Descriptor Approach in Design of Discrete-Time Fault Estimators

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Abstract: - Design conditions for linear discrete-time fault estimators are given in the paper, formulated by using the descriptor approach as a set of linear matrix inequalities in the bounded real lemma structures. Combining the state and the fault adapting equations, it is demonstrated that the conditions can be obtained for three basic fault estimator structures, when taking into design the constrain implying from the fault time-difference norm boundary. The result is illustrated by a numerical example.

Key-Words: - Discrete-time linear systems, state observers, discrete-time fault estimation, linear matrix inequalities, bounded real lemma.

1 Introduction

Capacity and behavior of real technological processes and systems are still exposed to possibility of failures. As well as limitations caused by faults and failures, control systems must be designed to operate within the acceptable range of system property degradation. A fault tolerant control (FTC) scheme allows to construct strategies which improve tolerance to many kind of faults. Since under special circumstances the fault estimation strategies are needed to realize a safe faulty system control, many sophisticated modifications of state-based fault estimators have been developed [6], [20]. These statements are supported by a number of bibliographical reviews [1], [10], giving a reader the actual practice state.

The principles based on adaptive observers are preferable used to estimate additive faults. The technique generally integrates the error between system and model output to estimate state and time-varying faults [18]. To construct fast enough adaptive observer algorithms, estimation error prediction by a derivative term is added in the observer structure [7], [8], [12]. The main design principle that is used for continuous-time fault estimation schemes is the Lyapunov function method. In the particular case of interest a parametric approach for the fault observer design is based on a solution of the generalized Sylvester matrix equation [16]. In this context, by application fields conditioned distinctions in the design conditions are given, e.g., in [2], [4], [5], [17], more theoretical details can be found in [11], [21]. It needs to be underlined that it is not possible to extrapolate formally the results obtained for linear continuous-time adaptive fault observers to discrete-time fault estimator structures. Exploiting the approaches presented in [9], [13], [15], the papers cover simultaneous principle for actuator fault estimation in descriptor systems [14].

Estimation of additive faults is analyzed in the paper at first, focusing mainly on the important advantages of discrete-time version of proportional-difference-integral (PDS) estimators. In contrast with continuous-time fault estimation schemes, the H_∞ norm constraint is established in design. Under assumption of the fault time-difference bounds, design procedure for the discrete-time linear PDS fault observer is provided and it is shown that the design can be formulated in terms of the LMI set in the Bounded Real Lemma (BRL) structure. Moreover, it is proven that from the design condition intended to discrete-time PDS fault observer, the reduced structures of linear matrix inequalities for the design of PD and P discrete-time fault estimators are constructible in the straight way. The aim is to assess the importance of using the discrete-time fault estimation update algorithm and the constraint implying from H_∞ norm od the fault estimator transfer function matrix.

Throughout the paper the following notation is used: $\boldsymbol{X} < 0$ conveys that a square matrix $\boldsymbol{X}$ is symmetric and negative definite, $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denote the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $\mathrm{diag}[\bullet]$ enters up a (block) diagonal matrix, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^n$, $\mathbb{R}^{n \times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

2 Fault State Estimator Structure

A linear discrete-time multi input, multi output (MIMO) system in presence of an unknown additive fault is described by the state-space equations in the following form

$$\boldsymbol{q}(i+1)=\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{G}\boldsymbol{u}(i)+\boldsymbol{H}\boldsymbol{f}(i)+\boldsymbol{D}\boldsymbol{d}(i) \tag{1}$$

$$\boldsymbol{y}(\mathrm{i})=\boldsymbol{C}\boldsymbol{q}(\mathrm{i}) \tag{2}$$

where $\boldsymbol{q}(i)\in\mathbb{R}^{n}$, $\boldsymbol{u}(i)\in\mathbb{R}^{r}$, $\boldsymbol{y}(i)\in\mathbb{R}^{m}$, $\boldsymbol{d}(i)\in\mathbb{R}^{p}$ are state, input, output and disturbance vectors, $\boldsymbol{F}\in\mathbb{R}^{n\times n}$, $\boldsymbol{G}\in\mathbb{R}^{n\times r}$, $\boldsymbol{C}\in\mathbb{R}^{p\times n}$, $\boldsymbol{D}\in\mathbb{R}^{n\times p}$ and $\boldsymbol{f}(i)\in\mathbb{R}^{s}$ is the unknown fault vector associated with the fault input matrix $\boldsymbol{H}\in\mathbb{R}^{n\times s}$.

In order to solve the fault estimation problem it is considered in the following that the couple $(\boldsymbol{F},\boldsymbol{C})$ is observable, the fault difference $\Delta\boldsymbol{f}(i)$ is bounded and the values of $\boldsymbol{f}(i)$ are zero in the fault-free regime.

In the following, the Luenberger PDS discrete-time state observer is considered in the form of the state-space equations

$$\begin{aligned}\boldsymbol{q}_{e}(i+1)\ &=\boldsymbol{F}\boldsymbol{q}_{e}(i)+\boldsymbol{G}\boldsymbol{u}(i)+\boldsymbol{H}\boldsymbol{f}_{e}(t)\\ &+\boldsymbol{J}(\boldsymbol{y}(i)-\boldsymbol{y}_{e}(i))\\ &+\boldsymbol{K}(\Delta\boldsymbol{y}(i)-\Delta\boldsymbol{y}_{e}(i))\\ &+\boldsymbol{L}\sum_{j=0}^{i-1}(\boldsymbol{y}(j)-\boldsymbol{y}_{e}(j))\end{aligned} \tag{3}$$

$$\boldsymbol{y}_{e}(\mathrm{i})=\boldsymbol{C}\boldsymbol{q}_{e}(\mathrm{i}) \tag{4}$$

where $\boldsymbol{q}_{e}(i)\in\mathbb{R}^{n}$ is the state of the of the observer, $\boldsymbol{f}_{e}(i)\in\mathbb{R}^{s}$ is an estimation of the unknown fault $\boldsymbol{f}(i)$, vector $\boldsymbol{y}_{e}(i)\in\mathbb{R}^{m}$ is the estimated output and $\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}\in\mathbb{R}^{n\times p}$ are the observer gain matrices to be designed.

To estimate additive faults, the observer equations (3), (4) are combined with the fault estimation update equation

$$\begin{aligned}\Delta\boldsymbol{f}_{e}(i)\ &=\boldsymbol{f}_{e}(i+1)-\boldsymbol{f}_{e}(i)\\ &=\boldsymbol{M}(\boldsymbol{y}(i)-\boldsymbol{y}_{e}(i))\\ &+\boldsymbol{N}(\Delta\boldsymbol{y}(i)-\Delta\boldsymbol{y}_{e}(i))\\ &+\boldsymbol{O}\sum_{j=0}^{i-1}(\boldsymbol{y}(i)-\boldsymbol{y}_{e}(i))\end{aligned} \tag{5}$$

where $\boldsymbol{M},\boldsymbol{N},\boldsymbol{O}\in\mathbb{R}^{s\times m}$ are the fault update model gain matrices to be other subjects of the design procedure.

The task is to design $\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}$ as well as $\boldsymbol{M},\boldsymbol{N},\boldsymbol{O}$ in such a way that the sequence $\boldsymbol{f}_{e}(i)$ approximates a time-varying additive fault $\boldsymbol{f}(i)$.

In order to apply the descriptor approach under such formulation the following properties are recalled.

Proposition 1: [16] *Considering an autono-mous linear descriptor system model of the form*

$$\boldsymbol{E}\boldsymbol{q}(i+1)=\boldsymbol{F}\boldsymbol{q}(i) \tag{6}$$

then the pair $(\boldsymbol{E},\boldsymbol{F})$ *is regular, causal and stable if and only if there exists an invertible symmetric matrix* $\boldsymbol{X}$ *of appropriate dimension that such that*

$$\boldsymbol{F}^{T}\boldsymbol{X}\boldsymbol{F}-\boldsymbol{E}^{T}\boldsymbol{X}\boldsymbol{E}<0 \tag{7}$$

$$\boldsymbol{E}^{T}\boldsymbol{X}\boldsymbol{E}\geq 0 \tag{8}$$

Proposition 2: [3] (quadratic performance) *If the system (1), (2) is stable and bounded then it yields*

$$\sum_{i=0}^{\infty}(\boldsymbol{y}^{T}(i)\boldsymbol{y}(i)-\gamma_{\infty}^{2}\boldsymbol{u}^{T}(i)\boldsymbol{u}(i))>0 \tag{9}$$

where $\gamma_{\infty}\in\mathbb{R}$ is the H_{∞} norm of the discrete-time system transfer function matrix.

Lemma 1: *The generalized descriptor form of the linear discrete-time observer description combi-ned with the fault estimation equations* (3), (4) *is*

$$\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i+1)=\boldsymbol{F}_{e}^{\circ}\boldsymbol{e}^{\circ}(i)+\boldsymbol{D}^{\circ}\boldsymbol{d}^{\circ}(i) \tag{10}$$

$$\boldsymbol{e}_{y}(i)=\boldsymbol{V}^{\circ}\boldsymbol{C}^{\circ}\boldsymbol{e}^{\circ}(i) \tag{11}$$

where

$$\begin{aligned}\boldsymbol{e}^{\circ T}(i)&=\begin{bmatrix}\boldsymbol{e}_{q}^{T}(i) & \boldsymbol{e}_{f}^{T}(i) & \boldsymbol{z}^{T}(i)\end{bmatrix}\\ \boldsymbol{d}^{\circ T}(i)&=\begin{bmatrix}\boldsymbol{d}^{T}(i) & \Delta\boldsymbol{f}^{T}(i)\end{bmatrix}\end{aligned} \tag{12}$$

$$\begin{aligned}\boldsymbol{E}^{\circ}&=\boldsymbol{I}^{\circ}+\boldsymbol{S}^{\circ}\boldsymbol{C}^{\circ}\\ \boldsymbol{I}^{\circ}&=\mathrm{diag}\begin{bmatrix}\boldsymbol{I}_{n} & \boldsymbol{I}_{s} & \boldsymbol{I}_{m}\end{bmatrix}\\ \boldsymbol{C}^{\circ}&=\mathrm{diag}\begin{bmatrix}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{I}_{m}\end{bmatrix}\\ \boldsymbol{S}^{\circ}&=\begin{bmatrix}\boldsymbol{K}^{\circ} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\\ \boldsymbol{K}^{\circ T}&=\begin{bmatrix}\boldsymbol{K}^{T} & \boldsymbol{N}^{T} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{F}_{e}^{\circ}&=\boldsymbol{F}^{\circ}-\boldsymbol{Q}^{\circ}\boldsymbol{C}^{\circ}\\ \boldsymbol{F}^{\circ}&=\begin{bmatrix}\boldsymbol{F} & \boldsymbol{H} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_{s} & \boldsymbol{0}\\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{I}_{m}\end{bmatrix}\\ \boldsymbol{Q}^{\circ}&=\begin{bmatrix}\boldsymbol{J}^{\circ}-\boldsymbol{K}^{\circ} & \boldsymbol{0} & \boldsymbol{L}^{\circ}\end{bmatrix}\\ \boldsymbol{J}^{\circ T}&=\begin{bmatrix}\boldsymbol{J}^{T} & \boldsymbol{M}^{T} & \boldsymbol{0}\end{bmatrix}\\ \boldsymbol{L}^{\circ T}&=\begin{bmatrix}\boldsymbol{L}^{T} & \boldsymbol{O} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{14}$$

$$\begin{aligned}\boldsymbol{D}^{\circ}&=\begin{bmatrix}\boldsymbol{D} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_{s}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\\ \boldsymbol{V}^{\circ}&=\begin{bmatrix}\boldsymbol{I}_{m} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{15}$$

while $\boldsymbol{F}^{\circ},\boldsymbol{F}_{e}^{\circ},\boldsymbol{E}^{\circ},\boldsymbol{C}^{\circ},\boldsymbol{I}^{\circ}\in\mathbb{R}^{(n+s+m)\times(n+s+m)}$ and likely $\boldsymbol{J}^{\circ},\boldsymbol{K}^{\circ},\boldsymbol{L}^{\circ}\in\mathbb{R}^{(n+s+m)\times m}$ and $\boldsymbol{D}^{\circ}\in\mathbb{R}^{(n+s+m)\times(p+s)}$.

Proof: To have a general enough approach to fault observer descriptions, the new vector variable $\boldsymbol{z}(i)$ is introduced as follows

$$\boldsymbol{z}(i)=\sum_{j=0}^{i-1}(\boldsymbol{y}(j)-\boldsymbol{y}_e(j)) \tag{16}$$

while, evidently, $\boldsymbol{z}(i)$ is a solution of the following difference equation

$$\begin{aligned}\boldsymbol{z}(i+1) &= \sum_{j=0}^{i}(\boldsymbol{y}(j)-\boldsymbol{y}_e(j))\\ &= \boldsymbol{z}(i)+\boldsymbol{y}(i)-\boldsymbol{y}_e(i)\end{aligned} \tag{17}$$

Therefore, (3) can be rewritten in an associated set of equations

$$\begin{aligned}\boldsymbol{q}_e(i+1) &= \boldsymbol{F}\boldsymbol{q}_e(i)+\boldsymbol{G}\boldsymbol{u}(i)+\boldsymbol{H}\boldsymbol{f}_e(t)\\ &+\boldsymbol{L}\boldsymbol{z}(i)+\boldsymbol{J}(\boldsymbol{y}(i)-\boldsymbol{y}_e(i))\\ &+\boldsymbol{K}(\Delta\boldsymbol{y}(i)-\Delta\boldsymbol{y}_e(i))\end{aligned} \tag{18}$$

$$\boldsymbol{z}(i+1)=\boldsymbol{z}(i)+\boldsymbol{y}(i)-\boldsymbol{y}_e(i) \tag{19}$$

Considering the following forward differences related with the system and observer output

$$\begin{aligned}\Delta\boldsymbol{y}(i) &= \boldsymbol{y}(i+1)-\boldsymbol{y}(i)\\ \Delta\boldsymbol{y}_e(i) &= \boldsymbol{y}_e(i+1)-\boldsymbol{y}_e(i)\end{aligned} \tag{20}$$

the equation (18) implies

$$\begin{aligned}\boldsymbol{q}_e(i+1) &= \boldsymbol{F}\boldsymbol{q}_e(i)+\boldsymbol{G}\boldsymbol{u}(i)+\boldsymbol{H}\boldsymbol{f}_e(t)\\ &+\boldsymbol{L}\boldsymbol{z}(i)+(\boldsymbol{J}-\boldsymbol{K})(\boldsymbol{y}(i)-\boldsymbol{y}_e(i))\\ &+\boldsymbol{K}(\boldsymbol{y}(i+1)-\boldsymbol{y}_e(i+1))\end{aligned} \tag{21}$$

Thus, introducing the set of estimation errors as

$$\begin{aligned}\boldsymbol{e}_q(i) &= \boldsymbol{q}(i)-\boldsymbol{q}_e(i)\\ \boldsymbol{e}_f(i) &= \boldsymbol{f}(i)-\boldsymbol{f}_e(i)\\ \boldsymbol{e}_y(i) &= \boldsymbol{C}\boldsymbol{e}_q(i)\end{aligned} \tag{22}$$

and subtracting (21) from the system state difference equation (1), after simple manipulations it is obtained

$$\begin{aligned}\boldsymbol{e}_q(i+1) &= \boldsymbol{H}\boldsymbol{e}_f(i)+\boldsymbol{D}\boldsymbol{d}(i)-\boldsymbol{L}\boldsymbol{z}(i)\\ &+(\boldsymbol{F}-(\boldsymbol{J}-\boldsymbol{K})\boldsymbol{C})\boldsymbol{e}_q(i)\\ &-\boldsymbol{K}\boldsymbol{C}\boldsymbol{e}_q(i+1)\end{aligned} \tag{23}$$

as well as

$$\boldsymbol{z}(i+1)=\boldsymbol{z}(i)+\boldsymbol{C}\boldsymbol{e}_q(i) \tag{24}$$

while (23), (24) define the equivalent equations of the fault state estimator.

Exploiting (22), it is easy to formulate that

$$\begin{aligned}\boldsymbol{e}_f(i+1) &= \boldsymbol{f}(i+1)-\boldsymbol{f}_e(i+1)+\boldsymbol{f}(i)-\boldsymbol{f}(i)\\ &+\boldsymbol{f}_e(i)-\boldsymbol{f}_e(i)\\ &= \boldsymbol{f}(i)-\boldsymbol{f}_e(i)-\Delta\boldsymbol{f}_e(i)+\Delta\boldsymbol{f}(i)\end{aligned} \tag{25}$$

where

$$\Delta\boldsymbol{f}(i)=\boldsymbol{f}(i+1)-\boldsymbol{f}(i) \tag{26}$$

is an unknown difference of the fault vector, and this difference has to be bounded.

Since, with the variable description (16), (17), the fault adapting equation (5) analogously gives

$$\Delta\boldsymbol{f}_e(i)=(\boldsymbol{M}-\boldsymbol{N})\boldsymbol{e}_y(i)+\boldsymbol{N}\boldsymbol{e}_y(i+1)+\boldsymbol{O}\boldsymbol{z}(i) \tag{27}$$

then, using the output relation from (22), it can obtain the equation corresponding to (27) in the form

$$\begin{aligned}\Delta\boldsymbol{f}_e(i) &= (\boldsymbol{M}-\boldsymbol{N})\boldsymbol{C}\boldsymbol{e}_q(i)\\ &+\boldsymbol{N}\boldsymbol{C}\boldsymbol{e}_q(i+1)+\boldsymbol{O}\boldsymbol{z}(i)\end{aligned} \tag{28}$$

Thus, substituting (28) into (25), the following relationship is resulted

$$\begin{aligned}\boldsymbol{e}_f(i+1) &= \boldsymbol{e}_f(i)+\Delta\boldsymbol{f}(i)-(\boldsymbol{M}-\boldsymbol{N})\boldsymbol{C}\boldsymbol{e}_q(i)\\ &-\boldsymbol{N}\boldsymbol{C}\boldsymbol{e}_q(i+1)-\boldsymbol{O}\boldsymbol{z}(i)\end{aligned} \tag{29}$$

Coupling the equations (23)-(25) and (29), the generalized fault observer equation form is obtained

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{I}_n+\boldsymbol{K}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{N}\boldsymbol{C} & \boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_q(i+1)\\ \boldsymbol{e}_f(i+1)\\ \boldsymbol{z}(i+1)\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{F}-(\boldsymbol{J}-\boldsymbol{K})\boldsymbol{C} & \boldsymbol{H} & -\boldsymbol{L}\\ -(\boldsymbol{M}-\boldsymbol{N})\boldsymbol{C} & \boldsymbol{I}_s & -\boldsymbol{O}\\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{I}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_q(i)\\ \boldsymbol{e}_f(i)\\ \boldsymbol{z}(i)\end{bmatrix}\\ &+\begin{bmatrix}\boldsymbol{D} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_s\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{d}(i)\\ \Delta\boldsymbol{f}(i)\end{bmatrix}\end{aligned} \tag{30}$$

$$\boldsymbol{e}_y(i)=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_q(i)\\ \boldsymbol{e}_f(i)\\ \boldsymbol{z}(i)\end{bmatrix} \tag{31}$$

It is clear from the equations (30), (31) that the block parameter matrices can be written as

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{F}-(\boldsymbol{J}-\boldsymbol{K})\boldsymbol{C} & \boldsymbol{H} & -\boldsymbol{L}\\ -(\boldsymbol{M}-\boldsymbol{N})\boldsymbol{C} & \boldsymbol{I}_s & -\boldsymbol{O}\\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{I}_m\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{F} & \boldsymbol{H} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{I}_m\end{bmatrix}\\ &-\begin{bmatrix}\boldsymbol{J}-\boldsymbol{K} & \boldsymbol{0} & \boldsymbol{L}\\ \boldsymbol{M}-\boldsymbol{N} & \boldsymbol{0} & \boldsymbol{O}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{C} & & \\ & \boldsymbol{0} & \\ & & \boldsymbol{I}_m\end{bmatrix}\end{aligned} \tag{32}$$

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{I}_n+\boldsymbol{K}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{N}\boldsymbol{C} & \boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{I}_n & & \\ & \boldsymbol{I}_n & \\ & & \boldsymbol{I}_n\end{bmatrix}\\ &+\begin{bmatrix}\boldsymbol{K} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{N} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{C} & & \\ & \boldsymbol{0} & \\ & & \boldsymbol{I}_m\end{bmatrix}\end{aligned} \tag{33}$$

Therefore, with the notations (12) then matrix structure (32), (33) gives (13), (14) and, consequently, the equations (30), (31) imply (10), (11), (15). This concludes the proof. ■

Corollary 1: *The matrix structures* (13)-(15) *do not allow application of structural matrix variables in LMI construction and, therefore, another structuring needs to be used. Introducing*

$$\begin{aligned}&\boldsymbol{F}^{\triangle}=\begin{bmatrix}\boldsymbol{F} & \boldsymbol{H}\\ \boldsymbol{0} & \boldsymbol{I}_s\end{bmatrix}\\ &\boldsymbol{C}^{\triangle}=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\\ &\boldsymbol{D}^{\triangle}=\begin{bmatrix}\boldsymbol{D} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_s\end{bmatrix}\\ &\boldsymbol{J}^{\triangle}=\begin{bmatrix}\boldsymbol{J}\\ \boldsymbol{M}\end{bmatrix}\\ &\boldsymbol{K}^{\triangle}=\begin{bmatrix}\boldsymbol{K}\\ \boldsymbol{N}\end{bmatrix}\\ &\boldsymbol{L}^{\triangle}=\begin{bmatrix}\boldsymbol{L}\\ \boldsymbol{O}\end{bmatrix}\end{aligned} \tag{34}$$

then

$$\begin{aligned}&\boldsymbol{E}^{\circ}=\boldsymbol{I}^{\circ}+\boldsymbol{S}^{\circ}\\ &\boldsymbol{I}^{\circ}=\operatorname{diag}\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{I}_s & \boldsymbol{I}_m\end{bmatrix}\\ &\boldsymbol{S}^{\circ}=\begin{bmatrix}\boldsymbol{K}^{\triangle}\boldsymbol{C}^{\triangle} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{35}$$

$$\begin{aligned}&\boldsymbol{F}_e^{\circ}=\boldsymbol{F}^{\circ}-\boldsymbol{Q}^{\circ}\\ &\boldsymbol{F}^{\circ}=\begin{bmatrix}\boldsymbol{F}^{\triangle} & \boldsymbol{0}\\ \boldsymbol{C}^{\triangle} & \boldsymbol{I}_m\end{bmatrix}\\ &\boldsymbol{Q}^{\circ}=\begin{bmatrix}(\boldsymbol{J}^{\triangle}-\boldsymbol{K}^{\triangle})\boldsymbol{C}^{\triangle} & \boldsymbol{L}^{\triangle}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{36}$$

$$\begin{aligned}&\boldsymbol{D}^{\circ}=\begin{bmatrix}\boldsymbol{D}^{\triangle}\\ \boldsymbol{0}\end{bmatrix}\\ &\boldsymbol{V}^{\circ}\boldsymbol{C}^{\circ}=\begin{bmatrix}\boldsymbol{C}^{\triangle} & \boldsymbol{0}\end{bmatrix}=\boldsymbol{C}^{\bullet}\end{aligned} \tag{37}$$

while, formally, (10), (11) *remain unchanged*.

The above defined structural matrix variables $\boldsymbol{J}^{\triangle},\boldsymbol{K}^{\triangle},\boldsymbol{L}^{\triangle}$ *can be applied into by LMIs given design conditions of discrete-time fault observers*.

Remark 1: *Because* $\boldsymbol{e}_q(i)$ *will always have a non-zero component caused by a disturbance seguence, its minimal impact on the dynamic property of the state observer can be reflected by including* H_∞ *norm of the disturbance transfer function matrix* $\boldsymbol{G}_d^{\circ}(z)$ *into design conditions using analogously the property (9) , i.e.,*

$$\sum_{i=0}^{\infty}(\boldsymbol{e}_y^T(i)\boldsymbol{e}_y(i)-\gamma_\infty^2\boldsymbol{d}^{\circ T}(i)\boldsymbol{d}^{\circ}(i))>0 \tag{38}$$

3 PDS Fault Estimator Design

In the above sense, the constructive proof is given for the following theorem to be immediately applied for synthesizing the parameters of PDS discrete-time fault estimator.

Theorem 1: *The PDS discrete-tim fault estimation structure is stable if there exist symmetric positive definite matrices* $\boldsymbol{P}_1^{\triangle}\in\mathbb{R}^{(n+s)\times(n+s)}$, $\boldsymbol{P}_2^{\triangle}\in\mathbb{R}^{m\times m}$, *matrices* $\boldsymbol{X}^{\triangle},\boldsymbol{Y}^{\triangle},\boldsymbol{Z}^{\triangle}\in\mathbb{R}^{(n+s)\times m}$ *and a positive scalar* $\gamma_\infty\in\mathbb{R}$ *and such that*

$$\begin{aligned}&\boldsymbol{P}_1^{\triangle}=\boldsymbol{P}_1^{\triangle T}>0\\ &\boldsymbol{P}_2^{\triangle}=\boldsymbol{P}_2^{\triangle T}>0\\ &\gamma_\infty>0\end{aligned} \tag{39}$$

$$\begin{bmatrix}\boldsymbol{\Xi}_{11} & *\\ \boldsymbol{\Xi}_{21} & \boldsymbol{\Xi}_{22}\end{bmatrix}<0 \tag{40}$$

where

$$\boldsymbol{\Xi}_{11}=\operatorname{diag}\begin{bmatrix}\boldsymbol{\Lambda}_{11} & -\boldsymbol{P}_2^{\triangle} & -\gamma_\infty\boldsymbol{I}_{p+s}\end{bmatrix} \tag{41}$$

$$\boldsymbol{\Xi}_{22}=\operatorname{diag}\begin{bmatrix}-\boldsymbol{P}_1^{\triangle} & -\boldsymbol{P}_2^{\triangle} & -\gamma_\infty\boldsymbol{I}_m & -\boldsymbol{P}_1^{\triangle}\end{bmatrix} \tag{42}$$

$$\boldsymbol{\Xi}_{21}=\begin{bmatrix}\boldsymbol{\Lambda}_{41} & \boldsymbol{Z}^{\triangle} & \boldsymbol{P}_1^{\triangle}\boldsymbol{D}^{\triangle}\\ \boldsymbol{P}_1^{\triangle}\boldsymbol{C}^{\triangle} & \boldsymbol{P}_2^{\triangle} & \boldsymbol{0}\\ \boldsymbol{C}^{\triangle} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{Y}^{\triangle}\boldsymbol{C}^{\triangle} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \tag{43}$$

$$\begin{aligned}&\boldsymbol{\Lambda}_{11}=-\boldsymbol{P}_1^{\triangle}-\boldsymbol{Y}^{\triangle}\boldsymbol{C}^{\triangle}-\boldsymbol{C}^{\triangle T}\boldsymbol{Y}^{\triangle T}\\ &\boldsymbol{\Lambda}_{41}=\boldsymbol{P}_1^{\triangle}\boldsymbol{F}^{\triangle}-\boldsymbol{X}^{\triangle}\boldsymbol{C}^{\triangle}+\boldsymbol{Y}^{\triangle}\boldsymbol{C}^{\triangle}\end{aligned} \tag{44}$$

When the above conditions hold

$$\begin{aligned}&\boldsymbol{J}^{\triangle}=(\boldsymbol{P}_1^{\triangle})^{-1}\boldsymbol{X}^{\triangle}\\ &\boldsymbol{K}^{\triangle}=(\boldsymbol{P}_1^{\triangle})^{-1}\boldsymbol{Y}^{\triangle}\\ &\boldsymbol{L}^{\triangle}=(\boldsymbol{P}_1^{\triangle})^{-1}\boldsymbol{Z}^{\triangle}\end{aligned} \tag{45}$$

while (34) *can be used to detach* $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{O}$ *and* $\boldsymbol{J}$, $\boldsymbol{K}$, $\boldsymbol{L}$, *respectively*.

Hereafter, $*$ *denotes the symmetric item in a symmetric matrix.*

Proof: Considering the discrete-time factors involved in the descriptor form of the design conditions (10), (11), $e_y(\mathrm{i})\neq 0$ for all i and benefit of the H_∞ norm constraint, the Lyapunov function is constructed as

$$\mathrm{v}(\boldsymbol{e}^{\circ}(i))=\boldsymbol{e}^{\circ T}(i)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i)+\mathrm{h}(i)>0 \tag{46}$$

where (38) implies

$$h(i)=\gamma_\infty^{-1}\sum_{i=0}^{i-1}(\boldsymbol{e}_y^T(i)\boldsymbol{e}_y(i)-\gamma_\infty^2\boldsymbol{d}^{\circ T}(i)\boldsymbol{d}^{\circ}(i))>0 \tag{47}$$

and $\gamma_\infty\in\mathbb{R}$ is the H_∞ norm of $\boldsymbol{G}_d^{\circ}(z)$.

Then it has to yield for an observer trajectory

$$\begin{aligned}\Delta\, \mathrm{v}(\boldsymbol{e}^{\circ}(i)) &= \boldsymbol{e}^{\circ T}(i+1)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}e^{\circ}(i+1) \\ &+ \gamma_{\infty}^{-1}\boldsymbol{e}_y^T(i)\boldsymbol{e}_y(i) - \gamma_{\infty}\boldsymbol{d}^{\circ T}(i)\boldsymbol{d}^{\circ}(i) \\ &- \boldsymbol{e}^{\circ T}(i)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i) < 0\end{aligned} \tag{48}$$

and, for the sake of completeness, it follows by solving with the output relation (11)

$$\begin{aligned}\Delta\, \mathrm{v}(\boldsymbol{e}^{\circ}(i)) &= \boldsymbol{e}^{\circ T}(i+1)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i+1) \\ &- \gamma_{\infty}\boldsymbol{d}^{\circ T}(i)\boldsymbol{d}^{\circ}(i) \\ &+ \gamma_{\infty}^{-1}\boldsymbol{e}^{\circ T}(i)\boldsymbol{C}^{\circ T}\boldsymbol{V}^{\circ T}\boldsymbol{V}^{\circ}\boldsymbol{C}^{\circ}\boldsymbol{e}^{\circ}(i) \\ &- \boldsymbol{e}^{\circ T}(i)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i) < 0\end{aligned} \tag{49}$$

Rendering the first element of the inequality (49) in the form

$$\begin{aligned}&\boldsymbol{e}^{\circ T}(i+1)\boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ}\boldsymbol{e}^{\circ}(i+1) \\ &= \boldsymbol{e}^{\circ T}(i)\boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ}\boldsymbol{e}^{\circ}(i) \\ &+ \boldsymbol{d}^{\circ T}(i)\boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ}\boldsymbol{d}^{\circ}(i) \\ &+ \boldsymbol{d}^{\circ T}(i)\boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ}\boldsymbol{e}^{\circ}(i) \\ &+ \boldsymbol{e}^{\circ T}(i)\boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ}\boldsymbol{d}^{\circ}(i)\end{aligned} \tag{50}$$

then, combining the vector variable into the composed variable vector

$$\boldsymbol{e}_o^{\circ T}(i) = \begin{bmatrix}\boldsymbol{e}^{\circ T}(i) & \boldsymbol{d}^{\circ T}(i)\end{bmatrix} \tag{51}$$

the inequality (49) can be interpreted equally as follows

$$\Delta\, \mathrm{v}(\boldsymbol{e}_o^{\circ}(i)) = \boldsymbol{e}_o^{\circ T}(i)\boldsymbol{P}_o^{\circ}\boldsymbol{e}_o^{\circ}(i) < 0 \tag{52}$$

where

$$\boldsymbol{P}_o^{\circ} = \begin{bmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} \\ \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & -\gamma\boldsymbol{I}_{p+s} + \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ}\end{bmatrix} < 0 \tag{53}$$

$$\boldsymbol{\Theta}_{11} = \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} + \boldsymbol{\Phi}_{11} \tag{54}$$

$$\boldsymbol{\Phi}_{11} = \gamma_{\infty}^{-1}\boldsymbol{C}^{\bullet T}\boldsymbol{C}^{\bullet} - \boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ} \tag{55}$$

To eliminate bilinear products, the following way is realized to separate the matrix inequality (53) in two matrix components as

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{\Phi}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{p+s}\end{bmatrix} \\ &+ \begin{bmatrix}\boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} \\ \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ}\end{bmatrix} < 0\end{aligned} \tag{56}$$

Since, according to the structure of the second matrix in the inequality (56), it can write

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} \\ \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ}\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{F}_e^{\circ T} \\ \boldsymbol{D}^{\circ T}\end{bmatrix}\boldsymbol{P}^{\circ}\begin{bmatrix}\boldsymbol{F}_e^{\circ} & \boldsymbol{D}^{\circ}\end{bmatrix}\end{aligned} \tag{57}$$

using the Schur complement property the matrix inequality (56) can be rearranged as

$$\begin{bmatrix}\boldsymbol{\Phi}_{11} & \boldsymbol{0} & \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ} \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{p+s} & \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ} \\ \boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} & -\boldsymbol{P}^{\circ}\end{bmatrix} < 0 \tag{58}$$

In the same way, applying the following matrix structure equivalency

$$\begin{aligned}&\begin{bmatrix}\gamma_{\infty}^{-1}\boldsymbol{C}^{\bullet T}\boldsymbol{C}^{\bullet} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{C}^{\bullet T} \\ \boldsymbol{0} \\ \boldsymbol{0}\end{bmatrix}\gamma_{\infty}^{-1}\boldsymbol{I}_m\begin{bmatrix}\boldsymbol{C}^{\bullet} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{59}$$

the matrix inequality (58) can be configured using the Schur complement property as

$$\begin{bmatrix}-\boldsymbol{\Omega}_{11} & \boldsymbol{0} & \boldsymbol{F}_e^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{C}^{\bullet T} \\ \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{p+s} & \boldsymbol{D}^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{0} \\ \boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} & \boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} & -\boldsymbol{P}^{\circ} & \boldsymbol{0} \\ \boldsymbol{C}^{\bullet} & \boldsymbol{0} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m\end{bmatrix} < 0 \tag{60}$$

where

$$\boldsymbol{\Omega}_{11} = \boldsymbol{E}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{E}^{\circ} \tag{61}$$

To apply the above defined without further restrictions, and with a symmetric positive definite matrix $\boldsymbol{P}^{\circ}$, it is obvious that

$$\boldsymbol{\Omega}_{11} = \boldsymbol{P}^{\circ} + \boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ} \tag{62}$$

$$\boldsymbol{P}^{\circ}\boldsymbol{F}_e^{\circ} = \boldsymbol{P}^{\circ}\boldsymbol{F}^{\circ} - \boldsymbol{P}^{\circ}\boldsymbol{Q}^{\circ} \tag{63}$$

and, to eliminate the bilinear element $\boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ}$ in (62), it can write with respect to its position in (60)

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{R}^{\circ} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0}\end{bmatrix}(\boldsymbol{P}^{\circ})^{-1}\begin{bmatrix}\boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{64}$$

Thus, applying the Schur complement property, the following inequality is obtained

$$\begin{bmatrix}\boldsymbol{\Pi}_{11} & * & * & * & * \\ 0 & -\gamma_{\infty}\boldsymbol{I}_{p+s} & * & * & * \\ \boldsymbol{\Pi}_{31} & \boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} & -\boldsymbol{P}^{\circ} & * & * \\ \boldsymbol{C}^{\bullet} & \boldsymbol{0} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m & * \\ \boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P}^{\circ}\end{bmatrix} < 0 \tag{65}$$

where

$$\begin{aligned}\boldsymbol{\Pi}_{11} &= -\boldsymbol{P}^{\circ} - \boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} - \boldsymbol{S}^{\circ T}\boldsymbol{P}^{\circ} \\ \boldsymbol{\Pi}_{31} &= \boldsymbol{P}^{\circ}\boldsymbol{F}^{\circ} - \boldsymbol{P}^{\circ}\boldsymbol{Q}^{\circ}\end{aligned} \tag{66}$$

Considering the block diagonal structure of the matrix $\boldsymbol{P}^{\circ}$ with the symmetric positive definite matrices $\boldsymbol{P}_1^{\vartriangle}$, $\boldsymbol{P}_1^{\vartriangle}$ as follows

$$\boldsymbol{P}^{\circ} = \operatorname{diag}\begin{bmatrix} \boldsymbol{P}_1^{\vartriangle} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \tag{67}$$

it can write with respect to the existing matrix block dimensions

$$\begin{aligned} \boldsymbol{P}^{\circ}\boldsymbol{S}^{\circ} &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle}\boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{68}$$

$$\begin{aligned} \boldsymbol{P}^{\circ}\boldsymbol{Q}^{\circ} &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \begin{bmatrix} (\boldsymbol{J}^{\vartriangle} - \boldsymbol{K}^{\vartriangle})\boldsymbol{C}^{\vartriangle} & \boldsymbol{L}^{\vartriangle} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle}\boldsymbol{J}^{\vartriangle}\boldsymbol{C}^{\vartriangle} - \boldsymbol{P}_1^{\vartriangle}\boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{P}_1^{\vartriangle}\boldsymbol{L}^{\vartriangle} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{69}$$

$$\begin{aligned} \boldsymbol{P}^{\circ}\boldsymbol{F}^{\circ} &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{C}^{\vartriangle} & \boldsymbol{I}_m \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle}\boldsymbol{F}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{P}_2^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \end{aligned} \tag{70}$$

$$\begin{aligned} \boldsymbol{P}^{\circ}\boldsymbol{D}^{\circ} &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}^{\vartriangle} \\ \boldsymbol{0} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle}\boldsymbol{D}^{\vartriangle} \\ \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{71}$$

Then, using (68)-(70) and applying the matrix variable substitutions

$$\begin{aligned} \boldsymbol{X}^{\vartriangle} &= \boldsymbol{P}_1^{\vartriangle}\boldsymbol{J}^{\vartriangle} \\ \boldsymbol{Y}^{\vartriangle} &= \boldsymbol{P}_1^{\vartriangle}\boldsymbol{K}^{\vartriangle} \\ \boldsymbol{Z}^{\vartriangle} &= \boldsymbol{P}_1^{\vartriangle}\boldsymbol{L}^{\vartriangle} \end{aligned} \tag{72}$$

the relations in (66) takes the structures of the block matrices

$$\begin{aligned} \boldsymbol{\Pi}_{11} &= \begin{bmatrix} -\boldsymbol{P}_1^{\vartriangle} - \boldsymbol{Y}^{\vartriangle}\boldsymbol{C}^{\vartriangle} - \boldsymbol{C}^{\vartriangle T}\boldsymbol{Y}^{\vartriangle T} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \\ \boldsymbol{\Pi}_{31} &= \begin{bmatrix} \boldsymbol{P}_1^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{X}^{\vartriangle}\boldsymbol{C}^{\vartriangle} + \boldsymbol{Y}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{Z}^{\vartriangle} \\ \boldsymbol{P}_2^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{P}_2^{\vartriangle} \end{bmatrix} \end{aligned} \tag{73}$$

Substituting (68), the the last row (column) block of (65) is zero matrix, except the main diagonal element $-\boldsymbol{P}_2^{\vartriangle}$. Evidently, they can be eliminated and so, with appropriately substitution of the rest abobe defined blocks, (65) gives (40). This concludes the proof. ■

It is obvious that (39), (40) imply a stable PDS discrete-time fault estimator. Thus, in terms of Lyapunov stability, both the state and fault estimation errors asymptotically converge to the associate equilibria.

4 PD Fault Estimator Design

Constructing the Luenberger PD discrete-time fault observer, (3)-(5) reduce to the equations

$$\begin{aligned} \boldsymbol{q}_e(i+1) &= \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{H}\boldsymbol{f}_e(t) \\ &\quad + \boldsymbol{J}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)) \\ &\quad + \boldsymbol{K}(\Delta\boldsymbol{y}(i) - \Delta\boldsymbol{y}_e(i)) \end{aligned} \tag{74}$$

$$\boldsymbol{y}_e(\mathrm{i}) = \boldsymbol{C}\boldsymbol{q}_e(\mathrm{i}) \tag{75}$$

$$\begin{aligned} \Delta\boldsymbol{f}_e(i) &= \boldsymbol{f}_e(i+1) - \boldsymbol{f}_e(i) \\ &= \boldsymbol{M}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)) \\ &\quad + \boldsymbol{N}(\Delta\boldsymbol{y}(i) - \Delta\boldsymbol{y}_e(i)) \end{aligned} \tag{76}$$

Consequently, this reduction leads to that (23), (29) take the following forms

$$\begin{aligned} \boldsymbol{e}_q(i+1) &= \boldsymbol{H}\boldsymbol{e}_f(i) + \boldsymbol{D}\boldsymbol{d}(i) \\ &\quad + (\boldsymbol{F} - (\boldsymbol{J} - \boldsymbol{K})\boldsymbol{C})\boldsymbol{e}_q(i) \\ &\quad - \boldsymbol{K}\boldsymbol{C}\boldsymbol{e}_q(i+1) \end{aligned} \tag{77}$$

$$\begin{aligned} \boldsymbol{e}_f(i+1) &= \boldsymbol{e}_f(i) + \Delta\boldsymbol{f}(i) \\ &\quad - (\boldsymbol{M} - \boldsymbol{N})\boldsymbol{C}\boldsymbol{e}_q(i) \\ &\quad - \boldsymbol{N}\boldsymbol{C}\boldsymbol{e}_q(i+1) \end{aligned} \tag{78}$$

and (30), (31) are modified as

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{I}_n + \boldsymbol{K}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{N}\boldsymbol{C} & \boldsymbol{I}_s \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q(i+1) \\ \boldsymbol{e}_f(i+1) \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{F} - (\boldsymbol{J} - \boldsymbol{K})\boldsymbol{C} & \boldsymbol{H} \\ -(\boldsymbol{M} - \boldsymbol{N})\boldsymbol{C} & \boldsymbol{I}_s \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q(i) \\ \boldsymbol{e}_f(i) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_s \end{bmatrix} \begin{bmatrix} \boldsymbol{d}(i) \\ \Delta\boldsymbol{f}(i) \end{bmatrix} \end{aligned} \tag{79}$$

$$\boldsymbol{e}_y(i) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q(i) \\ \boldsymbol{e}_f(i) \end{bmatrix} \tag{80}$$

Thus, it can write immediately for (79), (80)

$$\boldsymbol{E}^{\vartriangle}\boldsymbol{e}^{\vartriangle}(i+1) = \boldsymbol{F}_e^{\vartriangle}\boldsymbol{e}^{\vartriangle}(i) + \boldsymbol{D}^{\vartriangle}\boldsymbol{d}^{\vartriangle}(i) \tag{81}$$

$$\boldsymbol{e}_y(i) = \boldsymbol{C}^{\vartriangle}\boldsymbol{e}^{\vartriangle}(i) \tag{82}$$

where the matrix parameters $\boldsymbol{F}^{\vartriangle}$, $\boldsymbol{C}^{\vartriangle}$, $\boldsymbol{D}^{\vartriangle}$, $\boldsymbol{J}^{\vartriangle}$, $\boldsymbol{K}^{\vartriangle}$ are given in (34) and

$$\begin{aligned} \boldsymbol{e}^{\vartriangle T}(i) &= \begin{bmatrix} \boldsymbol{e}_q^T(i) & \boldsymbol{e}_f^T(i) \end{bmatrix} \\ \boldsymbol{d}^{\vartriangle T}(i) &= \begin{bmatrix} \boldsymbol{d}^T(i) & \Delta\boldsymbol{f}^T(i) \end{bmatrix} \end{aligned} \tag{83}$$

$$\begin{aligned} \boldsymbol{E}^{\vartriangle} &= \boldsymbol{I}^{\vartriangle} + \boldsymbol{S}^{\vartriangle} \\ \boldsymbol{I}^{\vartriangle} &= \operatorname{diag}\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{I}_s \end{bmatrix} \\ \boldsymbol{S}^{\vartriangle} &= \boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \end{aligned} \tag{84}$$

$$\begin{aligned} \boldsymbol{F}_e^{\vartriangle} &= \boldsymbol{F}^{\vartriangle} - \boldsymbol{Q}^{\vartriangle} \\ \boldsymbol{Q}^{\vartriangle} &= (\boldsymbol{J}^{\vartriangle} - \boldsymbol{K}^{\vartriangle})\boldsymbol{C}^{\vartriangle} \end{aligned} \tag{85}$$

Evidently, due to the specified structure of the Lyapunov matrix $\boldsymbol{P}^{\vartriangle}$, design condition formulation results also in this case in a linear LMI problem.

Theorem 2: *The PD discrete-tim fault estimation structure is stable if there exist symmetric positive definite matrix* $\boldsymbol{P}^{\vartriangle} \in \mathbb{R}^{(n+s)\times(n+s)}$, *a positive scalar* $\gamma_{\infty} \in \mathbb{R}$ *and matrices* $\boldsymbol{X}^{\vartriangle}, \boldsymbol{Y}^{\vartriangle} \in \mathbb{R}^{(n+s)\times m}$ *such that*

$$\begin{aligned} &\boldsymbol{P}^{\vartriangle} = \boldsymbol{P}^{\vartriangle T} > 0 \\ &\gamma_{\infty} > 0 \end{aligned} \tag{86}$$

$$\begin{bmatrix} \boldsymbol{\Sigma}_{11} & * \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} < 0 \tag{87}$$

where

$$\boldsymbol{\Sigma}_{11} = \mathrm{diag}\left[\boldsymbol{\Psi}_{11}^{\vartriangle} \;\; -\gamma_{\infty}\boldsymbol{I}_{p+s}\right] \tag{88}$$

$$\boldsymbol{\Sigma}_{22} = \mathrm{diag}\left[-\boldsymbol{P}^{\vartriangle} \;\; -\gamma_{\infty}\boldsymbol{I}_{m} \;\; -\boldsymbol{P}^{\vartriangle}\right] \tag{89}$$

$$\boldsymbol{\Sigma}_{21} = \begin{bmatrix} \boldsymbol{\Psi}_{31}^{\vartriangle} & \boldsymbol{P}_1^{\vartriangle}\boldsymbol{D}^{\vartriangle} \\ \boldsymbol{C}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{Y}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{0} \end{bmatrix} \tag{90}$$

$$\begin{aligned} &\boldsymbol{\Psi}_{11}^{\vartriangle} = -\boldsymbol{P}^{\vartriangle} - \boldsymbol{Y}^{\vartriangle}\boldsymbol{C}^{\vartriangle} - \boldsymbol{C}^{\vartriangle T}\boldsymbol{Y}^{\vartriangle T} \\ &\boldsymbol{\Psi}_{31}^{\vartriangle} = \boldsymbol{P}_1^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{X}^{\vartriangle}\boldsymbol{C}^{\vartriangle} + \boldsymbol{Y}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \end{aligned} \tag{91}$$

When the above conditions hold

$$\begin{aligned} &\boldsymbol{J}^{\vartriangle} = (\boldsymbol{P}^{\vartriangle})^{-1}\boldsymbol{X}^{\vartriangle} \\ &\boldsymbol{K}^{\vartriangle} = (\boldsymbol{P}^{\vartriangle})^{-1}\boldsymbol{Y}^{\vartriangle} \end{aligned} \tag{92}$$

while (34) *can be used to detach* $\boldsymbol{M}$, $\boldsymbol{N}$ *and* $\boldsymbol{J}$, $\boldsymbol{K}$, *respectively*.

Proof: The proof is similar to that of Theorem 2. Defining analogously the Lyapunov function as

$$\mathrm{v}(\boldsymbol{e}^{\vartriangle}(i)) = \boldsymbol{e}^{\vartriangle T}(i)\boldsymbol{E}^{\vartriangle T}\boldsymbol{P}^{\vartriangle}\boldsymbol{E}^{\vartriangle}\boldsymbol{e}^{\vartriangle}(i) + \mathrm{h}(i) > 0 \tag{93}$$

where $\mathrm{h}(i)$ is the same as in (47) since $\boldsymbol{d}^{\circ}(i) = \boldsymbol{d}^{\vartriangle}(i)$, then following the way above, the appropriate modifications of (60)-(63) are

$$\begin{bmatrix} -\boldsymbol{\Omega}_{11}^{\vartriangle} & \boldsymbol{0} & \boldsymbol{F}_e^{\vartriangle T}\boldsymbol{P}^{\vartriangle} & \boldsymbol{C}^{\vartriangle T} \\ \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{p+s} & \boldsymbol{D}^{\vartriangle T}\boldsymbol{P}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{P}^{\vartriangle}\boldsymbol{F}_e^{\vartriangle} & \boldsymbol{P}^{\vartriangle}\boldsymbol{D}^{\vartriangle} & -\boldsymbol{P}^{\vartriangle} & \boldsymbol{0} \\ \boldsymbol{C}^{\vartriangle} & \boldsymbol{0} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m \end{bmatrix} < 0 \tag{94}$$

where

$$\boldsymbol{\Omega}_{11}^{\vartriangle} = \boldsymbol{P}^{\vartriangle} + \boldsymbol{S}^{\vartriangle T}\boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} + \boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} + \boldsymbol{S}^{\vartriangle T}\boldsymbol{P}^{\vartriangle} \tag{95}$$

$$\boldsymbol{P}^{\vartriangle}\boldsymbol{F}_e^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{P}^{\vartriangle}\boldsymbol{Q}^{\vartriangle} \tag{96}$$

To eliminate the bilinear element $\boldsymbol{S}^{\vartriangle T}\boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle}$ in (95) by using the Shur complement property, it can rewrite (94) as

$$\begin{bmatrix} \boldsymbol{\Psi}_{11}^{\vartriangle} & * & * & * & * \\ \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{p+s} & * & * & * \\ \boldsymbol{\Psi}_{31}^{\vartriangle} & \boldsymbol{P}^{\vartriangle}\boldsymbol{D}^{\vartriangle} & -\boldsymbol{P}^{\vartriangle} & * & * \\ \boldsymbol{C}^{\vartriangle} & \boldsymbol{0} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m & * \\ \boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P}^{\vartriangle} \end{bmatrix} < 0 \tag{97}$$

where

$$\begin{aligned} &\boldsymbol{\Psi}_{11}^{\vartriangle} = -\boldsymbol{P}^{\vartriangle} - \boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} - \boldsymbol{S}^{\vartriangle T}\boldsymbol{P}^{\vartriangle} \\ &\boldsymbol{\Psi}_{31}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{P}^{\vartriangle}\boldsymbol{Q}^{\vartriangle} \end{aligned} \tag{98}$$

Since it can write for the following matrix elements

$$\boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \tag{99}$$

$$\boldsymbol{P}^{\vartriangle}\boldsymbol{Q}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{J}^{\vartriangle}\boldsymbol{C}^{\vartriangle} - \boldsymbol{P}^{\vartriangle}\boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \tag{100}$$

applying the notations

$$\begin{aligned} &\boldsymbol{X}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{J}^{\vartriangle} \\ &\boldsymbol{Y}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{K}^{\vartriangle} \end{aligned} \tag{101}$$

then with (91) the inequality (97) implies (87). This concludes the proof. ■

Corollary 2: *Considering the proportional (P) discrete-time fault estimator, then* (74)-(76) *imply*

$$\begin{aligned} \boldsymbol{q}_e(i+1) &= \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{H}\boldsymbol{f}_e(t) \\ &\quad + \boldsymbol{J}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)) \end{aligned} \tag{102}$$

$$\boldsymbol{y}_e(\mathrm{i}) = \boldsymbol{C}\boldsymbol{q}_e(\mathrm{i}) \tag{103}$$

$$\begin{aligned} \Delta\boldsymbol{f}_e(i) &= \boldsymbol{f}_e(i+1) - \boldsymbol{f}_e(i) \\ &= \boldsymbol{M}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)) \end{aligned} \tag{104}$$

and setting $\boldsymbol{Y}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{K}^{\vartriangle} = \boldsymbol{0}$ *then* (98)-(100) *imply*

$$\boldsymbol{P}^{\vartriangle}\boldsymbol{S}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{K}^{\vartriangle}\boldsymbol{C}^{\vartriangle} = \boldsymbol{0} \tag{105}$$

$$\boldsymbol{P}^{\vartriangle}\boldsymbol{Q}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{J}^{\vartriangle}\boldsymbol{C}^{\vartriangle} = \boldsymbol{X}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \tag{106}$$

$$\begin{aligned} &\boldsymbol{\Psi}_{11}^{\vartriangle} = -\boldsymbol{P}^{\vartriangle} \\ &\boldsymbol{\Psi}_{31}^{\vartriangle} = \boldsymbol{P}^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{X}^{\vartriangle}\boldsymbol{C}^{\vartriangle} \end{aligned} \tag{107}$$

Since under this setting the last row (column) block of the related inequality to (97) is zero matrix, except the main diagonal block $-\boldsymbol{P}^{\vartriangle}$, *they can be eliminated and, consequently*:

The P-type discrete-tim fault estimation structure is stable if there exist symmetric positive definite matrix $\boldsymbol{P}^{\vartriangle} \in \mathbb{R}^{(n+s)\times(n+s)}$, *a positive scalar* $\gamma_{\infty} \in \mathbb{R}$ *and a matrix* $\boldsymbol{X}^{\vartriangle} \in \mathbb{R}^{(n+s)\times m}$ *such that*

$$\begin{aligned} &\boldsymbol{P}^{\vartriangle} = \boldsymbol{P}^{\vartriangle T} > 0 \\ &\gamma_{\infty} > 0 \end{aligned} \tag{108}$$

$$\begin{bmatrix} -\boldsymbol{P}^{\vartriangle} & * & * & * \\ \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{p+s} & * & * \\ \boldsymbol{P}^{\vartriangle}\boldsymbol{F}^{\vartriangle} - \boldsymbol{X}^{\vartriangle}\boldsymbol{C}^{\vartriangle} & \boldsymbol{P}^{\vartriangle}\boldsymbol{D}^{\vartriangle} & -\boldsymbol{P}^{\vartriangle} & * \\ \boldsymbol{C}^{\vartriangle} & \boldsymbol{0} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_m \end{bmatrix} < 0 \tag{109}$$

When the above conditions hold

$$\boldsymbol{J}^{\vartriangle} = (\boldsymbol{P}^{\vartriangle})^{-1}\boldsymbol{X}^{\vartriangle} \tag{110}$$

while (34) *can be used to detach* $\boldsymbol{M}$ *and* $\boldsymbol{J}$.

Note, LMI of the structure (109) is the construction of BRL for P-type discrete-time fault estimator design. In the same sense, (40), (87) are BRLs for PID and PD discrete-time fault estimator design.

5 Illustrative Example

In the example, the input-output dynamics of the system (1), (2) is given by the state-space representation with the parameters

$$\boldsymbol{F}=\begin{bmatrix}1.0620 & -0.0045 & 0.2381 & -0.1961\\ -0.0220 & 0.8444 & -0.0021 & 0.0261\\ 0.0370 & 0.1546 & 0.7762 & 0.1967\\ 0.0011 & 0.1547 & 0.0454 & 0.9267\end{bmatrix}$$

$$\boldsymbol{G}=\begin{bmatrix}0.0003 & -0.0156\\ 0.2095 & 0.0001\\ 0.0630 & -0.1109\\ 0.0630 & -0.0030\end{bmatrix},\quad \boldsymbol{D}=\begin{bmatrix}0.0380\\ 0.0696\\ 0.2317\\ 0.0000\end{bmatrix}$$

$$\boldsymbol{C}=\begin{bmatrix}4 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad T_s=0.04s$$

while, for the actuator single faults, $\boldsymbol{G}=\boldsymbol{H}$.

Solving the inequalities (38), (39) using SeDuMi package for MATLAB, the design task is feasible and the LMIs matrix variables are

$$\boldsymbol{P}_1^{\vartriangle}=\begin{bmatrix}6.0360 & -0.3335 & 0.1372 & 0.1506 & -0.0826 & 0.8627\\ -0.3335 & 3.8222 & -0.6139 & -1.0472 & -1.2670 & -0.3541\\ 0.1372 & -0.6139 & 4.0746 & -0.5816 & -0.3917 & 1.1188\\ 0.1506 & -1.0472 & -0.5816 & 5.9277 & -1.0613 & -0.3412\\ -0.0826 & -1.2670 & -0.3917 & -1.0613 & 1.8110 & -0.0619\\ 0.8627 & -0.3541 & 1.1188 & -0.3412 & -0.0619 & 1.0385\end{bmatrix}$$

$$\boldsymbol{P}_2^{\vartriangle}=10^{-6}\begin{bmatrix}0.1894 & 0.0024\\ 0.0024 & 0.7898\end{bmatrix}$$

$$\boldsymbol{X}^{\vartriangle}=\begin{bmatrix}1.5302 & -0.9184\\ -0.0364 & 0.3609\\ 0.3568 & 0.8816\\ 0.0481 & 3.7117\\ -0.0193 & -0.2496\\ 0.0462 & 0.0741\end{bmatrix},\boldsymbol{Y}^{\vartriangle}=\begin{bmatrix}0.1114 & 0.0401\\ 0.0389 & 0.4222\\ 0.0585 & 0.2194\\ 0.0029 & -0.3076\\ 0.0128 & 0.3989\\ -0.0981 & 0.1311\end{bmatrix}$$

$$\boldsymbol{Z}^{\vartriangle}=10^{-7}\begin{bmatrix}-0.1262 & 0.1473\\ 0.0012 & 0.1927\\ -0.0018 & -0.0153\\ -0.0126 & -0.9835\\ 0.0012 & 0.0400\\ -0.0024 & -0.0067\end{bmatrix}$$

Immediately, the gain matrices are computed as

$$\boldsymbol{L}=10^{-7}\begin{bmatrix}-0.0243 & 0.0401\\ -0.0029 & -0.0809\\ -0.0073 & -0.0347\\ -0.0022 & -0.2224\end{bmatrix}$$

$$\boldsymbol{J}=\begin{bmatrix}0.3104 & -0.1805\\ 0.0414 & 0.9220\\ 0.2056 & 0.5299\\ 0.0171 & 1.0960\end{bmatrix},\quad \boldsymbol{K}=\begin{bmatrix}0.0476 & 0.0024\\ 0.0212 & 0.4069\\ 0.0767 & 0.1449\\ 0.0053 & 0.1595\end{bmatrix}$$

$$\boldsymbol{O}=10^{-7}\begin{bmatrix}-0.0045 & -0.1744\\ 0.0238 & -0.1135\end{bmatrix}$$

$$\boldsymbol{M}=\begin{bmatrix}0.0730 & 1.2695\\ -0.4109 & 0.4006\end{bmatrix},\quad \boldsymbol{N}=\begin{bmatrix}0.0368 & 0.6365\\ -0.2055 & 0.1973\end{bmatrix}$$

The obtained gain parameters force the stable discrete-time PDS fault observer, where the stable eigenvalue spectra are

$$\rho((\boldsymbol{E}^{\circ})^{-1}\boldsymbol{F}_e^{\circ})$$
$$=\{1, 1, 0.887, 0.729, 0.105, 0.010, 0.812\pm 0.177\mathrm{i}\}$$

$$\rho((\boldsymbol{I}_n+\boldsymbol{KC})^{-1}(\boldsymbol{F}-(\boldsymbol{J}-\boldsymbol{K})\boldsymbol{C}))$$
$$=\{-0.0060, 0.0887, 0.6677\pm 0.0743\mathrm{i}\}$$

For the sake of comparison, the inequalities (86), (87) satisfy the solutions

$$\boldsymbol{P}^{\vartriangle}=\begin{bmatrix}12.9918 & -0.7728 & 0.1657 & 0.2288 & -0.1336 & 1.8686\\ -0.7728 & 8.6910 & -1.3388 & -2.3797 & -2.8695 & -0.7877\\ 0.1657 & -1.3388 & 9.1245 & -1.3339 & -0.8470 & 2.4378\\ 0.2288 & -2.3797 & -1.3339 & 13.1358 & -2.2666 & -0.7547\\ -0.1336 & -2.8695 & -0.8470 & -2.2666 & 4.1500 & -0.1229\\ 1.8686 & -0.7877 & 2.4378 & -0.7547 & -0.1229 & 2.4082\end{bmatrix}$$

$$\boldsymbol{X}^{\vartriangle}=\begin{bmatrix}3.0096 & -2.0555\\ -0.0879 & 0.8890\\ 0.6983 & 1.9733\\ 0.0783 & 7.8500\\ -0.0353 & -0.5615\\ 0.0991 & 0.1545\end{bmatrix},\boldsymbol{Y}^{\vartriangle}=\begin{bmatrix}-0.0166 & 0.0519\\ 0.0958 & 0.9609\\ 0.0762 & 0.4609\\ -0.0013 & -0.7439\\ 0.0268 & 0.8036\\ -0.2298 & 0.2706\end{bmatrix}$$

and the discrete-time PD fault observer gain matrices are

$$\boldsymbol{J}=\begin{bmatrix}0.2795 & -0.1693\\ 0.0355 & 0.8367\\ 0.1730 & 0.5037\\ 0.0163 & 1.0129\end{bmatrix},\quad \boldsymbol{K}=\begin{bmatrix}0.0226 & 0.0034\\ 0.0177 & 0.3568\\ 0.0571 & 0.1270\\ 0.0041 & 0.1232\end{bmatrix}$$

$$\boldsymbol{M}=\begin{bmatrix}0.0595 & 1.1037\\ -0.3311 & 0.3330\end{bmatrix},\quad \boldsymbol{N}=\begin{bmatrix}0.0285 & 0.5386\\ -0.1622 & 0.1640\end{bmatrix}$$

The obtained discrete-time PD fault observer is stable since its eigenvalue spectrum is

$$\rho((\boldsymbol{I}_n+\boldsymbol{KC})^{-1}(\boldsymbol{F}-(\boldsymbol{J}-\boldsymbol{K})\boldsymbol{C}))$$
$$=\{-0.0181, 0.1354, 0.6710\pm 0.0759\mathrm{i}\}$$

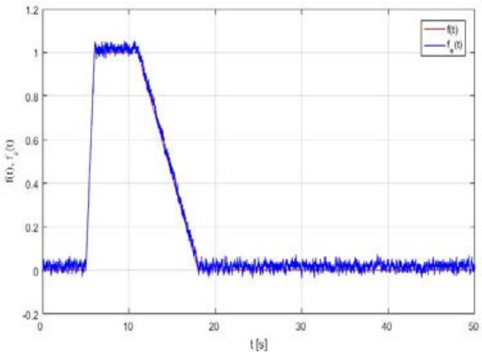

Fig. 1 Evolution of the first single actuator fault and its estimation with PDS fault observer

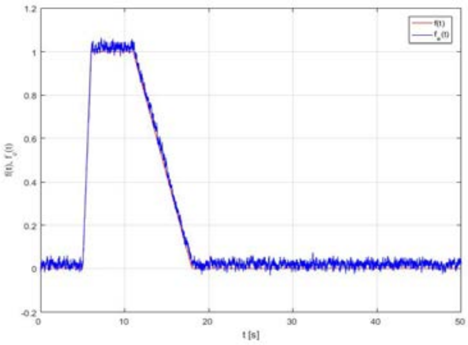

Fig. 2 Evolution of the first single actuator fault and its estimation with PD fault observer

Because $\boldsymbol{F}$ admits $\boldsymbol{q}(i)$ as a unstable state vector, the forced mode feedback control

$$\boldsymbol{u}(i) = -\boldsymbol{K}_c\boldsymbol{q}(i) + \boldsymbol{W}\boldsymbol{w}_o$$

is applied in simulation to stabilize asymptotically the system (1), (2), as well as to force the desired steady-state of the system output, where $\boldsymbol{W}$ is the signal gain matrix and $\boldsymbol{w}_o$ is desired output vector.

Applying the control law gain matrices as follows

$$\boldsymbol{K}_c = \begin{bmatrix} -1.6633 & 2.4844 & 0.1230 & 9.8708 \\ -17.6884 & 0.1286 & -8.1254 & 5.5029 \end{bmatrix}$$

$$\begin{aligned}\boldsymbol{W} &= (\boldsymbol{C}(\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}_c)^{-1}\boldsymbol{G})^{-1} \\ &= \begin{bmatrix} -0.3995 & 10.8885 \\ -3.9989 & 2.3916 \end{bmatrix}\end{aligned}$$

the system output steady-state vector $\boldsymbol{w}_o^T = [1 \;\; 0.5]$ is forced in the simulation, while the initial conditions are set as $\boldsymbol{q}(0) = \boldsymbol{q}_e(0) = \Delta\boldsymbol{q}_e(0) = \boldsymbol{0}$ and the gaussian noise variance of system disturbance is $\sigma_d^2 = 1.2 \times 10^{-2}$.

Applying the single trapezoidal fault entering directly at the first actuator

$$f(i,a,b,c,d) = \max\left(\min\left(\frac{i-a}{b-a}, 1, \frac{d-i}{d-c}\right), 0\right)$$

the simulation results are depicted in Fig. 1 and Fig. 2. The responses illustrates the evolution the fault estimation $f_e(i)$ in the PDS and PD structure due to a single fault $f(i)$ acting on the first actuator, which starts at the time instant $t_{fs} = a = 5\,\text{s}$ and ends at the time instant $t_{fe} = d = 17.5\,\text{s}$, where $b = 7\,\text{s}$, $c = 10\,\text{s}$.

These results characterize the discrete-time fault observer performances prescribed by the synthesis strategy, as well as applicability of the proposed approach to fault time-profile estimation in the presence of noisy perturbations. From Fig. 1 it is evident that the PDS estimator more suppresses the offset when estimating the additive fault shape with constant parts.

5 Concluding Remarks

In the paper, a modified design approach for a discrete-time implementation of PDS fault estimation is proposed. The considered system is supposed to be affected by system noise and additive faults. In this regards, new theorem is introduced, adjusting with a given degree also the bound of the difference of the additive faults and the observer integral action. The conditions improve the fault estimation consistency, guarantee the asymptotic stability of the proposed observer structure and suppresses the impact of system noise on additive faults appreciation and the system variables estimation. Moreover, there are derived reduced structures of linear matrix inequalities for the design of PD and P discrete-tim fault estimators. It also is shown that the solutions to the additive fault estimation via Luenberger-type observer with the derivative part can be obtained directly by solving LMIs, without interactions in the design step. The proposed approaches are completely model based and smoothly convenient in practical use.

To show validity of the method, estimation of a single actuator fault is demonstrated to illustrate how the proposed observer structure is able to estimate a fault time profile in the appropriate precision. Simulation results illustrate noise robustness of the PDS and PD fault estimation structure, when applied to the system with unstable behavior and pointing out the necessity of the forced mode strategy for control. In particular, this procedure has enhanced performances under assumption of norm bounded differences of additive faults.

Acknowledgement: The work presented in this paper was supported by VEGA, the Grant Agency of Ministry of Education and Academy of Science of Slovak Republic, under Grant No. 1/0608/17. This support is very gratefully acknowledged.

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