On Fuzzy Dependencies and g-generated Fuzzy Implications in Fuzzy Relations

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Abstract: In this paper we use the g-generated fuzzy implications to research the concept of automatization in the process of derivation of new fuzzy functional and fuzzy multivalued dependencies from some given set of fuzzy functional and fuzzy multivalued dependencies. The formal definitions of fuzzy functional and fuzzy multivalued dependencies that we apply are based on application of similarity relations and conformance values. In this context, the paper follows similarity based fuzzy relational database approach. In order derive and then apply our results, we identify fuzzy dependencies with fuzzy formulas. The obtained results are verified through the resolution principle.

Key–Words: Fuzzy conjunctions, disjunctions and implications, g-implications, inference rules, resolution principle, fuzzy formulas, fuzzy functional and multivalued dependencies

1 Preliminaries and introduction

Recall that a mapping $\mathcal{C} : [0,1]^2 \to [0,1]$ is called a conjunction of the unit interval if: $\mathcal{C}(0,0) = 0$, $\mathcal{C}(0,1) = 0$, $\mathcal{C}(1,0) = 0$, $\mathcal{C}(1,1) = 1$, and $\mathcal{C}(x,z) \leq \mathcal{C}(y,z)$, $\mathcal{C}(z,x) \leq \mathcal{C}(z,y)$ for x, y, $z \in [0,1]$ and $x \leq y$.

A mapping $\mathcal{T} : [0,1]^2 \to [0,1]$ (see, e.g., [10, p. 16]) is called a triangular norm or a t-norm if: $\mathcal{T}(x,1) = x$, $\mathcal{T}(x,y) \leq \mathcal{T}(x,z)$ for $y \leq z$, $\mathcal{T}(x,y) = \mathcal{T}(y,x)$, and $\mathcal{T}(x,\mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z)$, where x, y, $z \in [0,1]$.

It is known that a t-norm is a conjunction on the unit interval, i.e., it is known that t-norms are widely used to model conjunction operators.

Some additional requirements of t-norms are the following ones: $\mathcal{T}$ is continuous, the partial mappings of $\mathcal{T}$ are left-continuous, $\mathcal{T}(x,x) = x$ for $x \in [0,1]$, $\mathcal{T}(x,x) < x$ for $x \in (0,1)$, Archimedean property, $\mathcal{T}$ is continuous and $\mathcal{T}(x,y) < \mathcal{T}(x,z)$ for x, y, $z \in (0,1)$ with $0 < y < z < 1$, $\mathcal{T}$ is continuous and for $x \in (0,1)$, there is some $y \in (0,1)$ such that $\mathcal{T}(x,y) = 0$.

The minimum t-norm $\mathcal{T}_M(x,y) = \min(x,y)$ is very commonly used in literature. We shall apply this t-norm in the sequel.

A mapping $\mathcal{S} : [0,1]^2 \to [0,1]$ is called a triangular co-norm or a t-co-norm if: $\mathcal{S}(x,0) = x$, $\mathcal{S}(x,y) \leq \mathcal{S}(x,z)$ for $y \leq z$, $\mathcal{S}(x,y) = \mathcal{S}(y,x)$, and $\mathcal{S}(x,\mathcal{S}(y,z)) = \mathcal{S}(\mathcal{S}(x,y),z)$, where x, y, $z \in [0,1]$.

Some additional properties of t-co-norms are the following ones: $\mathcal{S}$ is continuous, $\mathcal{S}(x,x) = x$ for $x \in [0,1]$, $\mathcal{S}(x,x) > x$ for $x \in (0,1)$, Archimedean property, $\mathcal{S}$ is continuous and $\mathcal{S}(x,y) < \mathcal{S}(x,z)$ for x, y, $z \in (0,1)$ with $0 < y < z < 1$, $\mathcal{S}$ is continuous and for $x \in (0,1)$, there is some $y \in (0,1)$ such that $\mathcal{S}(x,y) = 1$.

The maximum t-co-norm $\mathcal{S}_M(x,y) = \max(x,y)$ is very commonly used in literature. We shall apply this t-co-norm through the rest of the paper.

There are various definitions of fuzzy implications.

A function $\mathcal{I} : [0,1]^2 \to [0,1]$ (see, e.g., [2, p. 2, Def. 1.1.1.]) is called a fuzzy implication if: $\mathcal{I}(0,0) = 1$, $\mathcal{I}(1,1) = 1$, $\mathcal{I}(1,0) = 0$, $\mathcal{I}(x_1,y) \geq \mathcal{I}(x_2,y)$ for $x_1 \leq x_2$, and $\mathcal{I}(x,y_1) \leq \mathcal{I}(x,y_2)$ for $y_1 \leq y_2$, where $x, x_1, x_2, y, y_1, y_2 \in [0,1]$.

Note that this definition of fuzzy implication is equivalent to the one proposed in [9] (see also, [6]).

Also, note that each fuzzy implication satisfies the following properties: $\mathcal{I}(0,y) = 1$ for $y \in [0,1]$, and $\mathcal{I}(x,1) = 1$ for $x \in [0,1]$. These properties are known as left and right boundary conditions, respectively.

In [13], Yager defined two new classes of fuzzy implications, called the f-generated and the g-generated implications. In this paper we pay attention to the g-generated implications.

The g-generated implications are obtained from strictly increasing functions.

Let $g : [0, 1] \to [0, \infty]$ be a strictly increasing and continuous function such that $g(0) = 0$. The function $\mathcal{I}_g : [0, 1]^2 \to [0, 1]$ (see, e.g., [2, p. 116]) defined by

$$\mathcal{I}_g(x, y) = g^{(-1)}\left(\frac{1}{x} g(y)\right),$$

x, $y \in [0, 1]$, with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g-generated implication.

Here, the function $g^{(-1)}$ is the pseudo-inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & x \in [0, g(1)], \\ 1, & x \in [g(1), \infty]. \end{cases}$$

The function g itself is called a g-generator (of $\mathcal{I}_g$).

If g is a g-generator, then $\mathcal{I}_g$ is a fuzzy implication. Namely, it is not hard to verify that in this case $\mathcal{I}_g$ satisfies all conditions given by the fuzzy implication definition.

Note that the definition of $\mathcal{I}_g$ may be written in the following form (without use of $g^{(-1)}$), i.e., as

$$\mathcal{I}_g(x, y) = g^{-1}\left(\min\left(\frac{1}{x} g(y), g(1)\right)\right),$$

where x, $y \in [0, 1]$.

Also note that for the g-generator $g(x) = -\frac{1}{\log x}$, one obtains well known Yager fuzzy implication $\mathcal{I}_{YG}(x, y) = y^x$, x, $y \in [0, 1]$ with the understanding $0^0 = 1$. Yager implication $\mathcal{I}_{YG}$ is also an f-implication with the f-generator $f(x) = -\log x$ (see, e.g., [2, p. 110]).

The g-generators of the g-generated implications are unique up to a positive multiplicative constant. More precisely, if $g_1, g_2 : [0, 1] \to [0, \infty]$ are any two g-generators, then, $\mathcal{I}_{g_1} = \mathcal{I}_{g_2}$ if and only if there is a constant $c \in (0, \infty)$ such that $g_2(x) = c g_1(x)$ for $x \in [0, 1]$.

If g is a g-generator, then either $g(1) = \infty$ or $g(1) < \infty$. If $g(1) < \infty$, then $g_1 : [0, 1] \to [0, \infty]$ defined by $g_1(x) = \frac{g(x)}{g(1)}$, $x \in [0, 1]$ is a g-generator. Namely, g is a strictly increasing and continuous function such that $g(0) = 0$. Therefore, $g(1) \neq 0$. Consequently, g_1 is a strictly increasing and continuous function such that $g_1(0) = 0$, i.e., g_1 is a g-generator. Note that $g_1(1) = 1$. Now, $g(x) = g(1) g_1(x)$, $x \in [0, 1]$ and $g(1) \in (0, \infty)$ yield that $\mathcal{I}_g = \mathcal{I}_{g_1}$. This actually means that it is enough to consider those g-generators for which $g(1) = \infty$ or $g(1) = 1$.

The fuzzy implication applied in this paper will be a g-generated implication $\mathcal{I}_g$.

The main purpose of this paper is to prove the following theorems.

Theorem 1. *Suppose that $R(U) = R(A_1, A_2, ..., A_n)$ is a scheme on domains D_1, D_2,..., D_n, where U is the set of all attributes A_1, A_2,..., A_n on D_1, D_2,..., D_n, respectively (U is the universal set of attributes), and D_i is a finite set for $i \in \{1, 2, ..., n\}$. Let C be a set whose elements are fuzzy functional dependencies on U of the form $K \xrightarrow{\theta_2}_F L$ and fuzzy multivalued dependencies on U of the form $K \to\xrightarrow{\theta_2}_F L$, where K and L are subsets of U, and θ_2 is the linguistic strength of the dependency. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. Let $\{t_1, t_2\} = r \subseteq 2^{D_1} \times 2^{D_2} \times ... \times 2^{D_n}$ be any two-element fuzzy relation instance on $R(U)$ and $\beta \in [0, 1]$ be any number. Denote by $i_{r,\beta}$ a valuation joined to r and β, where $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0, 1]$ is defined via conformance $\varphi(A_k[t_1, t_2])$ of the attribute $A_k \in \{A_1, A_2, ..., A_n\}$ on tuples t_1 and t_2 by $i_{r,\beta}(A_k) > \frac{1}{2}$ if $\varphi(A_k[t_1, t_2]) \geq \beta$ and $i_{r,\beta}(A_k) \leq \frac{1}{2}$ if $\varphi(A_k[t_1, t_2]) < \beta$. Let*

$$i_{r,\beta}(A_i \wedge A_j) = \mathcal{T}_M(i_{r,\beta}(A_i), i_{r,\beta}(A_j)),$$
$$i_{r,\beta}(A_i \vee A_j) = \mathcal{S}_M(i_{r,\beta}(A_i), i_{r,\beta}(A_j)),$$
$$i_{r,\beta}(A_i \Rightarrow A_j) = \mathcal{I}_g(i_{r,\beta}(A_i), i_{r,\beta}(A_j)),$$

for A_i, $A_j \in \{A_1, A_2, ..., A_n\}$. Furthermore, let

$$(\wedge_{A \in K} A) \Rightarrow (\wedge_{B \in L} B)$$

be the fuzzy formula (with respect to $i_{r,\beta}$) joined to $K \xrightarrow{\theta_2}_F L$, and

$$(\wedge_{A \in K} A) \Rightarrow ((\wedge_{B \in L} B) \vee (\wedge_{C \in M} C)),$$

the fuzzy formula (with respect to $i_{r,\beta}$) joined to $K \to\xrightarrow{\theta_2}_F L$, where $M = U \setminus (K \cup L)$. Denote by C' resp. c' the set of fuzzy formulas resp. the fuzzy formula joined to C resp. c. Then, the following two conditions are equivalent:

(a) *Any two-element, fuzzy relation instance on $R(U)$ which satisfies all dependencies in C, satisfies the dependency c.*

(*b*) $i_{r,\beta}\left(c^{'}\right) > \frac{1}{2}$ *for every* $i_{r,\beta}$ *such that* $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ *for all* $\mathcal{K} \in C^{'}$.

Theorem 2. *Let* $r = \{t_1, t_2\}$ *be any two-element, fuzzy relation instance on* $R(U)$, *and* $X \to\xrightarrow{\theta}_F Y$ *be any fuzzy multivalued dependency on* U. *Let* $Z = U \setminus (X \cup Y)$. *Then,* r *satisfies* $X \to\xrightarrow{\theta}_F Y$, *and for all* $A \in X$, $\varphi(A[t_1,t_2]) \geq \theta$ *holds true if and only if* $\varphi(X[t_1,t_2]) \geq \theta$ *and* $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$, *where* $\mathcal{H}$ *denotes the fuzzy formula* $(\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))$ *joined to* $X \to\xrightarrow{\theta}_F Y$, *and* $\varphi(X[t_1,t_2])$ *denotes the conformance of the attribute set* X *on tuples* t_1 *and* t_2.

Here, we point out that the conformances $\varphi(A[t_1,t_2])$ ($\varphi(A_k[t_1,t_2])$), $\varphi(X[t_1,t_2])$, etc. are taken in the sense of Definitions 3.1. and 3.2. in [11, pp. 165-166]. These definitions, however, are based on the concept of similarity relations (see, [11, p. 163, Def. 2.1.]). Furthermore, the formal definitions of fuzzy functional and fuzzy multivalued dependencies (which are given on the basis of conformance values) are taken in the sense of Definitions 3.3. and 4.1. in [11, pp. 167-172].

Obviously, $\varphi(A[t_1,t_2]) \geq \theta$ for all $A \in X$ and all $t_1, t_2 \in r$ if and only if $\varphi(X[t_1,t_2]) \geq \theta$ for all $t_1, t_2 \in r$, where r is a fuzzy relation instance on $R(U)$ and $X \subseteq U$. Moreover, if $r = \{t_1, t_2\}$, then r satisfies $X \to\xrightarrow{\theta}_F Y$, and for all $A \in X$, $\varphi(A[t_1,t_2]) \geq \theta$ holds true if and only if $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Y[t_1,t_2]) \geq \theta$ or $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Z[t_1,t_2]) \geq \theta$, where $X \to\xrightarrow{\theta}_F Y$ is a fuzzy multivalued dependency on U and $Z = U \setminus (X \cup Y)$.

Note that a variant of Theorem 1 as well as a variant of Theorem 2 are proved in [7] for arbitrary f-generated implication (see also, [5], [4]). Since the present paper deals with the class of g-generated implications, it represents a natural complement of the aforementioned papers.

2 Proofs of results

Proof. (of Theorem 2.)

($\Rightarrow$) Assume that r satisfies $X \to\xrightarrow{\theta}_F Y$, and that for all $A \in X$, $\varphi(A[t_1,t_2]) \geq \theta$ holds true.

Now,

$$\varphi(X[t_1,t_2]) \geq \theta, \quad \varphi(Y[t_1,t_2]) \geq \theta \quad \text{or}$$
$$\varphi(X[t_1,t_2]) \geq \theta, \quad \varphi(Z[t_1,t_2]) \geq \theta.$$

Assume that $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ hold true. $\varphi(Y[t_1,t_2]) \geq \theta$ yields that $\varphi(B[t_1,t_2]) \geq \theta$ for all $B \in Y$. Therefore, $i_{r,\theta}(B) > \frac{1}{2}$ for all $B \in Y$. We conclude, $i_{r,\theta}(\wedge_{B\in Y}B) > \frac{1}{2}$. Similarly, $i_{r,\theta}(\wedge_{A\in X}A) > \frac{1}{2}$.

We have,

$$\begin{aligned}
& i_{r,\theta}(\mathcal{H}) \\
&= i_{r,\theta}((\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))) \\
&= g^{-1}\Big(\min\Big(\frac{1}{i_{r,\theta}(\wedge_{A\in X}A)} \cdot \\
&\quad g\left(i_{r,\theta}((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))\right), g(1)\Big)\Big) \\
&= g^{-1}\Big(\min\Big(\frac{1}{i_{r,\theta}(\wedge_{A\in X}A)} \cdot \\
&\quad g\left(\max\left(i_{r,\theta}(\wedge_{B\in Y}B), i_{r,\theta}(\wedge_{C\in Z}C)\right)\right), g(1)\Big)\Big).
\end{aligned}$$

Let

$$\begin{aligned}
a &= i_{r,\theta}(\wedge_{A\in X}A), \\
b &= \max\left(i_{r,\theta}(\wedge_{B\in Y}B), i_{r,\theta}(\wedge_{C\in Z}C)\right).
\end{aligned}$$

Now,

$$i_{r,\theta}(\mathcal{H}) = g^{-1}\left(\min\left(\frac{1}{a}g(b), g(1)\right)\right).$$

$i_{r,\theta}(\wedge_{A\in X}A) > \frac{1}{2}$ yields that $a > \frac{1}{2}$. Furthermore, $i_{r,\theta}(\wedge_{B\in Y}B) > \frac{1}{2}$ yields that $b > \frac{1}{2}$.

Now, $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$ if and only if

$$\min\left(\frac{1}{a}g(b), g(1)\right) > g\left(\frac{1}{2}\right).$$

If $\min\left(\frac{1}{a}g(b), g(1)\right) = g(1)$, then $\min\left(\frac{1}{a}g(b), g(1)\right) > g\left(\frac{1}{2}\right)$ since $g(1) > g\left(\frac{1}{2}\right)$, i.e., $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$ holds true in this case.

Suppose that $\min\left(\frac{1}{a}g(b), g(1)\right) = \frac{1}{a}g(b)$. We have, $\frac{1}{a}g(b) > g\left(\frac{1}{2}\right)$ if and only if $ag\left(\frac{1}{2}\right) < g(b)$. Now, $a \leq 1$ and $\frac{1}{2} < b$ yield that

$$ag\left(\frac{1}{2}\right) \leq g\left(\frac{1}{2}\right) < g(b).$$

Therefore, $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$.

If we assume that $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Z[t_1,t_2]) \geq \theta$ hold true, then, reasoning as in the previous case, we obtain that $i_{r,\theta}(\wedge_{A\in X}A) > \frac{1}{2}$ and $i_{r,\theta}(\wedge_{C\in Z}C) > \frac{1}{2}$. Consequently, $a > \frac{1}{2}$ and $b > \frac{1}{2}$.

Now, reasoning in the same way as in the previous case, we conclude that $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$.

$(\Leftarrow)$ Assume that $\varphi(X[t_1,t_2]) \geq \theta$ and $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$ hold true.

Now,

$$i_{r,\theta}(\mathcal{H}) = g^{-1}\left(\min\left(\frac{1}{a}g(b), g(1)\right)\right) > \frac{1}{2},$$

where

$$a = i_{r,\theta}(\wedge_{A\in X}A),$$
$$b = \max\left(i_{r,\theta}(\wedge_{B\in Y}B), i_{r,\theta}(\wedge_{C\in Z}C)\right).$$

$\varphi(X[t_1,t_2]) \geq \theta$ yields that $a > \frac{1}{2}$. Our aim is to prove that $b > \frac{1}{2}$ holds also true. Namely, if $b > \frac{1}{2}$, then $i_{r,\theta}(\wedge_{B\in Y}B) > \frac{1}{2}$ or $i_{r,\theta}(\wedge_{C\in Z}C) > \frac{1}{2}$. In the first case, for example, one obtains that $i_{r,\theta}(B) > \frac{1}{2}$ for all $B \in Y$. This means that $\varphi(B[t_1,t_2]) \geq \theta$ for all $B \in Y$. Consequently, $\varphi(Y[t_1,t_2]) \geq \theta$.

Similarly, $i_{r,\theta}(\wedge_{C\in Z}C) > \frac{1}{2}$ yields that $\varphi(Z[t_1,t_2]) \geq \theta$.

Since, $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$, we have that

$$\min\left(\frac{1}{a}g(b), g(1)\right) > g\left(\frac{1}{2}\right).$$

If $\min\left(\frac{1}{a}g(b), g(1)\right) = \frac{1}{a}g(b)$, then $\frac{1}{a}g(b) > g\left(\frac{1}{2}\right)$.

If $\min\left(\frac{1}{a}g(b), g(1)\right) = g(1)$, then

$$\frac{1}{a}g(b) \geq g(1) > g\left(\frac{1}{2}\right).$$

Hence, $\frac{1}{a}g(b) > g\left(\frac{1}{2}\right)$ in any case. Now,

$$a < \frac{g(b)}{g\left(\frac{1}{2}\right)}.$$

Since $a > \frac{1}{2}$, the last inequality implies that

$$\frac{g(b)}{g\left(\frac{1}{2}\right)} > 1,$$

i.e., that $g(b) > g\left(\frac{1}{2}\right)$, i.e., that $b > \frac{1}{2}$.

Therefore, $b > \frac{1}{2}$, and then $\varphi(Y[t_1,t_2]) \geq \theta$ or $\varphi(Z[t_1,t_2]) \geq \theta$. In other words,

$$\varphi(X[t_1,t_2]) \geq \theta, \ \ \varphi(Y[t_1,t_2]) \geq \theta \quad \text{or}$$
$$\varphi(X[t_1,t_2]) \geq \theta, \ \ \varphi(Z[t_1,t_2]) \geq \theta.$$

As noted before, this means that r satisfies $X \to\xrightarrow{\theta}_F Y$, and that for all $A \in X$, $\varphi(A[t_1,t_2]) \geq \theta$ holds true. This completes the proof. $\square$

Proof. (of Theorem 1.)

We write $X \xrightarrow{\theta_1}_F Y$ resp. $X \to\xrightarrow{\theta_1}_F Y$ instead of c if c is a fuzzy functional resp. fuzzy multivalued dependency. Therefore, we write

$$(\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)$$

resp.

$$(\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{D\in Z}D))$$

instead of c', where $Z = U \setminus (X \cup Y)$.

We let the set $\{a,b\}$ to be the domain of each of the attributes in U.

Choose some $\theta'' \in \left[0,\theta'\right)$, where θ' is the minimum of the strengths of all dependencies that appear in $C \cup \{c\}$. If $\theta' = 1$, then one obtains a non-interesting case where each $c_1 \in C \cup \{c\}$ is of strength 1. Hence, we assume that $\theta' < 1$.

Put $s(a,b) = s(b,a) = \theta''$ to be a similarity relation on domains of the attributes in U.

$(a) \Rightarrow (b)$ Suppose that (b) is not valid.

Now, there is some $i_{r,\beta}$ such that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$ and $i_{r,\beta}\left(c'\right) \leq \frac{1}{2}$.

Here, $i_{r,\beta}$ is associated to some two-element, fuzzy relation instance r on $R(U)$, and some $\beta \in [0,1]$. We may take $r = \{t_1,t_2\}$.

We introduce $Z' = \left\{A \in U \mid i_{r,\beta}(A) > \frac{1}{2}\right\}$.

Let $Z' = \varnothing$.

This means that $i_{r,\beta}(A) \leq \frac{1}{2}$ for all $A \in U$.

$i_{r,\beta}\left(c'\right) \leq \frac{1}{2}$ yields that

$$i_{r,\beta}((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)) \leq \frac{1}{2}$$

resp.

$$i_{r,\beta}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\left(\wedge_{B\in Y}B\right)\vee\left(\wedge_{D\in Z}D\right)\right)\right)\leq\frac{1}{2}.$$

Hence,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2}$$

resp.

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\left(\wedge_{B\in Y}B\right)\vee\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2},$$

i.e.,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2}$$

resp.

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2},$$

i.e.,

$$\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)\leq g\left(\frac{1}{2}\right)$$

resp.

$$\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)\leq g\left(\frac{1}{2}\right).$$

The fact that g is a strictly increasing function implies that the condition $g\left(1\right)\leq g\left(\frac{1}{2}\right)$ is not satisfied. This means that $g\left(1\right)$ is not the minimum in any of the aforementioned cases. Therefore,

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\leq g\left(\frac{1}{2}\right)$$

resp.

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\leq g\left(\frac{1}{2}\right).$$

If $i_{r,\beta}\left(\wedge_{A\in X}A\right)=0$, then $\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}=\infty$. Since $g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\geq 0$ resp. $g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\geq 0$, and $\infty\cdot 0=\infty$, we obtain that

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)=\infty$$

resp.

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)=\infty.$$

Therefore, $g\left(\frac{1}{2}\right)\geq\infty$. This, however, contradicts the fact that g is a strictly increasing function. In other words, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>0$.

If $g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)=\infty$ resp. $g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)=\infty$, then $i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1$ resp. $\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)=1$, i.e., $i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1$ resp. $i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1$ or $i_{r,\beta}\left(\wedge_{D\in Z}D\right)=1$. This means that in any case there exists $A\in U$ such that $i_{r,\beta}\left(A\right)=1>\frac{1}{2}$. This is a contradiction. Hence, $g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)<\infty$ resp. $g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)<\infty$.

We have,

$$i_{r,\beta}\left(\wedge_{A\in X}A\right)\geq\frac{g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}$$

resp.

$$i_{r,\beta}\left(\wedge_{A\in X}A\right)\geq\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}.$$

Now, $i_{r,\beta}(A) \le \frac{1}{2}$, $A \in U$,
yields that $i_{r,\beta}(\wedge_{B\in Y} B) \le \frac{1}{2}$ resp.
$\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right) \le \frac{1}{2}$. Hence,

$$g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right) \le g\left(\frac{1}{2}\right)$$

resp.

$$g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right) \le g\left(\frac{1}{2}\right),$$

i.e.,

$$\frac{g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right)}{g\left(\frac{1}{2}\right)} \le 1$$

resp.

$$\frac{g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right)}{g\left(\frac{1}{2}\right)} \le 1.$$

These inequalities imply that the condition $i_{r,\beta}(\wedge_{A\in X} A) = 1$ must be satisfied. Consequently, $i_{r,\beta}(A) = 1$ for some $A \in X \subseteq U$. This is a contradiction.

We conclude, $Z' \neq \emptyset$.

Now, assume that $Z' = U$.

Then $i_{r,\beta}(A) > \frac{1}{2}$ for $A \in U$.

As earlier, $i_{r,\beta}\left(c'\right) \le \frac{1}{2}$ yields that

$$\frac{1}{i_{r,\beta}(\wedge_{A\in X} A)} \cdot g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right) \le g\left(\frac{1}{2}\right)$$

resp.

$$\frac{1}{i_{r,\beta}(\wedge_{A\in X} A)} \cdot g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right) \le g\left(\frac{1}{2}\right).$$

If $i_{r,\beta}(\wedge_{A\in X} A) = 0$, then there exists $A \in X \subseteq U$ such that $i_{r,\beta}(A) = 0$. This is a contradiction. Hence, $i_{r,\beta}(\wedge_{A\in X} A) > 0$. We obtain,

$$i_{r,\beta}(\wedge_{A\in X} A) \ge \frac{g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right)}{g\left(\frac{1}{2}\right)}$$

resp.

$$i_{r,\beta}(\wedge_{A\in X} A) \ge \frac{g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right)}{g\left(\frac{1}{2}\right)}.$$

Now, $i_{r,\beta}(A) > \frac{1}{2}$, $A \in U$
implies that $i_{r,\beta}(\wedge_{B\in Y} B) > \frac{1}{2}$ resp.
$\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right) > \frac{1}{2}$.
Consequently, $g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right) > g\left(\frac{1}{2}\right)$ resp.
$g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right) > g\left(\frac{1}{2}\right)$,
i.e.,

$$\frac{g\left(i_{r,\beta}(\wedge_{B\in Y} B)\right)}{g\left(\frac{1}{2}\right)} > 1$$

resp.

$$\frac{g\left(\max\left(i_{r,\beta}(\wedge_{B\in Y} B), i_{r,\beta}(\wedge_{D\in Z} D)\right)\right)}{g\left(\frac{1}{2}\right)} > 1.$$

Therefore, $i_{r,\beta}(\wedge_{A\in X} A) > 1$. This is a contradiction.

Consequently, $Z' \neq U$.

Choose $r' = \left\{t', t''\right\}$ to be the two-element, fuzzy relation instance given by Table 1.

Table 1:

	attributes of Z'	other attributes
t'	$a, a, \ldots, a$	$a, a, \ldots, a$
t''	$a, a, \ldots, a$	$b, b, \ldots, b$

Our goal is to prove that this fuzzy relation instance satisfies all elements in C, and violates c.

Let $K \xrightarrow{\theta_2}_F L$ be a fuzzy functional dependency from set C. Now,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}(\wedge_{A\in K} A)} \cdot g\left(i_{r,\beta}(\wedge_{B\in L} B)\right), g(1)\Big)\Big) > \frac{1}{2},$$

i.e.,

$$\min\Big(\frac{1}{i_{r,\beta}(\wedge_{A\in K} A)} \cdot g\left(i_{r,\beta}(\wedge_{B\in L} B)\right), g(1)\Big) > g\left(\frac{1}{2}\right).$$

If $\min\left(\frac{1}{i_{r,\beta}(\wedge_{A\in K}A)} g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right), g\left(1\right)\right) = g\left(1\right)$, then

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)} g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) \geq g\left(1\right) > g\left(\frac{1}{2}\right).$$

Otherwise, we immediately obtain that

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)} g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) > g\left(\frac{1}{2}\right).$$

In other words, the last inequality holds always true.

Suppose that $i_{r,\beta}\left(\wedge_{A\in K}A\right) \leq \frac{1}{2}$. Now, there is some $A \in K$ such that $i_{r,\beta}\left(A\right) \leq \frac{1}{2}$. Consequently, $A \notin Z^{'}$. Therefore, $\varphi\left(A\left[t^{'}, t^{''}\right]\right) = \theta^{''}$, and hence $\varphi\left(K\left[t^{'}, t^{''}\right]\right) = \theta^{''}$.

The fact that $s\left(a, b\right) = s\left(b, a\right) = \theta^{''}$ hold true gives us that $\varphi\left(Q\left[t^{'}, t^{''}\right]\right) \geq \theta^{''}$ for every $Q \subseteq U$. Therefore, $\varphi\left(L\left[t^{'}, t^{''}\right]\right) \geq \theta^{''}$, i.e.,

$$\varphi\left(L\left[t^{'}, t^{''}\right]\right) \geq \min\left(\theta_2, \varphi\left(K\left[t^{'}, t^{''}\right]\right)\right).$$

This means that $r^{'}$ satisfies the dependency $K \xrightarrow{\theta_2}_F L$.

Suppose that $i_{r,\beta}\left(\wedge_{A\in K}A\right) > \frac{1}{2}$. Now,

$$i_{r,\beta}\left(\wedge_{A\in K}A\right) < \frac{g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)}.$$

Consequently,

$$\frac{g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)} > 1,$$

i.e.,

$$g\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) > g\left(\frac{1}{2}\right),$$

i.e.,

$$i_{r,\beta}\left(\wedge_{B\in L}B\right) > \frac{1}{2}.$$

Now, $i_{r,\beta}\left(B\right) > \frac{1}{2}$ for every $B \in L$. Then, $B \in Z^{'}$ for $B \in L$, i.e., $L \subseteq Z^{'}$. We obtain, $\varphi\left(L\left[t^{'}, t^{''}\right]\right) = 1$, i.e.,

$$\varphi\left(L\left[t^{'}, t^{''}\right]\right) \geq \min\left(\theta_2, \varphi\left(K\left[t^{'}, t^{''}\right]\right)\right).$$

Thus, $r^{'}$ satisfies $K \xrightarrow{\theta_2}_F L$.

Let $K \to\xrightarrow{\theta_2}_F L$ be a fuzzy multivalued dependency from the set C. Now,

$$\begin{aligned}&g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right),\\&g\left(1\right)\Big)\Big)\\&> \frac{1}{2},\end{aligned}$$

i.e.,

$$\begin{aligned}&\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right), g\left(1\right)\Big)\\&> g\left(\frac{1}{2}\right),\end{aligned}$$

where $M = U \setminus \left(K \cup L\right)$.

If

$$\begin{aligned}&\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right), g\left(1\right)\Big)\\&= g\left(1\right),\end{aligned}$$

then

$$\begin{aligned}&\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)\\&\geq g\left(1\right) > g\left(\frac{1}{2}\right).\end{aligned}$$

Otherwise,

$$\begin{aligned}&\frac{1}{i_{r,\beta}\left(\wedge_{A\in K}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)\\&> g\left(\frac{1}{2}\right)\end{aligned}$$

holds immediately true. In any case, the last inequality is valid.

Assume that $i_{r,\beta}\left(\wedge_{A\in K}A\right)\le\frac{1}{2}$. Reasoning as before, we obtain that $\varphi\left(K\left[t',t''\right]\right)=\theta''$.

Now, there exists $t'''\in r'$, $t'''=t'$ such that

$$\varphi\left(K\left[t''',t'\right]\right)=1\ge\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),$$
$$\varphi\left(L\left[t''',t'\right]\right)=1\ge\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),$$
$$\varphi\left(M\left[t''',t''\right]\right)\ge\theta''=\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right).$$

Therefore, r' satisfies the dependency $K\rightarrow\xrightarrow{\theta_2}_F L$.

Now, assume that $i_{r,\beta}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$. It follows that $\varphi\left(K\left[t',t''\right]\right)=1$.

We have,

$$i_{r,\beta}\left(\wedge_{A\in K}A\right)$$
$$<\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}.$$

Since $i_{r,\beta}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$, we conclude that

$$\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}>1,$$

i.e.,

$$g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)$$
$$>g\left(\frac{1}{2}\right),$$

i.e.,

$$\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)>\frac{1}{2}.$$

Hence, $i_{r,\beta}\left(\wedge_{B\in L}B\right)>\frac{1}{2}$ or $i_{r,\beta}\left(\wedge_{D\in M}D\right)>\frac{1}{2}$. In the first case, $\varphi\left(L\left[t',t''\right]\right)=1$. In the second case, $\varphi\left(M\left[t',t''\right]\right)=1$.

Assume that $\varphi\left(L\left[t',t''\right]\right)=1$.

In this case, there is $t'''\in r'$, $t'''=t''$ such that

$$\begin{aligned}\varphi\left(K\left[t''',t'\right]\right)&=1\\&\ge\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),\\\varphi\left(L\left[t''',t'\right]\right)&=1\\&\ge\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),\\\varphi\left(M\left[t''',t''\right]\right)&=1\\&\ge\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right).\end{aligned}\tag{1}$$

Hence, r' satisfies the dependency $K\rightarrow\xrightarrow{\theta_2}_F L$.

Now, assume that $\varphi\left(M\left[t',t''\right]\right)=1$.

In this case, there exists $t'''\in r'$, $t'''=t'$ such that (1) holds true. Consequently, the instance r' satisfies the dependency $K\rightarrow\xrightarrow{\theta_2}_F L$.

It remains to prove that the instance r' does not satisfy $X\xrightarrow{\theta_1}_F Y$ resp. $X\rightarrow\xrightarrow{\theta_1}_F Y$.

First, let

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot$$
$$g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)\Big)\le\frac{1}{2}.$$

We have,

$$\min\left(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\right)$$
$$\le g\left(\frac{1}{2}\right).$$

Since $g\left(1\right)>g\left(\frac{1}{2}\right)$, we obtain that

$$\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\le g\left(\frac{1}{2}\right).$$

Assume that $g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\ne0,\infty$.

If $i_{r,\beta}\left(\wedge_{A\in X}A\right)=0$, then, as earlier, we obtain that $g\left(\frac{1}{2}\right)\ge\infty$. This contradicts the fact that g is a strictly increasing function. Therefore, $i_{r,\beta}\left(\wedge_{A\in X}A\right)$ >0. Hence,

$$\frac{g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}\le i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

If $i_{r,\beta}\left(\wedge_{A\in X}A\right)\leq\frac{1}{2}$, then

$$\frac{g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}=0,$$

i.e., $g\left(\frac{1}{2}\right)=\infty$, i.e., a contradiction. Hence, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$. Then, $\varphi\left(X\left[t^{'},t^{''}\right]\right)=1$.

Now,

$$\frac{g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}\leq i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

implies that

$$\frac{g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}\leq\frac{1}{2}<1.$$

Hence,

$$g\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)<g\left(\frac{1}{2}\right),$$

i.e.,

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right)<\frac{1}{2}.$$

We conclude, $\varphi\left(Y\left[t^{'},t^{''}\right]\right)=\theta^{''}$.

Now,

$$\begin{aligned}\varphi\left(Y\left[t^{'},t^{''}\right]\right)&=\theta^{''}<\theta^{'}\leq\theta_1=\min\left(\theta_1,1\right)\\&=\min\left(\theta_1,\varphi\left(X\left[t^{'},t^{''}\right]\right)\right),\end{aligned}$$

i.e., $r^{'}$ does not satisfy the dependency $X\xrightarrow{\theta_1}_F Y$.

Second, let

$$\begin{aligned}&g^{-1}\Big(\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right),\\&g\left(1\right)\Big)\Big)\\&\leq\frac{1}{2}.\end{aligned}$$

We have,

$$\begin{aligned}&\min\Big(\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)\\&\leq g\left(\frac{1}{2}\right).\end{aligned}$$

As earlier, $g\left(1\right)>g\left(\frac{1}{2}\right)$ yields that

$$\begin{aligned}&\frac{1}{i_{r,\beta}\left(\wedge_{A\in X}A\right)}\cdot\\&g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\\&\leq g\left(\frac{1}{2}\right).\end{aligned}$$

Assume that $g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\neq 0,\infty$.

If $i_{r,\beta}\left(\wedge_{A\in X}A\right)=0$, then $g\left(\frac{1}{2}\right)\geq\infty$. This is a contradiction, Hence, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>0$. Therefore,

$$\begin{aligned}&\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}\\&\leq i_{r,\beta}\left(\wedge_{A\in X}A\right).\end{aligned}$$

If $i_{r,\beta}\left(\wedge_{A\in X}A\right)\leq\frac{1}{2}$, then

$$\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}=0,$$

i.e., $g\left(\frac{1}{2}\right)=\infty$, i.e., a contradiction. Hence, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$. We obtain, $\varphi\left(X\left[t^{'},t^{''}\right]\right)=1$.

Now, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$ and

$$\begin{aligned}&\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}\\&\leq i_{r,\beta}\left(\wedge_{A\in X}A\right).\end{aligned}$$

yield that

$$\begin{aligned}&\frac{g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}{g\left(\frac{1}{2}\right)}\\&\leq\frac{1}{2}<1,\end{aligned}$$

i.e., that

$$g\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right) \leq g\left(\frac{1}{2}\right).$$

We obtain,

$$\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right) \leq \frac{1}{2}.$$

Hence, $i_{r,\beta}\left(\wedge_{B\in Y}B\right) < \frac{1}{2}$ and $i_{r,\beta}\left(\wedge_{D\in Z}D\right) < \frac{1}{2}$, i.e., $\varphi\left(Y\left[t',t''\right]\right) = \theta''$ and $\varphi\left(Z\left[t',t''\right]\right) = \theta''$.

Now, if $t''' \in r'$ and $t''' = t'$, then

$$\begin{aligned}\varphi\left(X\left[t''',t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Y\left[t''',t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Z\left[t''',t''\right]\right) &= \theta'' < \theta' \leq \theta_1\\ &= \min\left(\theta_1, 1\right)\\ &= \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right).\end{aligned}$$

Otherwise, if $t''' \in r'$ and $t''' = t''$, then

$$\begin{aligned}\varphi\left(X\left[t''',t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Y\left[t''',t'\right]\right) &= \theta'' < \theta' \leq \theta_1\\ &= \min\left(\theta_1, 1\right)\\ &= \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Z\left[t''',t''\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right).\end{aligned}$$

We conclude that the instance r' violates $X \rightarrow\xrightarrow{\theta_1}_F Y$.

$(b) \Rightarrow (a)$ Suppose that (a) is not valid.

Now, there is a two-element fuzzy relation instance on $R(U)$ which satisfies elements in C, and does not satisfy c. Denote it by $r' = \left\{t', t''\right\}$.

Put, $Z' = \left\{A \in U \mid \varphi\left(A\left[t',t''\right]\right) = 1\right\}$.

If we assume that $Z' = \emptyset$, then $\varphi\left(A\left[t',t''\right]\right) = \theta''$ for $A \in U$, and, consequently, $\varphi\left(Q\left[t',t''\right]\right) = \theta''$ for $Q \subseteq U$.

Suppose that the instance r' does not satisfy the dependency $X \xrightarrow{\theta_1}_F Y$. Now,

$$\begin{aligned}\theta'' = \varphi\left(Y\left[t',t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right)\\ &= \min\left(\theta_1, \theta''\right) = \theta''.\end{aligned}$$

This is a contradiction.

Suppose that the instance r' does not satisfy the dependency $X \rightarrow\xrightarrow{\theta_1}_F Y$. Now, the conditions

$$\begin{aligned}\varphi\left(X\left[t''',t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Y\left[t''',t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Z\left[t''',t''\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right)\end{aligned}$$

don't simultaneously hold for any $t''' \in r'$. In particular, the conditions

$$\begin{aligned}\varphi\left(X\left[t',t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Y\left[t',t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right),\\ \varphi\left(Z\left[t',t''\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right)\end{aligned}$$

don't simultaneously hold. Since the first and the second condition hold obviously true, we obtain that

$$\begin{aligned}\theta'' = \varphi\left(Z\left[t',t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right)\\ &= \min\left(\theta_1, \theta''\right) = \theta''.\end{aligned}$$

Hence, a contradiction.

Therefore, $Z' \neq \emptyset$.

If we assume that $Z' = U$, then $\varphi\left(A\left[t',t''\right]\right) = 1$ for $A \in U$, and then $\varphi\left(Q\left[t',t''\right]\right) = 1$ for $Q \subseteq U$.

Suppose that the instance r' does not satisfy the dependency $X \xrightarrow{\theta_1}_F Y$. We obtain,

$$\begin{aligned}1 = \varphi\left(Y\left[t',t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t',t''\right]\right)\right)\\ &= \min\left(\theta_1, 1\right) = \theta_1,\end{aligned}$$

i.e., a contradiction.

Suppose that the instance r' does not satisfy the dependency $X \rightarrow\xrightarrow{\theta_1}_F Y$. Now, the condition

$$\varphi\left(Z\left[t',t''\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)$$

does not hold, i.e., we have that

$$1=\varphi\left(Z\left[t',t''\right]\right)<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)$$
$$=\min\left(\theta_1,1\right)=\theta_1.$$

This is a contradiction.

We conclude, $Z'\neq U$.

Now, r' is a two-element fuzzy relation instance on $R(U)$ and $\beta=1\in[0,1]$, so we are able to introduce $i_{r',\beta}=i_{r',1}$ by putting $i_{r',1}(A)>\frac{1}{2}$ if $\varphi\left(A\left[t',t''\right]\right)=1$, and $i_{r',1}(A)\leq\frac{1}{2}$ if $\varphi\left(A\left[t',t''\right]\right)<1$.

Our aim is to prove that all fuzzy formulas in C' are valid under $i_{r',1}$. On the other side, we shall prove that c' is not valid under $i_{r',1}$.

Suppose that $\mathcal{K}\in C'$ corresponds to some fuzzy functional dependency $K\xrightarrow{\theta_2}_F L$ from the set C.

Moreover, suppose that the inequality $i_{r',1}(\mathcal{K})>\frac{1}{2}$ is not true. We have,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2},$$

i.e.,

$$\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right),g\left(1\right)\Big)\leq g\left(\frac{1}{2}\right).$$

Since $g(1)>g\left(\frac{1}{2}\right)$, we obtain

$$\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)\leq g\left(\frac{1}{2}\right).$$

Suppose that $g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)\neq 0,\infty$.

If $i_{r',1}\left(\wedge_{A\in K}A\right)=0$, then $g\left(\frac{1}{2}\right)\geq\infty$. This contradicts the fact g is a strictly increasing function. Hence, $i_{r',1}\left(\wedge_{A\in K}A\right)>0$, and then

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)}\leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

If $i_{r',1}\left(\wedge_{A\in K}A\right)\leq\frac{1}{2}$, we obtain that

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)}=0,$$

i.e., that $g\left(\frac{1}{2}\right)=\infty$. This is a contradiction.

Hence, $i_{r',1}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$, i.e., $\varphi\left(K\left[t',t''\right]\right)=1$.

Furthermore, $i_{r',1}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$ and

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)}\leq i_{r',1}\left(\wedge_{A\in K}A\right)$$

imply that

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}{g\left(\frac{1}{2}\right)}\leq\frac{1}{2}<1.$$

Thus,

$$g\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)<g\left(\frac{1}{2}\right),$$

i.e., $i_{r',1}\left(\wedge_{B\in L}B\right)<\frac{1}{2}$, i.e., $\varphi\left(L\left[t',t''\right]\right)=\theta''$.

Now, the fact that the instance r' satisfies the dependency $K\xrightarrow{\theta_2}_F L$, yields that

$$\theta''=\varphi\left(L\left[t',t''\right]\right)\geq\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right)$$
$$=\min\left(\theta_2,1\right)=\theta_2.$$

This is a contradiction.

Therefore, $i_{r',1}(\mathcal{K})>\frac{1}{2}$.

Now, suppose that $\mathcal{K}\in C'$ corresponds to some fuzzy multivalued dependency $K\rightarrow\xrightarrow{\theta_2}_F L$ from the set C. Put, $M=U\setminus(K\cup L)$.

Assume that $i_{r',1}(\mathcal{K})\leq\frac{1}{2}$, i.e., that

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right),g\left(1\right)\Big)\Big)\leq\frac{1}{2}.$$

We have,

$$\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right), g\left(1\right)\Big) \leq g\left(\frac{1}{2}\right).$$

As before, $g\left(1\right)>g\left(\frac{1}{2}\right)$ implies that

$$\frac{1}{i_{r',1}\left(\wedge_{A\in K}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right) \leq g\left(\frac{1}{2}\right).$$

Suppose that $g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)\neq 0,\infty$.

If $i_{r',1}\left(\wedge_{A\in K}A\right)=0$, then $g\left(\frac{1}{2}\right)\geq\infty$, i.e., a contradiction. Hence, $i_{r',1}\left(\wedge_{A\in K}A\right)>0$. We obtain,

$$\frac{g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)}{g\left(\frac{1}{2}\right)} \leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

As earlier, $i_{r',1}\left(\wedge_{A\in K}A\right)\leq\frac{1}{2}$ yields $g\left(\frac{1}{2}\right)=\infty$. This is not possible, however. Therefore, $i_{r',1}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$.

We obtain, $\varphi\left(K\left[t',t''\right]\right)=1$, and

$$\frac{g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)}{g\left(\frac{1}{2}\right)} \leq \frac{1}{2}<1,$$

i.e.,

$$g\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right) < g\left(\frac{1}{2}\right),$$

i.e.,

$$\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)<\frac{1}{2}.$$

Hence, $i_{r',1}\left(\wedge_{B\in L}B\right)<\frac{1}{2}$ and $i_{r',1}\left(\wedge_{D\in M}D\right)<\frac{1}{2}$, i.e., $\varphi\left(L\left[t',t''\right]\right)=\theta''$ and $\varphi\left(M\left[t',t''\right]\right)=\theta''$.

Since the inequalities

$$\begin{aligned}\varphi\left(M\left[t',t''\right]\right)&=\theta''<\theta'\leq\theta_2\\&=\min\left(\theta_2,1\right)\\&=\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right)\end{aligned}$$

and

$$\begin{aligned}\varphi\left(L\left[t'',t'\right]\right)&=\theta''<\theta'\leq\theta_2\\&=\min\left(\theta_2,1\right)\\&=\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right)\end{aligned}$$

hold true, the instance r' does not satisfy the dependency $K\rightarrow\xrightarrow{\theta_2}_F L$. This is a contradiction.

Therefore, $i_{r',1}\left(\mathcal{K}\right)>\frac{1}{2}$.

Now, we prove that c' is not valid under $i_{r',1}$.

Suppose that $i_{r',1}\left(c'\right)>\frac{1}{2}$, where c' corresponds to the dependency $X\xrightarrow{\theta_1}_F Y$. We have,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)\Big)>\frac{1}{2},$$

i.e.,

$$\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)>g\left(\frac{1}{2}\right).$$

If

$$\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right),g\left(1\right)\Big)=g\left(1\right),$$

then

$$\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\geq g\left(1\right)>g\left(\frac{1}{2}\right).$$

Otherwise,

$$\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)>g\left(\frac{1}{2}\right).$$

In any case, the last inequality holds true.

Assume that $g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\neq 0,\infty$.

If $i_{r',1}\left(\wedge_{A\in X}A\right)=0$, then $\varphi\left(X\left[t',t''\right]\right)=\theta''$. Now,

$$\varphi\left(Y\left[t',t''\right]\right)\geq\theta''=\min\left(\theta_1,\theta''\right)=\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right).$$

This means that r' satisfies $X\xrightarrow{\theta_1}_F Y$, i.e., this is a contradiction.

Therefore, $i_{r',1}\left(\wedge_{A\in X}A\right)>0$, and then

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}>i_{r',1}\left(\wedge_{A\in X}A\right).$$

If $i_{r',1}\left(\wedge_{A\in X}A\right)\leq\frac{1}{2}$, then $\varphi\left(X\left[t',t''\right]\right)=\theta''$. Hence,

$$\varphi\left(Y\left[t',t''\right]\right)\geq\theta''=\min\left(\theta_1,\theta''\right)=\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right).$$

This is a contradiction.

We obtain, $i_{r',1}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$.

The inequality $i_{r',1}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$ and the inequality

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}>i_{r',1}\left(\wedge_{A\in X}A\right)$$

yield that

$$\frac{g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)}{g\left(\frac{1}{2}\right)}>1,$$

i.e., $g\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)>g\left(\frac{1}{2}\right)$, i.e., $i_{r',1}\left(\wedge_{B\in Y}B\right)>\frac{1}{2}$. Hence, $\varphi\left(Y\left[t',t''\right]\right)=1$. We obtain,

$$\varphi\left(Y\left[t',t''\right]\right)=1\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right).$$

This is a contradiction.

We conclude, $i_{r',1}\left(c'\right)\leq\frac{1}{2}$.

Now, suppose that $i_{r',1}\left(c'\right)>\frac{1}{2}$, where c' corresponds to the dependency $X\rightarrow\xrightarrow{\theta_1}_F Y$. We have,

$$g^{-1}\Big(\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)\Big)>\frac{1}{2},$$

i.e.,

$$\min\Big(\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right),g\left(1\right)\Big)>g\left(\frac{1}{2}\right).$$

Reasoning as in the case of fuzzy functional dependency $X\xrightarrow{\theta_1}_F Y$, we obtain that

$$\frac{1}{i_{r',1}\left(\wedge_{A\in X}A\right)}\cdot g\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)>g\left(\frac{1}{2}\right).$$

Assume that $g\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)\neq 0,\infty$.

If $i_{r',1}(\wedge_{A\in X}A)=0$, then $\varphi\left(X\left[t',t''\right]\right)=\theta''$. Now,

$$\varphi\left(Z\left[t',t''\right]\right)\geq\theta''=\min\left(\theta_1,\theta''\right)$$
$$=\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)$$

and (or)

$$\varphi\left(Y\left[t'',t'\right]\right)\geq\theta''=\min\left(\theta_1,\theta''\right)$$
$$=\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right).$$

This means that r' satisfies $X\to\xrightarrow{\theta_1}_F Y$ (namely, the remaining inequalities required to r' satisfies $X\to\xrightarrow{\theta_1}_F Y$ with $t'''=t'$ resp. $t'''=t''$ are already fulfilled). This, however, contradicts the fact that r' does not satisfy the dependency $X\to\xrightarrow{\theta_1}_F Y$.

Hence, $i_{r',1}(\wedge_{A\in X}A)>0$. Now,

$$\frac{g\left(\max\left(i_{r',1}(\wedge_{B\in Y}B),i_{r',1}(\wedge_{D\in Z}D)\right)\right)}{g\left(\frac{1}{2}\right)}$$
$$>i_{r',1}(\wedge_{A\in X}A).$$

If $i_{r',1}(\wedge_{A\in X}A)\leq\frac{1}{2}$, then $\varphi\left(X\left[t',t''\right]\right)=\theta''$. We obtain,

$$\varphi\left(Z\left[t',t''\right]\right)\geq\theta''=\min\left(\theta_1,\theta''\right)$$
$$=\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right).$$

This is a contradiction.

Hence, $i_{r',1}(\wedge_{A\in X}A)>\frac{1}{2}$.

The inequality $i_{r',1}(\wedge_{A\in X}A)>\frac{1}{2}$ and the inequality

$$\frac{g\left(\max\left(i_{r',1}(\wedge_{B\in Y}B),i_{r',1}(\wedge_{D\in Z}D)\right)\right)}{g\left(\frac{1}{2}\right)}$$
$$>i_{r',1}(\wedge_{A\in X}A)$$

yield that

$$\frac{g\left(\max\left(i_{r',1}(\wedge_{B\in Y}B),i_{r',1}(\wedge_{D\in Z}D)\right)\right)}{g\left(\frac{1}{2}\right)}>1,$$

i.e., $g\left(\max\left(i_{r',1}(\wedge_{B\in Y}B),i_{r',1}(\wedge_{D\in Z}D)\right)\right)>g\left(\frac{1}{2}\right)$, i.e., $\max\left(i_{r',1}(\wedge_{B\in Y}B),i_{r',1}(\wedge_{D\in Z}D)\right)>\frac{1}{2}$. Therefore, $i_{r',1}(\wedge_{B\in Y}B)>\frac{1}{2}$ or $i_{r',1}(\wedge_{D\in Z}D)>\frac{1}{2}$, i.e., $\varphi\left(Y\left[t',t''\right]\right)=1$ or $\varphi\left(Z\left[t',t''\right]\right)=1$.

If $\varphi\left(Y\left[t',t''\right]\right)=1$, then

$$\varphi\left(Y\left[t',t''\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right),$$

i.e., r' satisfies $X\to\xrightarrow{\theta_1}_F Y$ with $t'''=t''$. This is a contradiction.

If $\varphi\left(Z\left[t',t''\right]\right)=1$, then

$$\varphi\left(Z\left[t',t''\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right),$$

i.e., r' satisfies $X\to\xrightarrow{\theta_1}_F Y$ with $t'''=t'$. This is a contradiction as well.

We conclude, $i_{r',1}\left(c'\right)\leq\frac{1}{2}$.

This completes the proof. □

3 Applications

The results derived in this paper can be applied in the following way.

First, we recall the inference rules [11].

IR1 Inclusive rule for $FFDs$: If $X\xrightarrow{\theta_1}_F Y$ holds, and $\theta_1\geq\theta_2$, then $X\xrightarrow{\theta_2}_F Y$ holds.

IR2 Reflexive rule for $FFDs$: If $X\supseteq Y$, then $X\to_F Y$ holds.

IR3 Augmentation rule for $FFDs$: If $X\xrightarrow{\theta}_F Y$ holds, then $XZ\xrightarrow{\theta}_F YZ$ holds.

IR4 Transitivity rule for $FFDs$: If $X\xrightarrow{\theta_1}_F Y$ holds and $Y\xrightarrow{\theta_2}_F Z$ holds, then $X\overset{\min(\theta_1,\theta_2)}{\to}_F Z$ holds.

IR5 Inclusive rule for $FMVDs$: If $X\to\xrightarrow{\theta_1}_F Y$ holds, and $\theta_1\geq\theta_2$, then $X\to\xrightarrow{\theta_2}_F Y$ holds.

IR6 Complementation rule for $FMVDs$: If $X\to\xrightarrow{\theta}_F Y$ holds, then $X\to\xrightarrow{\theta}_F U-XY$ holds.

IR7 Augmentation rule for $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta}_F Y$ holds, and $W \supseteq Z$, then $WX \rightarrow\!\!\xrightarrow{\theta}_F YZ$ holds.

IR8 Transitivity rule for $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds and $Y \rightarrow\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Z - Y$ holds.

IR9 Replication rule: If $X \xrightarrow{\theta}_F Y$ holds, then $X \rightarrow\!\!\xrightarrow{\theta}_F Y$ holds.

IR10 Coalescence rule for $FFDs$ and $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y we have $W \xrightarrow{\theta_2}_F Z$, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Z$ holds.

IR11 Union rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds and $X \xrightarrow{\theta_2}_F Z$ holds, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F YZ$ holds.

IR12 Pseudotransitivity rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds and $WY \xrightarrow{\theta_2}_F Z$ holds, then $WX \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Z$ holds.

IR13 Decomposition rule for $FFDs$: If $X \xrightarrow{\theta}_F Y$ holds and $Z \subseteq Y$, then $X \xrightarrow{\theta}_F Z$ holds.

IR14 Union rule for $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds and $X \rightarrow\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F YZ$ holds.

IR15 Pseudotransitivity rule for $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds and $WY \rightarrow\!\!\xrightarrow{\theta_2}_F Z$ holds, then $WX \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Z - WY$ holds.

IR16 Decomposition rule for $FMVDs$: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds and $X \rightarrow\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Y \cap Z$, $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Y - Z$, $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Z - Y$ hold.

IR17 Mixed pseudotransitivity rule: If $X \rightarrow\!\!\xrightarrow{\theta_1}_F Y$ holds and $XY \xrightarrow{\theta_2}_F Z$ holds, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Z - Y$ holds.

Here, $FFDs$ ($FMVDs$) means fuzzy functional dependencies (fuzzy multivalued dependencies).

Example 1. Let $U = \{Q_i \mid i \in \{1, 2, ..., 7\}\}$ be the universal set of attributes.

(a) If the fuzzy multivalued dependencies:

$$Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_1}_F Q_2Q_4Q_5Q_6,$$
$$Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_2}_F Q_3Q_4Q_6Q_7$$

hold true, then the fuzzy multivalued dependencies:

(i) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Q_4Q_6$,

(ii) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Q_2Q_5$,

(iii) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Q_3Q_7$

hold also true.

(b) If the fuzzy functional and the fuzzy multivalued dependencies:

$$Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_1}_F Q_2Q_4Q_5Q_6,$$
$$Q_1Q_2Q_3Q_4Q_5Q_6 \xrightarrow{\theta_2}_F Q_3Q_4Q_6Q_7$$

hold true, then the fuzzy functional dependency

$$Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Q_3Q_7$$

holds also true.

Proof. (I) We may apply the inference rules listed above.
(a) We obtain:

1) $Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_1}_F Q_2Q_4Q_5Q_6$ (input)

2) $Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_2}_F Q_3Q_4Q_6Q_7$ (input)

3) $Q_1Q_2Q_3Q_4 \rightarrow\!\!\xrightarrow{\theta_1}_F Q_1Q_2Q_3Q_4Q_5Q_6$ (IR7, 1) augment with $Q_1Q_2Q_3Q_4$)

4) $Q_1Q_2Q_3Q_4Q_5Q_6 \rightarrow\!\!\xrightarrow{\theta_2}_F Q_2Q_3Q_4Q_5Q_6Q_7$ (IR7, 2) augment with $Q_2Q_4Q_5Q_6$)

5) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Q_7$ (IR8, 3), 4))

6) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\!\!\rightarrow}_F Q_1Q_2Q_3Q_4Q_7$ (IR7, 5) augment with $Q_1Q_2Q_3Q_4$)

7) $Q_1Q_2Q_3Q_4Q_7 \rightarrow\xrightarrow{\theta_2}_F Q_3Q_4Q_6Q_7$ (IR7, 2) augment with Q_7)

8) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_F Q_6$ (IR8, 6), 7))

9) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_F Q_4Q_6$ (IR7, 8) augment with Q_4)

10) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_F Q_3Q_7$ (IR7, 5) augment with Q_3)

11) $Q_1Q_2Q_3Q_4 \rightarrow\xrightarrow{\theta_2}_F Q_1Q_2Q_3Q_4Q_6Q_7$ (IR7, 2) augment with $Q_1Q_2Q_3Q_4$)

12) $Q_1Q_2Q_3Q_4Q_6Q_7 \rightarrow\xrightarrow{\theta_1}_F Q_2Q_3Q_4Q_5Q_6Q_7$ (IR7, 1) augment with $Q_3Q_4Q_6Q_7$)

13) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_F Q_5$ (IR8, 11), 12))

14) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_F Q_2Q_5$ (IR7, 13) augment with Q_2)

Hence, *(i)* – *(iii)* hold true.
(b) We obtain:

1) $Q_1Q_2Q_3Q_4 \rightarrow\xrightarrow{\theta_1}_F Q_2Q_4Q_5Q_6$ (input)

2) $Q_1Q_2Q_3Q_4Q_5Q_6 \xrightarrow{\theta_2}_F Q_3Q_4Q_6Q_7$ (input)

3) $Q_1Q_2Q_3Q_4 \rightarrow\xrightarrow{\theta_1}_F Q_7$ (IR6, 1))

4) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Q_7$ (IR10, 2), 3))

5) $Q_1Q_2Q_3Q_4 \overset{\min(\theta_1,\theta_2)}{\rightarrow}_F Q_3Q_7$ (IR3, 4) augment with Q_3)

This completes the proof. □

Proof. (II) We may apply Theorem 1 (see also, [5, Cor. 8]), and the resolution principle.
(a) (i) First, we associate fuzzy formulas to given fuzzy dependencies following Theorem 1. We obtain:

$$\begin{aligned}
\mathcal{K}_1 &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow \\
&\quad ((Q_2 \wedge Q_4 \wedge Q_5 \wedge Q_6) \vee Q_7), \\
\mathcal{K}_2 &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow \\
&\quad ((Q_3 \wedge Q_4 \wedge Q_6 \wedge Q_7) \vee Q_5), \\
c' &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow \\
&\quad ((Q_4 \wedge Q_6) \vee (Q_5 \wedge Q_7)).
\end{aligned}$$

Second, we find their conjunctive normal forms (we consider $\neg c'$ following resolution principle). We have:

$$\begin{aligned}
\mathcal{K}_1 &\equiv (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7) \wedge \\
&\quad (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7), \\
\mathcal{K}_2 &\equiv (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6) \wedge \\
&\quad (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7), \\
\neg c' &\equiv Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge (\neg Q_4 \vee \neg Q_6) \wedge \\
&\quad (\neg Q_5 \vee \neg Q_7).
\end{aligned}$$

Here, conjunctive terms are the following ones: $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$, Q_1, Q_2, Q_3, Q_4, $\neg Q_4 \vee \neg Q_6$, $\neg Q_5 \vee \neg Q_7$.
Let G be the set of these terms.
Third, we apply the resolution principle to the elements of the set G. We deduce:

1) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (input)

2) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (input)

3) Q_1 (input)

4) Q_2 (input)

5) Q_3 (input)

6) Q_4 (input)

7) $\neg Q_4 \vee \neg Q_6$ (input)

8) $\neg Q_5 \vee \neg Q_7$ (input)

9) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 1) and 3))

10) $\neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 9) and 4))

11) $\neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 10) and 5))

12) $Q_6 \vee Q_7$ (resolvent from 11) and 6))

13) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 2) and 3))

14) $\neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 13) and 4))

15) $\neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 14) and 5))

16) $Q_5 \vee Q_6$ (resolvent from 15) and 6))

17) $\neg Q_6$ (resolvent from 7) and 6))

18) Q_7 (resolvent from 17) and 12))

19) Q_5 (resolvent from 16) and 17))

20) $\neg Q_5$ (resolvent from 8) and 18))

Now, a refutation of the negation $\neg c^{'}$ follows from 19) and 20). This actually means that the fuzzy formulas $\mathcal{K}_1$, $\mathcal{K}_2$ and $c^{'}$ hold at the same time. More precisely, the condition (b) of Theorem 1 is satisfied. Consequently, our claim follows from the assertion (a) of Theorem 1 (see also, [5, Cor. 8]).
(a) (ii) In this case, the fuzzy formulas $\mathcal{K}_1$ and $\mathcal{K}_2$ have the same form as in the previous case. Therefore, their conjunctive normal forms as well as their conjunctive terms remain the same. Now,

$$c^{'} \equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow ((Q_2 \wedge Q_5) \vee (Q_6 \wedge Q_7)) .$$

Hence,

$$\neg c^{'} \equiv Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge (\neg Q_2 \vee \neg Q_5) \wedge (\neg Q_6 \vee \neg Q_7) .$$

The elements of the set G are: $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$, Q_1, Q_2, Q_3, Q_4, $\neg Q_2 \vee \neg Q_5$, $\neg Q_6 \vee \neg Q_7$.
We obtain:

1) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (input)

2) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (input)

3) Q_1 (input)

4) Q_2 (input)

5) Q_3 (input)

6) Q_4 (input)

7) $\neg Q_2 \vee \neg Q_5$ (input)

8) $\neg Q_6 \vee \neg Q_7$ (input)

9) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 1) and 3))

10) $\neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 9) and 4))

11) $\neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 10) and 5))

12) $Q_5 \vee Q_7$ (resolvent from 11) and 6))

13) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 2) and 3))

14) $\neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 13) and 4))

15) $\neg Q_4 \vee Q_5 \vee Q_6$ (resolvent from 14) and 5))

16) $Q_5 \vee Q_6$ (resolvent from 15) and 6))

17) $\neg Q_5$ (resolvent from 7) and 4))

18) Q_7 (resolvent from 17) and 12))

19) Q_6 (resolvent from 16) and 17))

20) $\neg Q_7$ (resolvent from 8) and 19))

A refutation of $\neg c^{'}$ follows from 18) and 20).
(a) (iii) The proof is similar to the proofs of (i) and (ii).
(b) We have:

$$\begin{aligned} \mathcal{K}_1 &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow \\ &\quad ((Q_2 \wedge Q_4 \wedge Q_5 \wedge Q_6) \vee Q_7) , \\ \mathcal{K}_2 &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5 \wedge Q_6) \Rightarrow \\ &\quad (Q_3 \wedge Q_4 \wedge Q_6 \wedge Q_7) , \\ c^{'} &\equiv (Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4) \Rightarrow \\ &\quad (Q_3 \wedge Q_7) . \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{K}_1 &\equiv (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7) \wedge \\ &\quad (\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7) , \\ \mathcal{K}_2 &\equiv \big(\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee \neg Q_5 \vee \\ &\quad \neg Q_6 \vee Q_7\big), \\ \neg c^{'} &\equiv Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge (\neg Q_3 \vee \neg Q_7) . \end{aligned}$$

The elements of the set G are: $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$, $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee \neg Q_5 \vee \neg Q_6 \vee Q_7$, Q_1, Q_2, Q_3, Q_4, $\neg Q_3 \vee \neg Q_7$.
We deduce:

1) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (input)

2) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (input)

3) $\neg Q_1 \vee \neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee \neg Q_5 \vee \neg Q_6 \vee Q_7$ (input)

4) Q_1 (input)

5) Q_2 (input)

6) Q_3 (input)

7) Q_4 (input)

8) $\neg Q_3 \vee \neg Q_7$ (input)

9) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 1) and 4))

10) $\neg Q_3 \vee \neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 9) and 5))

11) $\neg Q_4 \vee Q_5 \vee Q_7$ (resolvent from 10) and 6))

12) $Q_5 \vee Q_7$ (resolvent from 11) and 7))

13) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 2) and 4))

14) $\neg Q_3 \vee \neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 13) and 5))

15) $\neg Q_4 \vee Q_6 \vee Q_7$ (resolvent from 14) and 6))

16) $Q_6 \vee Q_7$ (resolvent from 15) and 7))

17) $\neg Q_2 \vee \neg Q_3 \vee \neg Q_4 \vee \neg Q_5 \vee \neg Q_6 \vee Q_7$ (resolvent from 3) and 4))

18) $\neg Q_3 \vee \neg Q_4 \vee \neg Q_5 \vee \neg Q_6 \vee Q_7$ (resolvent from 17) and 5))

19) $\neg Q_4 \vee \neg Q_5 \vee \neg Q_6 \vee Q_7$ (resolvent from 18) and 6))

20) $\neg Q_5 \vee \neg Q_6 \vee Q_7$ (resolvent from 19) and 7))

21) $\neg Q_7$ (resolvent from 8) and 6))

22) Q_5 (resolvent from 21) and 12))

23) Q_6 (resolvent from 21) and 16))

24) $\neg Q_6 \vee Q_7$ (resolvent from 20) and 22))

25) Q_7 (resolvent from 24) and 23))

A refutation of $\neg c^{'}$ follows from 25) and 21).
This completes the proof. $\square$

It is worth to note that the authors in [12] first proposed the way of obtaining fuzzy implications from t-norms. In particular, the first fuzzy implication obtained in this way was the Yager fuzzy implication $\mathcal{I}_{YG}$.

Questions like: behavior of f-implications with respect to their distributivity over t-norms and t-conorms, the law of importation and contrapositive symmetry, algebraic properties of f- and g-generated fuzzy implications, description of their intersections with (S,N)-implications and R-implications, description of certain sub-families of such intersections, characterization of the intersections of f- and g-generated fuzzy implications with various families of fuzzy implications, partial characterization of such intersections, introducing of new classes of fuzzy implications, etc. are quite well understood in [8], [1] and [3].

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