A Short Note on the Asymptotically Stable Lagrange Solutions of 2-D Systems with Simultaneously Triangularizable Matrices

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Abstract: - This short note aims to investigate the asymptotic stability conditions for 2-d (two dimensional) discrete system whose state space representation is composed by matrices that can be simultaneously triangularizable. To accomplish it, the original system is transformed into a system with only triangular matrices. Then Lagrange solutions are assumed for this transformed system and conditions for asymptotic stability are sought.

Key-Words: - 2-d systems, simultaneously triangularizable matrices, asymptotic stability, Lagrange method, Lie algebra, Laffey's theorem

1 Introduction

Historically speaking, investigations of the stability of 2-d discrete systems from the engineering standpoint goes back to the 1960s [1] [2]. Then, the state space model as formalized in the ordinary automatic control theory was brought about in order to make it easier to carry out the analysis and synthesis of these kinds of systems [3] [4].

Mathematically, z-transform has been around since the early days and is still widely in use [5] – [8]. The energy method underlying the Lyapunov methods and LMIs came up in the 1980s and has been probed since then [9]- [10].

More recently, the relationship between the stability and eigenvalues of the matrices composing the state space model has been examined by Izuta on the basis of Lagrange method to solve partial difference equations [11]- [16]. The core reasoning in these papers has been finding a way to diagonalize the matrices composing the system, and then to figure out stability conditions by applying the Lagrange method on this this transformed system. Yet, in a very recent paper the authors studied this problem in the Lie algebra frame of reference in regard to simultaneously diagonalizable or triangularizable matrices [17]. It is worth noting that triangularizable matrices was investigated in connection with issues related to realization of systems [18].

Taking this background into account, this short note focus only on 2-d systems with simultaneously triangularizable matrices and probe the conditions for the systems to be asymptotically stable. The point here is that the simultaneously triangularization of matrices is not restricted to the standard Lie algebra as carried out in general; actually, this work considers Laffey's theorem, which covers a larger class of matrices beyond commutative ones.

Finally, this paper is organized as follows. Section 2 gives the definitions required in the sequel, section 3 provides some mathematical facts that are taken for granted to establish the results, and section 4 presents the results.

2 Preliminaries and Problem

In this section the kind of system handled in this paper and the definitions as well as the problem are presented. To begin with, let the system be given by the following definition.

Definition 1. Consider the 2-d discrete system be given by equation (1).

$$\overbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}}^{\mathbf{c}_{n\text{x}n}} \overbrace{\begin{bmatrix} x_1(i+1,j) \\ \vdots \\ x_k(i+1,j) \\ x_{k+1}(i,j+1) \\ \vdots \\ x_n(i,j+1) \end{bmatrix}}^{\mathbf{x}(i+1,j+1)} = \qquad (1)$$

$$\overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}}^{\mathbf{A}_{nxn}} \overbrace{\begin{bmatrix} x_1(i,j) \\ \vdots \\ x_k(i,j) \\ x_{k+1}(i,j) \\ \vdots \\ x_n(i,j)\end{bmatrix}}^{\mathbf{x}(i,j)}$$

$$+ \overbrace{\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn}\end{bmatrix}}^{\mathbf{B}_{nxn}} \overbrace{\begin{bmatrix} x_1(i-d_1,j) \\ \vdots \\ x_k(i-d_k,j) \\ x_{k+1}(i,j-d_{k+1} \\ \vdots \\ x_n(i,j-d_n)\end{bmatrix}}^{\mathbf{x}(i-1,j-1)}$$

in which the over-braced $\mathbf{x}(*, *)$, $\mathbf{A}_{nxn}$ and $\mathbf{B}_{nxn}$ are compact notations for the vectors and matrices, and the vectors in terms of x's represent the states of the system. Moreover, the matrices are all full ranked structures with real valued eigenvalues as well as real valued entries.

Next, we present the definition of asymptotic stability that is used throughout this paper and whose details can be found in [19].

Definition 2. System (1) is said to be asymptotically stable if its solutions $x_*(i, j)$'s (for all $_*$) vanish as the indices increase, namely the following conditions are satisfied.

$$\lim_{(i+j)\to\infty} |x_*(i,j)| \to 0 \text{, for all } * = 1, \ldots, n \quad (2)$$

In addition, the stability conditions are pursued on the grounds of the Lagrange method for solving partial difference equations. Basically, it consists in defining candidate solutions and then verifying that whether (2) is fulfilled.

Definition 3. A non-null Lagrange candidate solution is given by

$$x(i,j) = \sum_{k=1}^{n} I_k\, \alpha_k^i \beta_k^j \quad (3)$$

in which I_k's stand for the initial values and α_k's and β_k's are non-null real valued numbers.

Now the problem concerned with in this work is as following.

Problem 1. To investigate, on the basis of Lagrange method for solving partial difference equations, conditions under which there exists an asymptotically stable solution to (1) when A, B are simultaneously triangularizable in the sense of Laffey's theorem [18].

3 Mathematical Facts

In this section, we present for the sake of completeness of the paper what are the conditions for multiple matrices to be simultaneously triangularizable. The most well-known theory is in the scope of Lie algebra [20], which is defined as

Definition 4. Let the vector space of matrices $\mathcal{M}$ with bilinear transformation be defined by the operator $[*\ ,\ *]$: $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, in which **A** and **B** are two matrices of $\mathcal{M}$. Since $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$ and $[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = 0$, this algebra of matrices defines a Lie algebra $\mathcal{L}$.

Also, recalling the concept of solvability.

Definition 5. A Lie algebra $\mathcal{L}$ is said to be solvable if there exists an integer number k such that $\mathcal{L}^{(k)} = 0$ is satisfied for $\mathcal{L}^{(k)}$ defined recursively by the following expressions: $\mathcal{L}^{(1)} = [\mathcal{L}, \mathcal{L}]$, ..., and $\mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}]$.

This definition provides Lie's theorem, which is stated as next.

Proposition 1. The matrices of Lie algebra $\mathcal{L}$ are simultaneously triangularizable if and only if $\mathcal{L}$ is solvable.

A more general class of simultaneously triangularizable matrices is dealt with Laffey's theorem.

Proposition 2 (Laffey's theorem [21]). Let A_{nxn}, B_{nxn} be matrices over an algebraically closed field F of characteristic zero, and let C= AB - BA. Assume

(1) C has rank two, and

(2) $A^iB^jC^k$ is nilpotent for $0 \le i, j \le n-1$, $1 \le k < 2$.

Then A, B are simultaneously triangularizable over F.

In the sequel, we investigate the stability conditions in this framework.

4 Results

The main result is presented here.

Theorem 1. Let system (1) be such that matrices $\mathbf{A_{nxn}}$, $\mathbf{B_{nxn}}$ and $\mathbf{C_{nxn}}$ are simultaneously triangularizable. Furthermore, let the system obtained by means of similarity transformation be written in terms of variables z(i,j) with the vector represented by $\mathbf{z(i,j)}$, and whose triangular matrices are $\mathbf{U_A}$, $\mathbf{U_B}$ and $\mathbf{U_C}$. Then there exist asymptotically stable Lagrange solutions $\mathbf{z(i,j)}$ if there exist a non-null solution β_2 ($0 \neq |\beta_2| < 1$) for $\gamma_2\beta_2^{d_2+1} - \lambda_2\beta_2^{d_2} - \mu_2 = 0$, and at least one of the following conditions is fulfilled.

(1) $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. In addition, there exists a non-null real valued number α_1, $0 \neq |\alpha_1| < 1$, such that $\gamma_1\alpha_1^1 - \lambda_1 - \alpha_1^{-d_1} = \bar{a}_{12} + \bar{b}_{12}\beta_2^{-d_2} - \bar{c}_{12}\beta_2$.

(2) There exists an α_1 , $0 \neq |\alpha_1| < 1$, satisfying $\gamma_1\alpha_1^1 - \lambda_1 - \alpha_1^{-d_1}$ and the previously determined β_2 is such that $\bar{a}_{12} + \bar{b}_{12}\beta_2^{-d_2} - \bar{c}_{12}\beta_2$.

Proof. By proposition 1 and 2, matrices **A**, **B** and **C** are simultaneously triangularizable. Thus, there exists a matrix **T** such that

$$\mathbf{T^{-1}AT} = \mathbf{U_A}, \quad \text{and} \quad \mathbf{T^{-1}BT} = \mathbf{U_B}, \quad \text{and} \quad \mathbf{T^{-1}CT} = \mathbf{U_C} \tag{4}$$

in which $\mathbf{U_A}$, $\mathbf{U_B}$ and $\mathbf{U_C}$ are triangular matrices, which without loss of generality we assume here that they are upper triangular matrices with diagonal entries consisting of their eigenvalues $\lambda_1, \cdots, \lambda_n$; $\mu_1, \cdots, \mu_n$ and $\gamma_1, \cdots, \gamma_n$, respectively. It is also worth noting that $\mathbf{U_A}$, $\mathbf{U_B}$ and $\mathbf{U_C}$ are Jordan canonical forms when the matrices generate a solvable Lie algebra and these matrices have multiple eigenvalues.

Thus system (1) translates into

$$\begin{aligned}\mathbf{T^{-1}CTT^{-1}x(i+1, j+1)} = \\ \mathbf{T^{-1}ATT^{-1}x(i,j)} \\ + \mathbf{T^{-1}BTT^{-1}x(i-d, j-d)}.\end{aligned} \tag{5}$$

which by means of the transformation $\mathbf{z(i,j)} = \mathbf{T^{-1}x(i,j)}$, $\mathbf{z(i,j)}$ reads

$$\begin{aligned}\begin{bmatrix}\gamma_1 & \bar{c}_{12} \\ 0 & \gamma_2\end{bmatrix}\begin{bmatrix}z_1(i+1,j) \\ z_2(i,j+1)\end{bmatrix} = \begin{bmatrix}\lambda_1 & \bar{a}_{12} \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}z_1(i,j) \\ z_2(i,j)\end{bmatrix} \\ + \begin{bmatrix}\mu_1 & \bar{b}_{12} \\ 0 & \mu_2\end{bmatrix}\begin{bmatrix}z_1(i-d_1,j) \\ z_2(i,j-d_2)\end{bmatrix}.\end{aligned} \tag{6}$$

Now considering the Lagrange candidate solutions given by equation (3), (6) becomes.

$$\begin{aligned}\begin{bmatrix}\gamma_1 & \bar{c}_{12} \\ 0 & \gamma_2\end{bmatrix}\begin{bmatrix}\alpha_1^{i+1}\beta_1^j \\ \alpha_2^i\beta_2^{j+1}\end{bmatrix} = \begin{bmatrix}\lambda_1 & \bar{a}_{12} \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\alpha_1^i\beta_1^j \\ \alpha_2^i\beta_2^j\end{bmatrix} \\ + \begin{bmatrix}\mu_1 & \bar{b}_{12} \\ 0 & \mu_2\end{bmatrix}\begin{bmatrix}\alpha_1^{i-d_1}\beta_1^j \\ \alpha_2^i\beta_2^{j-d_2}\end{bmatrix},\end{aligned} \tag{7}$$

from which we obtain

$$\begin{aligned}\gamma_2\beta_2 - \lambda_2 - \mu_2\beta_2^{-d_2} = 0, \\ \text{and} \\ \alpha_1^i\beta_1^j\left(\gamma_1\alpha_1^1 - \lambda_1 - \alpha_1^{-d_1}\right) = \\ \alpha_2^i\beta_2^j(\bar{a}_{12} + \bar{b}_{12}\beta_2^{-d_2} - \bar{c}_{12}\beta_2).\end{aligned} \tag{8}$$

Note that the first equation in (8) provides the first part of the theorem since it indicates that as far as the absolute value of non-null solution β_2 is less than 1, we can establish an α_2 $(0 \neq |\alpha_1| < 1)$ in order to get an asymptotically stable Lagrange solution of $z_2(i,j)$. On the other hand, the second equation says that there are two possibilities of establishing the remaining solution: first, taking into account the fact that the equality must be valid for all of the indices *i* and *j*; second, requiring the terms independent of indices to be null. In any case, these give the claims of theorem.

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