FUZZY REGRESSION BASED ON NON-PARAMETRIC METHODS

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Abstract: In recent years, a number of methods have been proposed to construct fuzzy regression models based the fuzzy distance. Most of the researches that have been proposed have used the parametric methods specifying the form of the relationship between the dependent and independent variables. In this talk, we introduce nonparametric fuzzy regression methods such as Rank transform method, Theil's method, Kernel method, k-nearest neighborhood method and Median smoothing method and discuss the efficiency of the proposed methods.

1 Introduction

Tanaka et al.(1982) first introduced a fuzzy regression model to construct a functional relationship between fuzzy explanatory and response variables. He suggested a fuzzy regression model as follows:

$$Y_i = F(\mathbf{A}, \mathbf{X}_i), \qquad i = 1, \cdots, n, \tag{1}$$

where $\mathbf{X}_i = (X_{i0}, X_{i1}, \cdots, X_{ip})$ is a $(p+1)$-dimensional vector of known predictors, $\mathbf{A} = (A_0, A_1, \cdots, A_p)$ is a $(p+1)$-dimensional vector of unknown coefficients, $F(\mathbf{A}, \mathbf{X}_i)$ is a function about the vector $\mathbf{A}$ and Y_i is a predicted variable corresponding to the input vector $\mathbf{X}_i$. The coefficient A_i, the predictor X_{ip} and the predicted value Y_i are LR-fuzzy numbers in the regression equation (1).

The membership function of an LR-fuzzy number A, denoted by $(a, l_a, r_a)_{LR}$, with a mode a and a right(left) spread $r_a > 0 (l_a > 0)$ is

$$\mu_A(x) = \begin{cases} L_A\left(\dfrac{a-x}{l_a}\right) \text{if} & 0 \le a - x \le l_a, \\ R_A\left(\dfrac{x-a}{r_a}\right) \text{if} & 0 \le x - a \le r_a, \\ 0 & \text{otherwise,} \end{cases}$$

where the functions L and R are continuous and strictly decreasing functions on $[0,1]$ with $L_A(1) = R_A(1) = 0$ and $L_A(0) = R_A(0) = 1$ (Zadeh, 1965). Specially, if the left and right spread are same, we denote the symmetric fuzzy number as $(a, s)_{LR}$. And if $L_A(x) = R_A(x) = 1 - x$, we call the LR-fuzzy number a triangular fuzzy number and denote as $(a, l_a, r_a)_T$. An alpha-level set of the fuzzy number A with the membership function μ_A, denoted by $A(\alpha)$, is defined by $A(\alpha) = \{x \in R | \mu_A(x) \ge \alpha\}$ for all $\alpha \in (0,1]$. The α-level set of the fuzzy number A can be represented as follows:

$$A(\alpha) = [l_A(\alpha), r_A(\alpha)],$$

where $l_A(\alpha) = a - l_a L_A^{-1}(\alpha)$ and $r_A(\alpha) = a + r_a R_A^{-1}(\alpha)$. The 0-level set $A(0)$ is defined as the closure of the set $\{x \in R | \mu_A(x) > 0\}$.

Fuzzy regression models can be classified into two types based on the functional relationship between the dependant and independent variables, which is expressed by response functions. If the relationship is unknown, the model is called a nonparametric fuzzy regression model (Cheng & Lee 1999, Kao & Lin 2005, Kim & Chen 1997, Nevitt & Tam 1998, Wang et al. 2007, Calonico et al. 2014, Razzaghnia et al. 2015, Jung et al. 2015). And if the relationship is known, it is called a parametric fuzzy regression model (Diamond 1988, Bargiela 2007, Wu 2008, Kim et al. 2008, Yoon & Choi 2013, Jung et al. 2014, Lee et al. 2015, Yoon et al. 2016, Lee et al. 2017). Various fuzzy regression methods using fuzzy distance have suggested to analyze the fuzzy regression models.

In this paper, we introduce nonparametric fuzzy regression methods such as Rank transform method (Iman & Cononver, 1979), Theil's method (Theil, 1950), Kernel method (Cheng & Lee, 1999), k-nearest neighborhood method(k-NN), and Median smoothing method based on the basis of an alpha level set of a fuzzy data, and discuss the efficiency of the proposed nonparametric methods.

2 Nonparametric Fuzzyreg Ression

In this section, we apply nonparametric methods, which use equations of the components of the α-level set of the observed fuzzy numbers, to estimate the value Y_i based on $\{X_{i1}(\alpha),\cdots,X_{ip}(\alpha),Y_i(\alpha)\}$ where $X_{ik}(\alpha)$ and $Y_i(\alpha)$ are the α-level sets of X_{ik} and Y_i, respectively.

Zadeh introduced resolution identity theorem (Zadeh, 1975) which can define a fuzzy set using α-level sets family. Based on the resolution identity, the membership function of a fuzzy set can be estimated using a finite number of α-level sets. This study introduces several non-parametric fuzzy regression method such as fuzzy Kernel method, k-nearest neighborhood method(k-NN), and Median smoothing method which use weights. In addition,Theil's method and the Rank transformation method are introduced. Rank transform method is a simple procedure where the data are merely replaced with their corresponding ranks i.e. assign rank 1 to the smallest observation and continue to rank n for the largest observation. Also Theils method is introduced in this paper which involves basically calculating the regression coefficients for all possible pairs of the sets. In order to estimate a non-parametric fuzzy regression model using a finite number of α-level sets of independent and dependent variables, we propose following procedures. The five estimation methods that are proposed in this paper can be conducted similarly based on following 5 steps.

(i) Estimate the mode y_i of Y_i based on the nonparametric method and the set

$$\{(l_{X_i}(1),l_{Y_i}(1)) : i=1,\cdots,n\}$$

and

$$\{(r_{X_i}(1),r_{Y_i}(1)) : i=1,\cdots,n\},$$

where $l_{X_i}(1)=r_{X_i}(1)=(x_{i1}(1),\cdots,x_{ip}(1))$.

(ii) Estimate the endpoints $\hat{l}_{Y_i}(\alpha_o)$ and $\hat{r}_{Y_i}(\alpha_o)$ of the output Y_i from the nonparametric method based on the set

$$\{(l_{X_i}(\alpha_o),l_{Y_i}(\alpha_o)) : i=1,\cdots,n\}$$

and

$$\{(r_{X_i}(\alpha_o),r_{Y_i}(\alpha_o)) : i=1,\cdots,n\},$$

where

$$l_{X_i}(\alpha_o)=(l_{x_{i1}}(\alpha_o),\cdots,l_{x_{ip}}(\alpha_o))$$

and

$$r_{X_i}(\alpha_o)=(r_{x_{i1}}(\alpha_o),\cdots,r_{x_{ip}}(\alpha_o))$$

for some $\alpha_o\in(0,1)$.

(iii) Estimate the pseudo endpoints $\bar{l}_{Y_i}(\alpha^*)$ and $\bar{r}_{Y_i}(\alpha^*)$ of the output Y_i from the nonparametric method based on the set

$$\{(l_{X_i}(\alpha^*),l_{Y_i}(\alpha^*)) : i=1,\cdots,n\}$$

and

$$\{(r_{X_i}(\alpha^*),r_{Y_i}(\alpha^*)) : i=1,\cdots,n\},$$

where

$$l_{X_i}(\alpha^*)=(l_{x_{i1}}(\alpha^*),\cdots,l_{x_{ip}}(\alpha^*))$$

and

$$r_{X_i}(\alpha^*)=(l_{x_{i1}}(\alpha^*),\cdots,l_{x_{ip}}(\alpha^*))$$

for $\alpha^*\neq\alpha_o$.

(iv) The estimation of endpoints of $\hat{Y}_i(\alpha^*)$ are proposed by

$$\hat{l}_{Y_i}(\alpha^*)=\begin{cases}\mathrm{Max}\{\hat{l}_{Y_i}(\alpha_o),\mathrm{Min}\{\bar{l}_{Y_i}(\alpha^*),\hat{l}_{Y_i}(1)\}\} & \text{if } \alpha^*\geq\alpha_o\\ \mathrm{Min}\{\bar{l}_{Y_i}(\alpha_o),\hat{l}_{Y_i}(\alpha^*)\} & \text{if } \alpha^*<\alpha\end{cases}$$

and

$$\hat{r}_{Y_i}(\alpha^*)=\begin{cases}\mathrm{Min}\{\hat{r}_{Y_i}(\alpha_o),\mathrm{Min}\{\bar{r}_{Y_i}(\alpha^*),\hat{r}_{Y_i}(1)\}\} & \text{if } \alpha^*\geq\alpha_o\\ \mathrm{Max}\{\bar{r}_{Y_i}(\alpha_o),\hat{r}_{Y_i}(\alpha^*)\} & \text{if } \alpha^*<\alpha\end{cases}$$

from based on the results in (i)-(iii).

(v) Estimate the reference functions $L_{\hat{Y}_i}(\cdot)$ and $R_{\hat{Y}_i}(\cdot)$ of the output Y_i applying the least squares method based on

$$\{(\hat{l}_{Y_i}(\alpha_j),\alpha_j)|j=1,\cdots,s\}$$

and

$$\{(\hat{r}_{Y_i}(\alpha_j),\alpha_j)|j=1,\cdots,s\}.$$

Using above procedure several fuzzy nonparametric estimators are proposed.

2.1 k-nearest neighborhood method(k-NN)

The k-nearest neighbors(k-NN) method is a nonparametric method used for classification and regression (Altman, 1992). The k-nearest neighbor estimate is the weighted average in a varying neighborhood based on the observation $\{(X_i,Y_i) : i = 1,\cdots,n\}$. The neighborhood is defined as the k-nearest neighbors of x in Euclidean distance. The k-nearest neighbor smoother is defined as (Altman, 1992)

$$\hat{Y}_i = S(x = X_i) = \sum_{j=1}^{n} w_j(x)Y_j,$$

where $\hat{Y}_i$ is the estimate of Y_i and $w_j(x)$, is the weight sequence defined through the set of indexes $J_x = \{j : X_j$ is one of the k-nearest observations to $x\}$. Here,

$$w_j(x) = \begin{cases} \frac{1}{k} & \text{if } j \in J_x, \\ 0 & \text{o.w} \end{cases}$$

For fuzzy non-parametric k-NN estimation, let us define the set $J_{l_{X_i}}(\alpha)$ for $l_{X_i}(\alpha)$ as follows.
$J_{l_{X_i}(\alpha)} = \{j : l_{X_j(\alpha)}$ is the k-nearest observation of $l_{X_i}(\alpha)\}$, where $l_{X_i}(\alpha) = \{(l_{x_{i1}}(\alpha),\cdots,l_{x_{ip}}(\alpha)) : i = 1,\cdots,n\}$.
Based on k-NN method, a fuzzy non-parametric regression estimator estimator of left endpoint is proposed as follows:

$$\hat{l_{Y_i}}(\alpha) = \sum_{j=1}^{n} w_j(l_{x_i}(\alpha))l_{Y_j}(\alpha).$$

From the similar procedure, the right endpoint can be estimated as

$$\hat{r_{Y_i}}(\alpha) = \sum_{j=1}^{n} w_j(l_{x_i}(\alpha))r_{Y_j}(\alpha).$$

2.2 Median smoothing method

This method is similar to the K-nearest neighbor smoothing. The neighborhood is defined as the K-nearest neighbors of x in Euclidean distance. The Median smoother is defined as

$$\hat{Y}_i = S(x = X_i) = \text{Med}_{j\in J_x}(Y_j),$$

Based on Median smoothing method, a fuzzy non-parametric regression estimator of left endpoint is proposed as follows:

$$\hat{l_{Y_i}}(\alpha) = \text{Med}\{l_{Y_j}(\alpha) : j \in J_{l_{X_i}(\alpha)}\},$$

where $J_{l_{X_i}(\alpha)}$ is defined above.

From the similar procedure, the right endpoint can be estimated as

$$\hat{r_{Y_i}}(\alpha) = \text{Med}\{r_{Y_j}(\alpha) : j \in J_{r_{X_i}(\alpha)}\}.$$

2.3 Kernel method

A sample approach to represent the weight sequence in the local averaging method is to represent the weight distribution by a density function, which contains a scale parameter that adjusts the size and the form of the weights according to the location of the point with respect to the point estimation x. This density function is known as the kernel function. The kernel estimate, $S(x)$, is defined as a weighted average of the response variable in a fixed neighborhood around x, determined in a shape by the kernel function K and the bandwidth h (Cheng and Lee, 1999). Kernel estimate, $S(x)$, is defined as

$$\hat{Y}_i = S(x = X_i) = \sum_{j=1}^{n} w_j(x)Y_j$$

and the weight sequence is

$$w_j(x) = \frac{K_h(x - X_j)}{P_h(x)},$$

where $P_h(x) = \sum_{j=1}^{n} K_h(x - X_j)$ and $K_h(u) = \frac{1}{h}K(\frac{u}{h})$ in which $K(\frac{u}{h})$ is the kernel with scale factor h. In this paper the kernel function is defined as follow:

$$K(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{if } |x| < 1 \\ 0 & o.w. \end{cases}$$

For the fuzzy non-parametric Kernel estimator, the left endpoint is proposed:

$$l_{Y_i}(\alpha) = \frac{1}{S_{l_{Y_i}(\alpha)}} \sum_{k=1}^{n} K\left(\frac{l_{X_k}(\alpha) - l_{X_i}(\alpha)}{h}\right) l_{Y_k}(\alpha),$$

where

$$S_{l_{Y_i}(\alpha)} = \sum_{j=1}^{n} K\left(\frac{l_{X_j}(\alpha) - l_{X_i}(\alpha)}{h}\right)$$

and

$$r_{Y_i}(\alpha) = \frac{1}{S_{r_{Y_i}(\alpha)}} \sum_{k=1}^{n} K\left(\frac{r_{X_k}(\alpha) - r_{X_i}(\alpha)}{h}\right) r_{Y_k}(\alpha),$$

where

$$S_{r_{Y_i}(\alpha)} = \sum_{j=1}^{n} K\left(\frac{r_{X_j}(\alpha) - r_{X_i}(\alpha)}{h}\right)$$

for the estimator of the right endpoint.

2.4 Theil's method

Theil's estimation method is widely used nonparametric regression method. Theil's estimation method can be applied to fuzzy regression using median in accordance with mode and end points of the alpha-level sets.(Choi et.al.2016)

For the fuzzy non-parametric Theil's estimator, the left endpoint of the slope is proposed as follows (Choi et.al.2016):

$$\text{Med}\left\{ \frac{l_{Y_i}(\alpha) - l_{Y_j}(\alpha)}{|l_{X_i}(\alpha) - l_{X_j}(\alpha)|} : 1 \le i < j \le n \right\}.$$

And from the relation

$$l_{Y_i}(\alpha) = l_{A_0}(\alpha) + l_{A_1}(\alpha) \cdot l_{x_{i1}}(\alpha),$$

estimator of the left end point of the constant term can be obtained as follows:

$$\hat{l_{A_0}}(\alpha) = \bar{l_{Y_i}}(\alpha) - \hat{l_{A_1}}(\alpha) \cdot \bar{l_{x_{i1}}}(\alpha)$$

From the similar procedure, the right endpoint can be estimated as

$$\text{Med}\left\{ \frac{r_{Y_i}(\alpha) - r_{Y_j}(\alpha)}{|r_{X_i}(\alpha) - r_{X_j}(\alpha)|} : 1 \le i < j \le n \right\}.$$

and

$$\hat{r_{A_0}}(\alpha) = \bar{r_{Y_i}}(\alpha) - \hat{r_{A_1}}(\alpha) \cdot \bar{r_{x_{i1}}}(\alpha)$$

For multiple fuzzy regression based on Theil's method, see (Choi et.al.2016).

2.5 Rank transform method

Iman and Conover (Iman and Conover, 1979) introduced Rank tranform method and showed that it is a robust and powerful procedure in hypothesis testing with respect to experimental designs. Assume that we have data $(x_i, y_i), i = 1, \cdots, n$. The dependent variable y_i is replaced by each corresponding rank. $R(y_i)$ is the rank assigned to i_{th} value of Y. Similarly, independent variable x_i is also replaced by each rank. $R(x_i)$ is the rank assigned to i_{th} value of X. For arbitrary given independent variable x^*, to get the predicted value $\hat{y}$ the $R(x^*)$ is defined as follows:

1)If $x^* < x_{(1)}$, then $R(x^*) = R(x_{(1)})$
2)If $x^* > x_{(n)}$, then $R(x^*) = R(x_{(n)})$
3)If $x^* = x_{(i)}$, then $R(x^*) = R(x_{(i)})$
4)If $x_{(i)} < x^* < x_{(i+1)} (i = 1, \cdots, n-1)$, then $R(x^*) = R(x_{(i)}) + [R(x_{(i+1)}) - R(x_{(i)})] \times \frac{x^* - x_{(i)}}{x_{(i+1)} - x_{(i)}}$

The fuzzy Rank transform estimator of left endpoint can be obtained using following rank which is proposed in (Jung et.al.,2015).

$$R(l_{Y_i}(\alpha)) = \frac{n+1}{2} + \beta_1(\alpha)(R(l_{x_{i1}}(\alpha)) - \frac{n+1}{2})$$

From the similar procedure, the right endpoint can be estimated using

$$R(r_{Y_i}(\alpha)) = \frac{n+1}{2} + \beta_1(\alpha)(R(r_{x_{i1}}(\alpha)) - \frac{n+1}{2})$$

For the estimators of multiple fuzzy regression, see (Jung et.al.,2015).

3 Numerical Example

In order to compare the efficiencies of the fuzzy regression models obtained by using five nonparametric methods, we use the data introduced by Diamond(1988), which has fuzzy input and fuzzy output. Also, we use a performance measure, $d\left(Y_i, \widehat{Y_i}\right)$ which is the sum of the difference area between the observed and the estimated fuzzy data, to compare the accuracy of the constructed fuzzy regression models (Jung et.al.,2015).

$$d\left(Y_i, \widehat{Y_i}\right) = \frac{\int_{-\infty}^{\infty} |\mu_{Y_i}(x) - \mu_{\widehat{Y_i}}(x)| dx}{\int_{-\infty}^{\infty} \mu_{Y_i}(x) dx} + h_d(Y_i(0), \widehat{Y_i}(0)), \tag{2}$$

where $h_d(A, B) = sup_{a \in A} inf_{b \in B} |a - b|$.

A data (Diamond, 1988) is introduced to compare proposed five fuzzy non-parametric method.

Table 2.1 Data given by Diamond

Input $(x_i, l_{x_i}, r_{x_i})_T$	Output $(y_i, l_{y_i}, r_{y_i})_T$
$(21, 4.20, 2.10)_T$	$(4.0, 0.6, 0.8)_T$
$(15, 2.25, 2.25)_T$	$(3.0, 0.3, 0.3)_T$
$(15, 1.50, 2.25)_T$	$(3.5, 0.35, 0.35)_T$
$(9, 1.35, 1.35)_T$	$(2.0, 0.4, 0.4)_T$
$(12, 1.20, 1.20)_T$	$(3.0, 0.3, 0.45)_T$
$(18, 3.60, 1.80)_T$	$(3.5, 0.53, 0.7)_T$
$(6, 0.60, 1.20)_T$	$(2.5, 0.25, 0.38)_T$
$(12, 1.80, 2.40)_T$	$(2.5, 0.5, 0.5)_T$

Using the performance measure defined above, the errors are obtained in Table 2.2 to compare the accuracies.

Table 2.2 Estimation Errors

Methods	3-NN	MSM	Kernel	Theil's	RTM
Error	2.206	2.069	2.226	1.756	2.124
	1.200	1.200	1.005	1.155	1.200
	1.319	1.400	1.297	1.110	1.434
	1.190	1.600	1.564	1.550	1.423
	1.500	1.500	0.813	0.811	1.189
	1.887	1.800	1.698	1.008	1.412
	1.286	1.288	1.319	0.776	1.154
	1.907	1.674	1.622	1.791	1.907
Sum	12.495	12.528	11.544	9.958	12.115

From Table 2.2, it is confirmed that Theil's method is more efficient than other methods with given data.

This paper can be extended for more properties of the estimators such as consistency, rate of convergence and other asymptotic properties, which show the properties of each estimators mathematically. And more efficient conditions which explain the properties of non-parametric methods used in fuzzy regression analysis can be dealt with in our next research.

4 Conclusions

In this paper, in order to construct a fuzzy regression model, we proposed five nonparametric methods, which are known as the distribution free methods and not dependent on the error distribution. The example given showed that Theil's Method is more efficient than other methods. Further studies such as consistency, rate of convergence, and other asymptotic theory are needed for more efficient conditions which explain the properties of nonparametric methods used in the fuzzy regression analysis.

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