On Existence of Vanishing at Infinity Solutions to Second-Order Linear Differential Equations with Low-Degree Polynomial Coefficients

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Abstract: We study a second order linear differential equation with low-degree polynomial coefficients arising while studying the Bellman equation for the investment portfolio control problem. Our purpose is to determine whether there exists a non-trivial solution vanishing at infinity. We prove an existence criterion for such solutions according to the signs of the coefficients. By the way, the same methods produce an existence criterion for non-trivial bounded solutions. Instead of a verbose formulation, a united criterion is presented in a table form admitting simple computer realization.

Key–Words: Linear differential equations, polynomial coefficients, solutions vanishing at infinity, investment portfolio control problem.

1 Introduction

In this paper we study the asymptotic behavior of solutions to the linear differential equation

$$y'' + ay' + by = 0 \tag{1}$$

with the polynomial coefficients $a(x) = a_1x + a_0$ and $b(x) = b_2x^2 + b_1x + b_0$.

This equation arises while studying the Bellman equation for the investment portfolio control problem with assets determined by solving a system of stochastic differential equations. General principles of optimal investment portfolio management based on the dynamic programming method are given, for example, by B. Øksendal in [1].

In [2] and [3], T. Bielecki, S. Pliska, and M. Sherris use a model with trends for assets depending on economic macro-indicators (macrofactors) also satisfying a system of stochastic differential equations. The authors study the problem of optimal portfolio management to maximize a so-called risk-sensitive functional, which includes the value of the interest rate of the portfolio over long periods and the risk premium. They established that if the macrofactor system has a single, so-called main, factor, then the asymptotic behavior of the Bellman function is related to that of solutions to the second-order differential equation with polynomial coefficients considered in the present paper.

Important are non-trivial solutions to this equation vanishing at infinities. Constant-sign solutions of this type are often called basic states. Conditions of the existence, uniqueness, and asymptotic behavior of basic states are studied in a lot of work, see [4] and the bibliography cited there.

Mainly, the multidimensional case is investigated. In the one-dimensional case the above questions can be studied in more detail.

To obtain the results of this paper, we use, in particular, some methods of [5]–[15]. In some simple special cases the results coincide with those of [16] and [17].

2 Notation

To formulate shortly the results on the problem considered we need the following notation and definition.

For any non-zero tuple $(c_0, c_1, \dots, c_n)$ we write $(c_0, \dots, c_n) \prec 0$ (corr. $(c_0, \dots, c_n) \succ 0$) if its first non-zero component is negative (corr. positive).

We write

$$(c_0, \dots, c_n) \preceq 0 \quad \text{(corr. } (c_0, \dots, c_n) \succeq 0)$$

if either $c_0 = \dots = c_n = 0$ or $(c_0, \dots, c_n) \prec 0$ (corr. $(c_0, \dots, c_n) \succ 0$).

Definition 1 *A tuple* $(c_0, \dots, c_n)$ *with components depending on the coefficients* a_j *and* b_j *will be said to* resolve the problem *of existence of vanishing at infinity solutions to equation* (1) *if*

1) $(c_0, \dots, c_n) \succ 0$ *iff equation* (1) *has a non-trivial solution vanishing at infinity;*

2) $(c_0, \dots, c_n) \succeq 0$ *iff equation* (1) *has a non-trivial solution bounded near infinity;*

3) $(c_0, \dots, c_n) \prec 0$ *iff all non-trivial solutions to* (1) *are unbounded near infinity.*

If this holds under some conditions fixing the sign of the expressions $E_1, \dots, E_k$ depending on a_j and b_j, then we express such a rule by a table of the following type.

	E_1	E_2	E_3	E_4			
1	$+$	$-$	0		c_0	c_1	c_2

The left-hand one-column part of the above table contains just the rule number, for reference.

The central part contains the signs of the related expressions. The $+$ and $-$ signs stand for strictly positive and negative values. The empty cell shows no influence of E_4 in the case considered.

The right-hand part contains the components of the tuple resolving the problem under the conditions defined in the central part. Logical expressions like $b_0 < 0$ can be used as components of such a tuple. They are treated as 1 if true and as 0 if false.

The table can contain several rows for several rules describing different sign combinations of E_j.

We use several modified Landau symbols (without traditional parentheses to make formulae less cumbersome).

Thus, the O symbol, instead of traditional $O(1)$, denotes any function bounded near infinity (O_+ if the function is also bounded below by a positive constant).

The $\widehat{O}$ symbol denotes any function having at infinity a finite non-zero limit ($\widehat{O}_+$ if the limit is positive).

The O' symbol can denote the derivative of any bounded function, i. e. it denotes a function with the bounded integral ($\widehat{O}'$ if the integral converges). E. g., $\sin x = O'$ and $x^\alpha \sin x = \widehat{O}'$ iff $\alpha < 0$.

The $\widetilde{O}$ symbol denotes any function bounded near infinity, but not tending to 0. E. g., $\sin \sqrt{x} = \widetilde{O}$, but $x^{-1} \sin x \neq \widetilde{O}$.

The following table summarizes the conditions for these O symbols to replace a function f.

O	$-\infty < \varliminf\limits_{x \to x_*} f(x) \le \varlimsup\limits_{x \to x_*} f(x) < \infty$
O_+	$0 < \varliminf\limits_{x \to x_*} f(x) \le \varlimsup\limits_{x \to x_*} f(x) < \infty$
$\widehat{O}$	$0 < \lim_{x \to x_*} \lvert f(x) \rvert < \infty$
$\widehat{O}_+$	$0 < \varliminf\limits_{x \to x_*} f(x) = \varlimsup\limits_{x \to x_*} f(x) < \infty$
O'	$-\infty < \varliminf\limits_{x \to x_*} \int_{x_0}^{x} f(s)\, ds \le \varlimsup\limits_{x \to x_*} \int_{x_0}^{x} f(s)\, ds < \infty$
$\widehat{O}'$	$-\infty < \int\limits_{x_0}^{x_*} f(s)\, ds < \infty$
$\widetilde{O}$	$0 < \varlimsup\limits_{x \to x_*} \lvert f(x) \rvert < \infty$

Several identical O symbols can denote different appropriate functions, even in the same expression.

Any inequality followed by the expression "as $x \to \infty$" is supposed to hold ultimately, i. e. for all sufficiently big x.

Any inequality followed by "as $x \dashrightarrow \infty$" is supposed to hold at least for a sequence $x_j \to \infty$.

The domain of an arbitrary function y is denoted by $\operatorname{dom} y$. So, y is defined on the interval $(\inf \operatorname{dom} y,\ \sup \operatorname{dom} y)$.

3 Main results and the first step of the proof

Theorem 2 *The problem of the existence of vanishing solutions to equation* (1) *is resolved by the tuples according to the following table.*

	a_1	b_2	b_1	a_0			
1	$+$				1		
2	$-$				$-b_2$	$-b_1$	$-b_0$
3	0	$-$			1		
4	0	$+$			a_0	1	
5	0	0	$-$		1		
6	0	0	$+$		a_0	1	
7	0	0	0	$+$	1		
8	0	0	0	0	$b_0<0$		
9	0	0	0	$-$	$-b_0$		

Before proving the theorem, note that the substitution $x \mapsto -x$ transforms equation (1) to itself with the only difference consisting in changing the signs of a_0 and b_1 to the opposite ones. So, we have the following immediate corollary.

Corollary 3 *The problem of the existence of vanishing at $-\infty$ solutions to equation* (1) *is resolved by the tuples according to the following table.*

	a_1	b_2	b_1	a_0			
1	$+$				1		
2	$-$				$-b_2$	b_1	$-b_0$
3	0	$-$			1		
4	0	$+$			$-a_0$	1	
5	0	0	$+$		1		
6	0	0	$-$		$-a_0$	1	
7	0	0	0	$-$	1		
8	0	0	0	0	$b_0<0$		
9	0	0	0	$+$	$-b_0$		

To prove the theorem, we use the substitution $y(x) = z(x)\, \exp g(x)$ with

$$g(x) = -\frac{1}{2}\int_0^x a(\xi)d\xi = -\frac{a_1x^2}{4} - \frac{a_0x}{2} \tag{2}$$

to transform equation (1) into

$$z''(x) = P(x)\, z(x) \tag{3}$$

with

$$P(x) = \frac{a'(x)}{2} + \frac{a(x)^2}{4} - b(x) = \sum_{j=0}^{2} P_j\, x^j,$$

$$P_2 = \frac{a_1^2}{4} - b_2, \tag{4}$$

$$P_1 = \frac{a_1a_0}{2} - b_1, \tag{5}$$

$$P_0 = \frac{a_1}{2} + \frac{a_0^2}{4} - b_0. \tag{6}$$

3.1 The case $P = 0$

Consider the most simple case of the equation (3) with $P = 0$, i. e. $P_2 = P_1 = P_0 = 0$. Equations (3) and then (1) can be resolved explicitly:

$$z(x) = C_0 + C_1x,$$
$$y(x) = (C_0 + C_1x)\, \exp\left(-\frac{a_1x^2}{4} - \frac{a_0x}{2}\right).$$

The last expression shows that a non-trivial vanishing solution to (1) exists iff either $a_1 > 0$ or $a_0 > 0 = a_1$. If neither of these conditions is satisfied, then a non-trivial bounded solution exists iff $a_1 = a_0 = 0$. So, if $P = 0$, the problem is resolved by the tuple (a_1, a_0). This can be shown by the following table.

	P_2	P_1	P_0		
1	0	0	0	a_1	a_0

4 The case $(P_2, P_1, P_0) \succ 0$

In this case ultimately $P(x) > 0$ and $P'(x) \geq 0$. So, we consider the behavior of solutions to (3) only for x with the above inequalities satisfied.

Put $p(x) = \sqrt{P(x)} > 0$ and notice that

$$\frac{p'(x)}{p(x)} = \frac{P'(x)}{2P(x)} = \frac{x^{-1}\deg P}{2} + Ox^{-2},$$

whence

$$p'(x) = x^{-1}p(x)\left(\frac{\deg P}{2} + Ox^{-1}\right).$$

Any non-trivial solution z to (3), if vanishing at some point x_0, must be strictly monotone on (x_0, ∞) and have a constant sign there. So, we will consider only solutions positive near ∞. Immediate calculations show that a positive function z is a solution to (3) iff the function $w = (\ln z)'$ is a solution to the equation

$$w' = p^2 - w^2. \tag{7}$$

Note that contrary to equation (3) with all solutions extensible onto the whole axis $(-\infty, \infty)$, equation (7) has both solutions defined on $(-\infty, \infty)$ and solutions unbounded near a finite point.

Indeed, the first type solutions correspond to globally positive solutions to equation (3), which do exist (e. g. all those with initial data $z(x_0) > 0$, $z'(x_0) = 0$).

The second type solutions correspond to the positive parts of non-trivial, but vanishing at some finite point x_0 solutions to equation (3). If $z'(x_0) > 0$, then $w(x) \to +\infty$ as $x \to x_0 = \inf \operatorname{dom} w$. If $z'(x_0) < 0$, then $w(x) \to -\infty$ as $x \to x_0 = \sup \operatorname{dom} w$.

Because of the uniqueness theorem, if w_1 and w_2 are two maximally extended solutions to equation (7) and the inequality $w_1 > w_2$ holds at some point, then it also holds at any point where these functions are defined. Hence, if $w_1(x) \to -\infty$ as $x \to \sup \operatorname{dom} w_1 < \infty$, then $w_2(x) \to -\infty$ as $x \to \sup \operatorname{dom} w_2 \le \sup \operatorname{dom} w_1 < \infty$.

Further, if w_3 is another solution to (7) having at some point x_0 a value sufficiently close to $w_1(x_0)$, then $w_3(x) \to -\infty$ as $x \to \sup \operatorname{dom} w_3 < \infty$. Indeed, consider the corresponding maximally extended solutions z_1 and z_3 to (3) with $z_1(x_0) = z_3(x_0) = 1$. The solution z_1 vanishes at $\sup \operatorname{dom} w_1 > x_0$, surely with the negative derivative, and becomes negative itself at some point $x_1 > \sup \operatorname{dom} w_1$. Then, by continuity, $z_3(x_1) < 0$ whenever the value $z_3'(x_0)$ is close enough to $z_1'(x_0)$. Hence, z_3 vanishes at some point $x_3 \in (x_0, x_1)$, which is just $\sup \operatorname{dom} w_3$, and $w_3(x) \to -\infty$ as $x \to x_3$.

Due to the above monotony and continuity, we can introduce the function $\widehat{w}$ as the least maximally extended solution to (7) defined near ∞.

Note that $\widehat{w}(x) < 0$ as $x \to \infty$. Indeed, if $\widehat{w}(x_0) \ge 0$, then, because of (7), $\widehat{w} > 0$ on (x_0, ∞). By continuity, any solution w to (7) with the value $w(x_0)$ sufficiently close to $\widehat{w}(x_0)$ is also positive as $x \to \infty$, even if $w(x_0) < \widehat{w}(x_0)$. But this contradicts to the definition of $\widehat{w}$.

The function $\widehat{w}$, or rather R defined by

$$R(x) = g(x) + \int_{x_0}^{x} \widehat{w}(s)ds, \qquad (8)$$

appears to be crucial for our problem in the case $(P_2, P_1, P_0) \succ 0$:

if $R(x) \to -\infty$ as $x \to \infty$, then equation (1) has a vanishing at infinity solution equal to $\exp R$;

if $R(x)$ is bounded near infinity, then equation (1) has a bounded solution equal to $\exp R$;

if $R(x) \to +\infty$ as $x \to \infty$, then equation (1) has no bounded solution.

4.1 The subcase $P_2 > 0$

Put $\lambda_2 = \sqrt{P_2} > 0$. Then

$$\begin{aligned} p(x) &= x\lambda_2 \left(1 + \frac{P_1}{\lambda_2^2 x} + \frac{P_0}{\lambda_2^2 x^2}\right)^{1/2} \\ &= x\lambda_2 + \frac{P_1}{2\lambda_2} + \frac{P_0}{2\lambda_2 x} - \frac{P_1^2}{8\lambda_2^3 x} + \frac{O}{x^2} \\ &= x\lambda_2 + \frac{P_1}{2\lambda_2} + Ox^{-1}, \end{aligned}$$

$$p'(x) = \lambda_2 + Ox^{-2} < \lambda_2 + 1, \ x \to \infty.$$

For any constant k we define the function w_k by

$$w_k(x) = -p(x) - \frac{x^{-1}}{2} + kx^{-2}.$$

Suppose $\widehat{w}(x) > w_k(x)$ as $x \dashrightarrow \infty$. Then at the points with the last inequality satisfied we have

$$\begin{aligned} &(\widehat{w} - w_k)' = p^2 - \widehat{w}^2 - w_k' > p^2 - w_k^2 - w_k' \\ &= p^2 - p^2 - x^{-1}p + 2kx^{-2}p + Ox^{-2} + p' \\ &= -\left(\lambda_2 + \frac{P_1}{2\lambda_2}x^{-1} + Ox^{-2}\right) + 2k\lambda_2 x^{-1} + p' \\ &= x^{-1}\left(2k\lambda_2 - \frac{P_1}{2\lambda_2} + Ox^{-1}\right) = \widehat{O}_+ x^{-1} \end{aligned}$$

whenever $k > \varkappa = \dfrac{P_1}{4\lambda_2^2}$. Hence either $\widehat{w}$ becomes positive or, due to the above inequalities, $\widehat{w} - w_k$ is ultimately increasing and $\widehat{w}$ is ultimately greater than w_k as well as any other solution to (7) sufficiently close to $\widehat{w}$ at some sufficiently big x. This contradicts to the minimum property of $\widehat{w}$.

Now suppose $\widehat{w}(x) < w_k(x)$ as $x \dashrightarrow \infty$. The similar estimates show that $(\widehat{w} - w_k)' < -\widehat{O}_+ x^{-1}$ whenever $k < \varkappa$. Hence

$$\widehat{w} - w_k = w + p + \frac{x^{-1}}{2} - kx^{-2} \to -\infty$$

and ultimately $\widehat{w} + p < -\sqrt{\lambda_2 + 1}$. Consider the positive function $\eta = -p - \widehat{w} - \sqrt{\lambda_2 + 1}$. We have

$$\begin{aligned} \left(\frac{1}{\eta}\right)' &= \frac{-\eta'}{\eta^2} = \frac{p' + p^2 - \widehat{w}^2}{\eta^2} \\ &= \frac{p' + p^2 - \left(p + \sqrt{\lambda_2 + 1} + \eta\right)^2}{\eta^2} \\ &< \frac{\lambda_2 + 1 + p^2 - p^2 - \lambda_2 - 1 - \eta^2}{\eta^2} = -1, \end{aligned}$$

whence $\dfrac{1}{\eta}$ vanishes at some finite point x_* and η, as well as $\widehat{w}$, is unbounded near x_*. This is impossible for the function $\widehat{w}$ defined near ∞.

The two above suppositions denied show that

$$\begin{aligned}\widehat{w} &= w_{\varkappa} + \widehat{O}x^{-2}\\ &= -\lambda_2 x - \frac{P_1}{2\lambda_2} + x^{-1}\left(\frac{P_1^2}{8\lambda_2^3} - \frac{P_0}{2\lambda_2} - \frac{1}{2}\right) + O',\end{aligned}$$

whence, taking into account (2), (4)–(6), and (8), we obtain

$$\begin{aligned}R' &= g' + \widehat{w} = \widehat{w} - \frac{a_1 x}{2} - \frac{a_0}{2}\\ &= O' - x\,\frac{a_1 + 2\lambda_2}{2} - \left(\frac{a_1 a_0 - 2b_1}{4\lambda_2} + \frac{a_0}{2}\right)\\ &- x^{-1}\left(\frac{1}{2} - \frac{(a_1 a_0 - 2b_1)^2}{32\lambda_2^3} + \frac{2a_1 + a_0^2 - 4b_0}{8\lambda_2}\right).\end{aligned}$$

This gives a tuple resolving our problem with $P_2 > 0$:

$$\begin{aligned}&\big(a_1 + 2\lambda_2,\\ &a_1 a_0 - 2b_1 + 2a_0\lambda_2,\\ &16\lambda_2^3 - (a_1 a_0 - 2b_1)^2 + 4\lambda_2^2\left(2a_1 + a_0^2 - 4b_0\right)\big).\end{aligned}$$

It looks rather cumbersome. So, we will try to simplify it depending on $\operatorname{sgn} a_1$.

While simplifying, we may replace any component by an expression having the same sign under the condition that all previous components equal zero. Besides, if a component cannot equal zero in the case considered, then we may remove all components following it. E. g., $(c_0, c_1, c_0 + c_1 - 12, 13) \sim (c_0, c_1, -1)$.

If $P_2 > 0$ and $a_1 \geq 0$, then the first component of the tuple obtained is positive and we obtain the next two rules shown in the following table.

	P_2	a_1	
2	+	+	1
3	+	0	1

Now consider the case with $P_2 > 0$ and $a_1 < 0$. We have the following sequence of equivalent inequalities:

$$\begin{aligned}a_1 + 2\lambda_2 \lesseqgtr 0 &\iff \sqrt{a_1^2 - 4b_2} \lesseqgtr -a_1\\ &\iff a_1^2 - 4b_2 \lesseqgtr a_1^2 \iff -b_2 \lesseqgtr 0.\end{aligned}$$

So, the first component of the tuple obtained may be replaced by $-b_2$.

If $b_2 = 0$ and therefore $2\lambda_2 = -a_1$, then the second component is equal to $-2b_1$ and may be replaced by $-b_1$.

If $b_2 = b_1 = 0$, then the third component equals

$$\begin{aligned}&16\lambda_2^3 - (a_1 a_0)^2 + 4\lambda_2^2\left(2a_1 + a_0^2 - 4b_0\right)\\ &= -2a_1^3 - (a_1 a_0)^2 + a_1^2\left(2a_1 + a_0^2 - 4b_0\right)\\ &= -4a_1^2 b_0\end{aligned}$$

and may be replaced by $-b_0$.

Thus, the tuple is equivalent to $(-b_2, -b_1, -b_0)$ and we obtain the next rule.

	P_2	a_1			
4	+	−	$-b_2$	$-b_1$	$-b_0$

4.2 The subcase $P_2 = 0,\ P_1 > 0$

Put $\lambda_1 = \sqrt{P_1}$. Then

$$\begin{aligned}p &= x^{1/2}\lambda_1\left(1 + \frac{P_0 x^{-1}}{\lambda_1^2}\right)^{1/2}\\ &= x^{1/2}\lambda_1 + \frac{P_0 x^{-1/2}}{2\lambda_1} + Ox^{-3/2}\\ p' &= \frac{\lambda_1 x^{-1/2}}{2} + Ox^{-3/2} < 1,\ \ x \to \infty.\end{aligned}$$

Similarly to the previous case, consider the function

$$w_k(x) = -p(x) - \frac{x^{-1}}{4} + kx^{-3/2}$$

and show that if $\widehat{w} \gtreqless w_k$ as $x \dashrightarrow \infty$, then ultimately $(\widehat{w} - w_k)' \gtreqless 2k\lambda_1 x^{-1} + Ox^{-3/2} \gtreqless 0$ and $\widehat{w} \gtreqless w_k$ whenever $k \gtreqless 0$.

The case $\widehat{w} > w_k$ contradicts to the minimum property of $\widehat{w}$. In the case $\widehat{w} < w_k$ we have

$$\left(\frac{1}{-p - \widehat{w} - 1}\right)' < -1,$$

which invokes unboundedness of $\widehat{w}$ near a finite x.

So,

$$\begin{aligned}\widehat{w} &= w_0 + Ox^{-3/2}\\ &= -\lambda_1 x^{1/2} - \frac{P_0 x^{-1/2}}{2\lambda_1} - \frac{x^{-1}}{4} + O',\end{aligned}$$

whence

$$\begin{aligned}R' &= \widehat{w} - \frac{a_1 x}{2} - \frac{a_0}{2}\\ &= -\frac{a_1 x}{2} - \lambda_1 x^{1/2} - \frac{a_0}{2} + Ox^{-1/2}.\end{aligned}$$

This yields a tuple resolving our problem with $P_2 = 0$ and $P_1 > 0$, which gives, after simplification, the following rule.

	P_2	P_1		
5	0	+	a_1	1

4.3 The subcase $P_2 = P_1 = 0,\ P_0 > 0$

In this case all solutions to equation (7) can be found explicitly. One of them is $w \equiv -\lambda_0$ with $\lambda_0 = \sqrt{P_0}$. All solutions less than $-\lambda_0$ has the form

$$w = \lambda_0 \coth \lambda_0 (x - x^*)$$

and tend to $-\infty$ as $x \to x^* - 0$. Hence $\widehat{w} = -\lambda_0$ and

$$\begin{aligned} R' &= -\frac{a_1 x}{2} - \frac{a_0}{2} - \lambda_0 \\ &= -\frac{a_1 x}{2} - \frac{a_0}{2} - \sqrt{\frac{a_1}{2} + \frac{a_0^2}{4} - b_0}. \end{aligned}$$

This gives a tuple resolving our problem with $P_0 > 0$, $P_2 = P_1 = 0$, namely

$$\left(a_1,\ a_0 + \sqrt{2a_1 + a_0^2 - 4b_0}\right),$$

which is equivalent to

$$\left(a_1,\ a_0 + \sqrt{a_0^2 - 4b_0}\right).$$

Its further simplification depends on $\operatorname{sgn} a_0$.

If $a_0 \geq 0$, then the second component is always positive in the case considered.

If $a_0 < 0$, then the inequality

$$a_0 + \sqrt{a_0^2 - 4b_0} \gtrless 0$$

is equivalent to the inequality $-b_0 \gtrless 0$.

Thus, we obtain three new rules as shown in the following table.

	P_2	P_1	P_0	a_0		
6	0	0	+	+	a_1	1
7	0	0	+	0	a_1	1
8	0	0	+	−	a_1	$-b_0$

5 The case $(P_2,\ P_1,\ P_0) \prec 0$

In this case we may assume P and P' to be negative and put

$$Q(x) = -P(x) = \sum_{j=0}^{2} Q_j\, x^j > 0$$

with

$$Q_2 = b_2 - \frac{a_1^2}{4}, \tag{9}$$

$$Q_1 = b_1 - \frac{a_1 a_0}{2}, \tag{10}$$

$$Q_0 = b_0 - \frac{a_1}{2} - \frac{a_0^2}{4}. \tag{11}$$

For any non-trivial solution z to (3) there exist defined near infinity continuous functions

$$\begin{aligned} S &= z^2 + \frac{(z')^2}{Q} > 0, \\ \sigma &= \ln S, \end{aligned}$$

and φ and such that

$$z = \sqrt{S} \sin \varphi, \quad \frac{z'}{\sqrt{Q}} = \sqrt{S} \cos \varphi. \tag{12}$$

Since equation (3) can be written now as

$$z'' + Qz = 0, \tag{13}$$

we have

$$\begin{aligned} S' &= 2zz' - \frac{2z'Qz}{Q} - \frac{(z')^2\, Q'}{Q^2} = -\frac{SQ' \cos^2 \varphi}{Q}, \\ \sigma' &= -\frac{Q' \cos^2 \varphi}{Q}, \\ \varphi' &= \left(\arctan \frac{z\sqrt{Q}}{z'}\right)' \\ &= \frac{(z')^2 \sqrt{Q} + \frac{1}{2} zz'Q'Q^{-1/2} + z^2 Q^{3/2}}{SQ} \\ &= \sqrt{Q} \cos^2 \varphi + \frac{SQ' \sin \varphi \cos \varphi}{2SQ} + \sqrt{Q} \sin^2 \varphi \\ &= \sqrt{Q} + \frac{Q' \sin \varphi \cos \varphi}{2Q}. \end{aligned}$$

Note that near the points with $z' = 0$ the same formula for φ' can be also obtained by using

$$\varphi' = \left(\operatorname{arccot} \frac{z'}{z\sqrt{Q}}\right)'.$$

For the cases with ultimately positive φ' we can treat φ as an independent variable and write

$$\begin{aligned} \frac{d\sigma}{d\varphi} &= -\frac{2Q' \cos^2 \varphi}{2Q^{3/2} + Q' \sin \varphi \cos \varphi} \\ &= -\frac{1 + \cos 2\varphi}{2Q^{3/2}\,(Q')^{-1} + \sin \varphi \cos \varphi.} \end{aligned}$$

5.1 The subcase $Q_2 > 0$

Put $\omega_2 = \sqrt{Q_2}$. Then $Q' = 2\omega_2^2 x + Q_1$ and

$$\begin{aligned} \sqrt{Q} &= x\omega_2 \left(1 + \frac{Q_1}{\omega_2^2 x} + \frac{Q_0}{\omega_2^2 x^2}\right)^{1/2} \\ &= x\omega_2 + \frac{Q_1}{2\omega_2} + Ox^{-1}, \end{aligned}$$

whence $\varphi' = \sqrt{Q} + \dfrac{Q' \sin\varphi\cos\varphi}{2Q} = \omega_2 x + O$ and $\varphi = \dfrac{\omega_2 x^2}{2} + Ox$. Thus, $\varphi \to \infty$ and $\varphi' > 0$ as $x \to \infty$, which, according to (12), means that any non-trivial solution z to (3) is oscillatory and φ may be treated as an independent variable related to x by

$$x = \widehat{O}\varphi^{1/2}, \quad \omega_2 x^2 = 2\varphi + Ox = 2\varphi + O\varphi^{1/2}.$$

Further, we have

$$\begin{aligned} Q^{3/2}\left(Q'\right)^{-1} &= \omega_2^3 x^3 \left(2\omega_2^2 x\right)^{-1}\left(1 + Ox^{-1}\right) \\ &= \varphi\left(1 + O\varphi^{-1/2}\right), \end{aligned}$$

whence

$$\begin{aligned} \frac{d\sigma}{d\varphi} &= -\frac{1+\cos 2\varphi}{2\varphi\left(1 + O\varphi^{-1/2}\right)} \\ &= -\frac{1}{2\varphi} - \frac{\cos 2\varphi}{2\varphi} + O\varphi^{-3/2}, \\ \sigma &= -\frac{\ln\varphi}{2} + \widehat{O}, \\ S &= e^{\sigma} = \widehat{O}_+\varphi^{-1/2} = \widehat{O}_+ x^{-1}, \\ z^2 &= \widetilde{O}x^{-1}, \quad z = \widetilde{O}x^{-1/2}. \end{aligned}$$

So, any non-trivial solution y to (1) satisfies

$$\begin{aligned} y(x) &= \widetilde{O}x^{-1/2}\exp\left(-\frac{a_1 x^2}{4} - \frac{a_0 x}{2}\right) \\ &= \widetilde{O}\exp\left(-\frac{a_1 x^2}{4} - \frac{a_0 x}{2} - \frac{\ln x}{2}\right), \end{aligned}$$

which shows the tuple $(a_1, a_0, 1)$ to resolve our problem if $Q_2 = -P_2 > 0$ and we obtain another related rule.

	P_2			
9	$-$	a_1	a_0	1

5.2 The subcase $Q_2 = 0, \quad Q_1 > 0$

Put $\omega_1 = \sqrt{Q_1}$. Then $Q' = \omega_1^2$ and

$$\sqrt{Q} = \omega_1 x^{1/2}\sqrt{1 + \frac{Q_0}{\omega_1^2 x}} = \omega_1 x^{1/2}\left(1 + Ox^{-1}\right),$$

whence

$$\varphi' = \sqrt{Q} + \frac{Q'\sin\varphi\cos\varphi}{2Q} = \omega_1 x^{1/2} + Ox^{-1/2}$$

and $\varphi = \dfrac{2\omega_1 x^{3/2}}{3} + Ox^{1/2}$. This shows again that any non-trivial solution z to (3) is oscillatory and φ may be treated as an independent variable related to x by $x = \widehat{O}\varphi^{2/3}$ and

$$2\omega_1 x^{3/2} = 3\varphi + Ox^{1/2} = 3\varphi + O\varphi^{1/3}.$$

Further, we have

$$\begin{aligned} 2Q^{3/2}\left(Q'\right)^{-1} &= 2\omega_1^3 x^{3/2}\cdot\frac{1}{\omega_1^2}\left(1 + Ox^{-1}\right) \\ &= 3\varphi\left(1 + O\varphi^{-2/3}\right), \end{aligned}$$

whence

$$\begin{aligned} \frac{d\sigma}{d\varphi} &= -\frac{1+\cos 2\varphi}{2Q^{3/2}\left(Q'\right)^{-1} + \sin\varphi\cos\varphi} \\ &= -\frac{1+\cos 2\varphi}{3\varphi + O\varphi^{1/3}} \\ &= \left(-\frac{1}{3\varphi} - \frac{\cos 2\varphi}{3\varphi}\right)\left(1 + O\varphi^{-2/3}\right) \\ &= -\frac{1}{3\varphi} - \frac{\cos 2\varphi}{3\varphi} + O\varphi^{-5/3}, \\ \sigma &= -\frac{\ln\varphi}{3} + \widehat{O}, \\ S &= e^{\sigma} = \widehat{O}_+\varphi^{-1/3} = \widehat{O}_+ x^{-1/2}, \\ z^2 &= \widetilde{O}x^{-1/2}, \quad z = \widetilde{O}x^{-1/4}. \end{aligned}$$

So, any non-trivial solution y to (1) satisfies

$$\begin{aligned} y(x) &= z(x)\exp\left(-\frac{a_1 x^2}{4} - \frac{a_0 x}{2}\right) \\ &= \widetilde{O}\exp\left(-\frac{a_1 x^2}{4} - \frac{a_0 x}{2} - \frac{\ln x}{4}\right) \end{aligned}$$

and we obtain another related rule.

	P_2	P_1			
10	0	$-$	a_1	a_0	1

5.3 The subcase $Q_2 = Q_1 = 0, \quad Q_0 > 0$

If $Q \equiv Q_0$, then each non-trivial solution z to (3), which can be found explicitly as a trigonometrical function, satisfies $z = \widetilde{O}$. Hence each non-trivial solution y to (1) satisfies

$$y(x) = \widetilde{O}\exp\left(-\frac{a_1 x^2}{4} - \frac{a_0 x}{2}\right)$$

and we obtain one more rule.

	P_2	P_1	P_0		
11	0	0	$-$	a_1	a_0

6 Common simplification

Now the rules obtained in previous sections can be put all together into a single table showing, in all possible cases, whether equation (1) has non-trivial solutions vanishing at infinity or bounded.

	P_2	P_1	P_0	a_1	a_0			
1	0	0	0			a_1	a_0	
2	$+$			$+$		1		
3	$+$			0		1		
4	$+$			$-$		$-b_2$	$-b_1$	$-b_0$
5	0	$+$				a_1	1	
6	0	0	$+$		$+$	a_1	1	
7	0	0	$+$		0	a_1	1	
8	0	0	$+$		$-$	a_1	$-b_0$	
9	$-$					a_1	a_0	1
10	0	$-$				a_1	a_0	1
11	0	0	$-$			a_1	a_0	

It differs from the smaller table in the statement of Theorem 2, which needs no calculation of P_2, P_1, P_0. So, we begin common simplification of the rules.

First, we consider three copies of the above table for three signs of a_1 and simplify them separately. The condition columns related to a_1 are filled with the corresponding signs. If the conditions of a rule contradict to the case considered, then the rule is removed from the related table (this can be seen by numbers skipped). In the case $a_1 = 0$, the components equal to a_1 are removed from the resolving tuples. For $a_1 > 0$ and $a_1 < 0$, the components equal to a_1 are replaced by 1 and -1 while all the following components of the tuple are removed. Thus, we obtain three tables.

	P_2	P_1	P_0	a_1	a_0			
1	0	0	0	$+$		1		
2	$+$			$+$		1		
5	0	$+$		$+$		1		
6	0	0	$+$	$+$	$+$	1		
7	0	0	$+$	$+$	0	1		
8	0	0	$+$	$+$	$-$	1		
9	$-$			$+$		1		
10	0	$-$		$+$		1		
11	0	0	$-$	$+$		1		

	P_2	P_1	P_0	a_1	a_0			
1	0	0	0	0		a_0		
3	$+$			0		1		
5	0	$+$		0		1		
6	0	0	$+$	0	$+$	1		
7	0	0	$+$	0	0	1		
8	0	0	$+$	0	$-$	$-b_0$		
9	$-$			0		a_0	1	
10	0	$-$		0		a_0	1	
11	0	0	$-$	0		a_0		

	P_2	P_1	P_0	a_1	a_0			
1	0	0	0	$-$		-1		
4	$+$			$-$		$-b_2$	$-b_1$	$-b_0$
5	0	$+$		$-$		-1		
6	0	0	$+$	$-$	$+$	-1		
7	0	0	$+$	$-$	0	-1		
8	0	0	$+$	$-$	$-$	-1		
9	$-$			$-$		-1		
10	0	$-$		$-$		-1		
11	0	0	$-$	$-$		-1		

One can see from the first of these tables that a vanishing non-trivial solution to (1) exists whenever $a_1 > 0$. This can be written by a single row in the table. The third table can be replaced by three non-header rows corresponding to different signs of P_2. The combined renumerated table looks like follows.

	P_2	P_1	P_0	a_1	a_0			
1				$+$		1		
2	0	0	0	0		a_0		
3	$+$			0		1		
4	0	$+$		0		1		
5	0	0	$+$	0	$+$	1		
6	0	0	$+$	0	0	1		
7	0	0	$+$	0	$-$	$-b_0$		
8	$-$			0		a_0	1	
9	0	$-$		0		a_0	1	
10	0	0	$-$	0		a_0		
11	$-$			$-$		-1		
12	0			$-$		-1		
13	$+$			$-$		$-b_2$	$-b_1$	$-b_0$

Under the conditions of rules 11 and 12, i. e.,

$$a_1 < 0 \text{ and } P_2 = \frac{a_1^2}{4} - b_2 \le 0,$$

the coefficient b_2 must be positive. Hence the tuple (-1) is equivalent, under these conditions, to $(-b_2)$ and even to $(-b_2, -b_1, -b_0)$. Such a replacement does not look like a simplification, but it allows to replace rules 11–13 by the one with the last mentioned tuple and the only condition $a_1 < 0$.

Further, if $a_1 = 0$, which is a common condition of rules 2–10, then $P_2 = -b_2$ and $P_1 = -b_1$. Other rules, after the previous simplification, ignore P_2 and P_1. So, we may replace, in the table header, P_2 (corr. P_1) by b_2 (corr. b_1) and, in the table body, all non-zero signs in the related columns by the opposite ones.

	b_2	b_1	P_0	a_1	a_0			
1				$+$		1		
2	0	0	0	0		a_0		
3	$-$			0		1		
4	0	$-$		0		1		
5	0	0	$+$	0	$+$	1		
6	0	0	$+$	0	0	1		
7	0	0	$+$	0	$-$	$-b_0$		
8	$+$			0		a_0	1	
9	0	$+$		0		a_0	1	
10	0	0	$-$	0		a_0		
11				$-$		$-b_2$	$-b_1$	$-b_0$

Now consider rules 2, 5–7, and 10. Note that, if $a_1 = b_2 = b_1 = 0$ (their common condition), then (1) becomes a linear equation with constant coefficients and has non-trivial solutions vanishing at infinity iff the real part of at least one of the characteristic roots

$$-\frac{a_0}{2} \pm \sqrt{\frac{a_0^2}{4} - b_0}$$

is negative. This always holds if $a_0 > 0$. If $a_0 \le 0$, then this holds iff $b_0 < 0$.

As for non-trivial bounded solutions, they exist iff the real part of at least one characteristic root is non-positive. So, concerning the situation when among the non-trivial solutions there is no vanishing one, but bounded solutions do exist, it is possible if either $a_0 = 0, b_0 \ge 0$ or $a_0 < 0, b_0 = 0$.

Thus, the rules shown in the table

	b_2	b_1	a_1	a_0	
1*	0	0	0	$+$	1
2*	0	0	0	0	$b_0 < 0$
3*	0	0	0	$-$	$-b_0$

may replace, in the previous one, rules 2, 5–7, and 10 producing the following table.

	b_2	b_1	P_0	a_1	a_0			
1				$+$		1		
1*	0	0		0	$+$	1		
2*	0	0		0	0	$b_0 < 0$		
3*	0	0		0	$-$	$-b_0$		
3	$-$			0		1		
4	0	$-$		0		1		
7	$+$			0		a_0	1	
8	0	$+$		0		a_0	1	
11				$-$		$-b_2$	$-b_1$	$-b_0$

After the above simplification, all the rules ignore $\operatorname{sgn} P_0$. Hence the column related may be removed. Besides, according to the great influence of $\operatorname{sgn} a_1$, the column related may be chosen as the first condition one. Finally, by a permutation of the rows, one can simplify the if-else structure of a computer program realizing the table.

	a_1	b_2	b_1	a_0			
1	$+$				1		
11	$-$				$-b_2$	$-b_1$	$-b_0$
3	0	$-$			1		
7	0	$+$			a_0	1	
4	0	0	$-$		1		
8	0	0	$+$		a_0	1	
1*	0	0	0	$+$	1		
2*	0	0	0	0	$b_0 < 0$		
3*	0	0	0	$-$	$-b_0$		

This table, after renumeration, coinsides with the table in the statement of Theorem 2.

7 How does it work?

Thus, we have obtained the resulting table of rules and proved Theorem 2. Now, given the coefficients a_0, a_1, b_0, b_1, b_2, how will we get to know whether equation (1) has a solution vanishing at infinity? And what about solutions bounded as $x \to \infty$?

The answer is generated as follows. Hereafter, by solutions we mean only non-trivial ones. Their vanishing and boundedness are supposed at/near infinity.

First, we look at a_1. If it is positive, then, according to rule 1 from the above table and its "very positive" tuple (1), there exists a solution vanishing at infinity (and therefore bounded).

If a_1 is negative, then, according to rule 11 (the next one), the answer depends on the leading non-zero coefficient of the polynomial $b(x) = b_2x^2 + b_1x + b_0$. Namely, if this coefficient is positive (i. e. $b_2 > 0$, or $b_1 > 0 = b_2$, or $b_0 > 0 = b_2 = b_1$), then there is no bounded solution to (1). If negative, then there exists a vanishing solution. If all b_j equal 0, then there exists a bounded solution, but no vanishing one.

Now the answer to the question posed is clear for $a_1 \neq 0$. The following considerations concern the case $a_1 = 0$.

If $b_2 < 0$, then, according to rule 3, there exists a vanishing solution. According to rule 7, the same is true if $b_2 > 0$ and $a_0 \geq 0$, but there is no bounded solution if $b_2 > 0$ and $a_0 < 0$.

We have not considered a half of all possible combinations proposed by the table (those with $a_1 = b_2 = 0$). But our partial explanation took more place than the whole table of rules. So, we cease our considerations still hoping the algorythm is clear enough.

Remind that in rule 2*, according to our notation, the tuple consists of a single 1 if b_0 is negative (therefore providing the existence of vanishing solutions). If b_0 is non-negative, then the tuple consists of a single 0 providing the existence of bounded solutions, which cannot vanish at infinity.

8 Conclusion

The technique proposed in this paper can help to investigate more accurately the profitability of the investment portfolio in the model of Bielecki and Pliska. Namely, the problem of determining the profitability is related to determining the eigenvalue of a spectral problem for an ordinary differential equation on $\mathbf{R}$ with polynomial or piecewise polynomial coefficients. These coefficients are picewise linear/quadratic under some restrictions on the shares of assets in the portfolio. By using the technique developed in this paper for analysis of the asymptotic behavior of solutions to a second-order equation with polynomial coefficients, one can estimate the magnitude of the fall in profitability due to the restrictions mentioned.

References:

[1] Øksendal B., *Stochastic Differential Equations*, New-York, Springer-Verlag Heidelberg, 2000.

[2] Bielecki T., Pliska S., Risk-sensitive dynamic asset management, *J. Appl. Math. and Optimiz. Math. Nachr.* 1999. 37. 337–360.

[3] Bielecki T., Pliska S., Sherris M., Risk sensitive asset allocation, *J. Econ. Dynamics and Contr.*, 2000. 24. 1145–1177.

[4] Pyatnitskii A. L., Shamaev A. S., On the asymptotic behavior of eigenvalues and eigenfunctions of non-self-adjoint elliptic operators, *Journal of Mathematical Sciences*, Vol. 120, No. 3, 2004.

[5] Astashova I., Qualitative properties of solutions to quasilinear ordinary differential equations (in Russian), in: I. V. Astashova (ed.), *Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis*: scientific edition, UNITY-DANA, Moscow (2012), 22–290.

[6] Astashova I. V., Asymptotic Classification of Solutions of Singular 4th-Order EmdenFowler Equations with a Constant Negative Potential, *J. Math. Sci.* (2018) 234:385. (Translated from *Trudy Seminara imeni I. G. Petrovskogo*, No. 31, pp. 3-21, 2016.)

[7] Astashova I. V., Asymptotics of Oscillating Solutions to Equations with Power Nonlinearities, *J. Math. Sci.* (2018) 230:651.

[8] Astashova I., On asymptotic classification of solutions to fourth-order differential equations with singular power nonlinearity. *Mathematical Modeling and Analysis*, 21(4):502-521, 2016.

[9] Astashova I. V., On quasi-periodic solutions to a higher-order Emden-Fowler type differential equation. *Boundary Value Problems*, 174:1-8, 2014.

[10] Astashova I. V., On qualitative properties and asymptotic behavior of solutions to higher-order nonlinear differential equations. *WSEAS Transactions on Mathematics*, 16(5):39-47, 2017.

[11] Astashova I. V., On asymptotic classification of solutions to nonlinear regular and singular third- and fourth-order differential equations with power nonlinearity. In *Differential and Difference Equations with Applications*, vol. 164 of *Springer Proceedings in Mathematics & Statistics*, pp. 191-204. Springer International Publishing, 2016.

[12] Astashova I. V., On asymptotic equivalence of n-th order nonlinear differential equations. *Tatra mountains mathematical publications*, 63:3138, 2015.

[13] Astashova I. V., On asymptotic behavior of solutions to a quasi-linear second order differential equations, *Functional Differential Equations*, 16(1):93-115, 2009.

[14] Korchemkina T., On the behavior of solutions to second-order differential equation with general power-law nonlinearity, *Memoirs on Differential Equations and Mathematical Physics*. (2018) Vol. 73, pp. 101–111.

[15] Dulina K. M., On Asymptotic Behavior of Solutions to the Second-Order Emden–Fowler Type

Differential Equations with Unbounded Negative Potential, *Functional Differential Equations*, 2016. Vol. 23. No 1–2. pp. 3–8.

[16] Bellman R., *Stability theory of differential equations*, McGRAW-HILL Book Company Inc., New-York–Toronto–London, 1953.

[17] Kiguradze I. T., Chanturia T. A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluver Academic Publishers, Dordreht-Boston-London, 1993.