-differential splines, which were suggested recently by one of the authors (see [5]-[7], [9]).

3.1 First option

We can construct the polynomial on $[t_i, t_{i+1}]$ in the form:

$$P(t_j + th) = \sum_{k=j-2}^{j+1} \det(F(t_k))\, w_k(t) + \int_{t_j}^{t_{j+1}} \det(F(t))\, dt\, w_j^{<0,1>}(t),$$

$$w_k(t) = \frac{(t-1)(5t-2)(t+2)(t+1)}{4},$$

$$w_{k+1}(t) = \frac{t(135t-97)(t+2)(t+1)}{228},$$
$$w_{k-1}(t) = -\frac{t(t-1)(25t-13)(t+2)}{76},$$
$$w_{k-2}(t) = \frac{t(t-1)(15t-8)(t+1)}{228},$$
$$w_k^{<0,1>}(t) = -\frac{30t(t-1)(t+2)(t+1)}{19h}.$$

It can be proved that

$\|P-F\| \le K\, h^5/5!\ \|F^{(5)}\|_{[t_{i-2},t_{i+1}]}$, $K>0$. Of cause we don't know the value of the integral. Using (2) we obtain:

$\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{-h}{720}(264 \det\left(F\left(t_{j-1}\right)\right) - 251 \det(F(t_{j+1})) - 646 \det(F(t_j)) - 106 \det(F(t_{j-2})) + 19 \det(F(t_{j-3}))).$

Using (5) we obtain: $\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{h}{720}(251 \det\left(F\left(t_j\right)\right) + 646 \det(F(t_{j+1})) - 264 \det(F(t_{j+2})) + 106 \det(F(t_{j+3})) - 19 \det(F(t_{j+4}))).$

Using (3) we obtain:
$\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{h}{720}(456 \det\left(F\left(t_j\right)\right) - 19 \det(F(t_{j+2})) + 346 \det(F(t_{j+1})) - 74 \det(F(t_{j-1})) + 11 \det(F(t_{j-2}))).$

3.2 The next option

$$P(x_j+th) = \sum_{k=j-1}^{j+2} \det\left(F(t_k)\right) v_k(t) + \int_{t_j}^{t_{j+1}} \det\left(F(t)\right) dt\, v_j^{<0,1>}(t),$$

where

$$v_j(t) = -(65\,t-22)(t-1)(t-2)(t+1)/44,$$
$$v_{j+1}(t) = -t(65\,t-43)(t-2)(t+1)/44,$$
$$v_{j-1}(t) = t(t-1)(t-2)(15\,t-7)/132,$$
$$v_{j+2}(t) = t(t-1)(15\,t-8)(t+1)/132,$$
$$v_j^{<0,1>}(t) = \frac{30}{11\,h}\, t(t-1)(t-2)(t+1).$$

It can be proved that

$\|P-F\| \le K\, h^5/5!\, \|F^{(5)}\|_{[t_{j-1},t_{j+2}]}$, $K>0$.

Using (3) we obtained the relation:

$\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{h}{720}(346 \det\left(F\left(t_j\right)\right) + 456 \det(F(t_{j+1})) - 74 \det(F(t_{j+2})) + 11 \det(F(t_{j+3})) - 19 \det(F(t_{j-1}))).$

We can also use the quadrature:

$\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{h}{1440}(-258 \det\left(F\left(t_{j+2}\right)\right) - 27 \det(F(t_{j-1})) + 637 \det(F(t_j)) + 77 \det(F(t_{j+3})) + 1022 \det(F(t_{j+1})) - 11 \det(F(t_{j+4}))).$

Example. If we obtain the root in the interval $[x_1, x_2]$, $x_1 = 1.899768$, $x_2 = 1.928536$, and other nodes are the following: $x_j = x_1 + j\,h$, $j = -1,0,1,2,3,4$, $h = 0.028768$, then we get the value of the largest eigenvalue 1.9071355.

3.3 The next option

$$P(x_j+th) = \sum_{k=j}^{j+3} \det\left(F(t_k)\right) v_k(t) + \int_{t_j}^{t_{j+1}} \det\left(F(t)\right) dt\, v_j^{<0,1>}(t),$$

where the basic splines are the following:

$$v_j(t) = (t-1)(t-2)(t-3)(135\,t-38)/228,$$
$$v_{j+1}(t) = t\,(t-2)(t-3)(5\,t-3)/4,$$
$$v_{j+2}(t) = -t(t-1)(t-3)(25\,t-12)/76,$$
$$v_{j+3}(t) = t(t-1)(t-2)(15\,t-7)/228,$$
$$v_j^{<0,1>}(t) = -\frac{30}{19\,h}\, t(t-1)(t-2)(t-3).$$

It can be shown that

$\|P-F\| \le K\, h^5/5!\ \|F^{(5)}\|_{[t_j,t_{j+3}]}$, $K>0$.

Using (5) we obtain:

$\int_{t_j}^{t_{j+1}} \det(F(t))dt \approx \frac{h}{720}(251 \det\left(F\left(t_j\right)\right) + 646 \det(F(t_{j+1})) - 264 \det(F(t_{j+2})) + 106 \det(F(t_{j+3})) - 19 \det(F(t_{j+4}))).$

3.4 One more option

$$P(x_j + th) = \sum_{k=j-3}^{j+1} \det(F(t_k))\, v_k(t) + \int_{t_j}^{t_{j+1}} \det(F(t))\, dt\, v_j^{<0,1>}(t),$$

where

$$v_j(t) = \frac{(323\,t - 135)(t+3)(t+2)(t^2-1)}{810},$$
$$v_{j+1}(t) = t\frac{(502\,t - 367)(t+3)(t+2)(t+1)}{3240},$$
$$v_{j-1}(t) = -\frac{t(t-1)(t+3)(t+2)(88\,t-47)}{540},$$
$$v_{j-2}(t) = \frac{t(t-1)(t+3)(53\,t-29)(t+1)}{810},$$
$$v_{j-3}(t) = -\frac{t(t-1)(38\,t-21)(t+2)(t+1)}{3240},$$
$$v_j^{<0,1>}(t) = -4\frac{t(t-1)(t+3)(t+2)(t+1)}{9h}.$$

It can be shown that

$\|P-F\| \le K\,h^6/6!\,\|F^{(6)}\|_{[t_{i-3},t_{i+1}]},\ K>0.$

3.5 One more option

$$P(x_j + th) = \sum_{k=j-2}^{j+2} \det(F(t_k))\, v_k(t) + \int_{t_j}^{t_{j+1}} \det(F(t))\, dt\, v_j^{<0,1>}(t),$$

where

$$v_j(t) = -\frac{(152\,t-55)(t^2-1)(t^2-4)}{220},$$
$$v_{j+1}(t) = \frac{t(173\,t-118)(t-2)(t+2)(t+1)}{330},$$
$$v_{j-1}(t) = \frac{t(t-1)(t-2)(t+2)(37t-18)}{330},$$
$$v_{j-2}(t) = -\frac{t(t-1)(t-2)(2t-1)(t+1)}{120},$$
$$v_{j+2}(t) = \frac{t(t-1)(38\,t-21)(t+2)(t+1)}{1320},$$
$$v_j^{<0,1>}(t) = 12\,t\frac{(t-1)(t-2)(t+2)(t+1)}{11h}.$$

It can be shown that

$\|P-F\| \le K\,h^6/6!\,\|F^{(6)}\|_{[t_{i-2},t_{i+2}]},\ K>0.$

3.6 One more option

$$P(x_j + th) = \sum_{k=j-1}^{j+3} \det(F(t_k))\, v_k(t) + \int_{t_j}^{t_{j+1}} \det(F(t))\, dt\, v_j^{<0,1>}(t),$$

$$v_j(t) = \frac{(173\,\mathrm{t}-55)(\mathrm{t}-2)(\mathrm{t}-3)(\mathrm{t}^2-1)}{330},$$
$$v_{j+1}(t) = \frac{1}{220}\mathrm{t}\,(152\,\mathrm{t}-97)(\mathrm{t}-2)(\mathrm{t}-3)(\mathrm{t}+1),$$
$$v_{j-1} = -\frac{1}{1320}\mathrm{t}(\mathrm{t}-1)(\mathrm{t}-2)(\mathrm{t}-3)(38\,\mathrm{t}-17)$$
$$v_{j+3}(t) = \frac{1}{120}\mathrm{t}(\mathrm{t}-1)(\mathrm{t}-2)(2\,\mathrm{t}-1)(\mathrm{t}+1),$$
$$v_{j+2}(t) = \frac{-1}{330}\mathrm{t}(\mathrm{t}-1)(37\,\mathrm{t}-19)(\mathrm{t}-3)(\mathrm{t}+1),$$
$$v_j^{<0,1>}(t) = -\frac{12}{11\mathrm{h}}\mathrm{t}\,(\mathrm{t}-1)(\mathrm{t}-2)(\mathrm{t}-3)(\mathrm{t}+1).$$

It can be shown that

$\|P-F\| \le K\,h^6/6!\,\|F^{(6)}\|_{[t_{i-1},t_{i+3}]},\quad K>0.$ We can also use the quadrature:

$\int_{t_j}^{t_{j+1}} \det(F(t))\,dt \approx \frac{h}{1440}(-258\det(F(t_{j+2})) - 27\det(F(t_{j-1})) + 637\det(F(t_j)) + 77\det(F(t_{j+3})) + 1022\det(F(t_{j+1})) - 11\det(F(t_{j+4})) + \frac{F^{(6)}(\mu)}{6!}\frac{271}{84}h^7,$ where $\mu\epsilon[t_{j-1},t_{j+4}], h = x_{j+1} - x_j = \text{const}.$ Using this formula we obtain 1.907136, while using Maple we obtain 1.907135.

4 Application for the Search for Eigenvalues of Matrices with Real Elements

In this section, we consider the calculation of the eigenvalues of the Le Verrier matrix M [13]:

$$M = \begin{pmatrix} -5.509882 & 1.870086 & 0.422908 & 0.008814 \\ 0.287865 & 11.811654 & 5.711900 & 0.058717 \\ 0.049099 & 4.308033 & 12.970687 & 0.229326 \\ 0.006235 & 0.269851 & 1.397369 & 17.596207 \end{pmatrix}.$$

It is not difficult to calculate the eigenvalues of this matrix. They have the following values: $-5.298698, -7.574043, -17.152427, -17.863261$. Applying the power method $Y_{k+1} = MY_k$ after 150 iterations with initial vector $Y_0 = (1, 0, 0, 0)$, we obtain a vector with components $\frac{(Y_{150})_i}{(Y_{149})_i}$, $i = 1,2,3,4$. Each of these components is the approximation to the eigenvalue largest in absolute value of matrix M:
$(-17.858871, -17.859182, -17.859907, -17.865201)$.
Thus we have obtained the largest eigenvalue with two precise digits after the decimal point. We performed 2400 multiplicative operations.

One can directly calculate the characteristic polynomial degree $n = 4$ of matrix M: $\varphi(t) = \det(M - tE) = t^4 - p_1 t^3 - p_2 t^2 - p_3 t - p_4$. Each p_k is the sum (taken with the sign $(-1)^{k-1}$) of all minors of order k supported on the main diagonal of the matrix M. The number of the minors is equal to the number of combinations from $n = 4$ to k, $p_4 = (-1)^3 \det(M)$. Thus, in order to calculate the eigenvalue, which is greatest in absolute value, it is necessary to perform about 100 multiplicative operations and solve the equation of the fourth degree.

Calculating the eigenvalue with the interpolation method, using formula (2), we perform about 300 multiplicative operations. Let us describe this process in more detail. Using Gershgorin's theorem, we find the interval in which there is an eigenvalue: $[a, b] = [-19.269662, -15.922752]$. In this interval, we construct a uniform grid of nodes $\{t_j\}, j = 0,1,2,3,4,5$, with step $h = \frac{b-a}{5} = 0.669382$. Next, we calculate determinants $\det\left(F\left(t_j\right)\right)$ of the fourth order at the grid nodes and successively find the roots of the polynomials $P_4^j(t)$ on the subintervals $[a_j, b_j]$. Using (1) and (3) we construct $P_4^0(t)$ and $P_4^1(t)$. There are no real roots of $P_4^j(t)$ on the intervals $\left[a_j, b_j\right], j = 0,1$. For $j = 2$ using (2) we obtain polynomial

$$P_4^2(t) = t^4 + 47.888430\, t^3 + 797.278765\, t^2 + 5349.455515\, t + 12296.550566.$$

Solving this equation we find the roots:
$-17.863261, -17.152427, -7.574043, -5.2986$, one of them, which equals -17.863261 is in the considered interval
$[a_2, b_2] = [-17.930898, -17.261516]$. Thus, the required eigenvalue is equal -17.863261 and the error of the method in this case is equal 0.

We note that with increasing order of the matrix, the degree of the polynomial for which it is necessary to find the root does not increase, but remains equal to 4. This is in contrast to finding the eigenvalues as the roots of the characteristic polynomial.

Thus, the difference between finding the eigenvalues of the matrix by the interpolation method from finding the eigenvalues by the method of finding the roots of the characteristic polynomial consists in solving an equation of a low degree. The number of multiplicative operations performed by the interpolation method will be 2.5 times greater than in the construction of the characteristic polynomial (except for the process of finding the roots).

4 Conclusion

At present, there are effective algorithms for finding the eigenvalues of the matrix. The power method and the method of scalar products are among the most commonly used. Nevertheless the interpolation method may also be used for finding eigenvalues. The most laborious part in the proposed interpolation method is the calculation of determinants. Therefore, according to the number of multiplicative and additive operations, the proposed method will be more efficient in comparison with the methods traditionally used for matrices which aren't very big, for example not more than $n = 10$. Nevertheless, the interpolation method can be used to calculate the real eigenvalues of matrices with real elements of any order.

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