Group consensus tracking control of dynamical multi-agent systems with time delays via pinning leader-follower approach

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Abstract: In this paper, we focus on studying the group consensus tracking issue of single-integrator and second-integrator multi-agent systems with fixed communication topology and time delays under a pinning control protocol, respectively. For the former, We aim to propose some necessary and/or sufficient group consensus tracking conditions by using Lyapunov-Krasovskii function. For the latter, the observer-based bounded group consensus tracking control problem of second-order multi-agent systems in a disturbance environment is investigated, and some sufficient bounded group consensus tracking criteria are established. Moreover, this paper proposes a method of graph refactoring to find the relationship between the communication topology graph and matrix. Finally, numerical simulations are given to verify the effectiveness of our theoretical results.

Key–Words: Multi-agent systems, group consensus tracking, pinning control, Lyapunov-Krasovskii function, distributed observer.

1 Introduction

In recent years, consensus problem of multi-agent systems (MASs) has attracted a great deal of attention because of its broad applications including scheduling of automated highway systems, formation control of satellite clusters, cooperative control of mobile autonomous robots, flocking control of multi-agent systems, spacecraft formation flying and so on. As far as we know, consensus problem can be divided into leaderless consensus and leader-following consensus according to the existence/nonexistence of a leader. Leaderless consensus means to design appropriate protocols and algorithms such that a group of agents can converge to a common value. Leader-following consensus (i.e., coordinated tracking problem or consensus tracking problem) represents that in a multi-agent team, there exists a leader or a reference signal which specifies an objective for all other agents (i.e., follower) to follow.

Recently, many generalized consensus problems such as consensus tracking problem and group consensus were addressed in the literature. [1] investigated the consensus tracking for a class of second-order multi-agent dynamic systems with disturbances and unmodeled agent dynamic. Consensus tracking problems with, respectively, bounded control effort and directed switching interaction topologies were considered in [2]. [3] considered a leader-following consensus problem of a group of autonomous agents with time-varying coupling delays. The observer-based bounded tracking control problem for multi-agent systems in a disturbance environment via sampled-data control was investigated in [6]. Group consensus problem aims to design appropriate protocols and algorithms such that agents in a network reach more than one consistent state, that is, the group consensus problem considers a network which is divided into multiple sub-networks, and each sub-network reach a different consistent value at last. In [9], group consensus in multi-agent systems with switching topologies and communication delays was discussed by introducing double-tree-form transformation. Group consensus of multi-agent systems with undirected topology was considered in [10] and several criteria were established based on graph theories and matrix theories. When the information exchange was directed, a novel consensus protocol was designed to solve the group consensus problem in [11]. [12] investigated a group consensus problem of multi-agent systems with sampled data, and a distributed linear consensus protocol was first designed to solve the group consensus problem. In [15-18], all the agents took the form of single-integrator dynamics, while [13] studied the group consensus of double-integrator dynamic, and presented convergence analysis by applying two different consensus protocols. On the other hand, due to the absence of global information that can be used to achieve group coordination, it is necessary to consider the distributed **observers design** issue for multi-agent systems (See [6], [7] and [8]).

Moreover, nowadays **group consensus tracking issue** of multi-agent systems has been pointed out and attracted the researcher's attention, which can be regarded as a generalization of consensus tracking problem and group consensus. Due to the complexity of group consensus tracking issue, few papers has been published in this topic. The group consensus tracking issue of second-order nonlinear multi-agent systems via pinning control scheme were addressed in [14], where the pinning protocol was designed according to the agent's property, that is, the inter-act agent and the intra-act agent. [15] investigated the group consensus tracking problem for multi-agent networks with directed topology and static virtual leader by designing a novel pinning control protocol, that is, the nodes with zero in-degree should be pinned.

Inspired by the above analysis, in this paper, we will investigate the group consensus tracking issue of continuous-time first-order and second-order multi-agent systems with active virtual leaders and time delays, respectively. Firstly, we use a novel pinning scheme, where the node with zero in-degree will be pinned. Secondly, we apply Lyapunov function method to solve the group consensus tracking issue and give the upper bound of delay τ, respectively. Thirdly, we adopt the method of graph refactoring to find the relationship between the matrix $L+D$ and a new refactored graph, which provides an easy way to determine whether $L+D$ is positive stable.

The outline of this paper is as follows. Some basic definitions and supporting results are presented in Section 2. Our main results are given in Section 3. Numerical examples are given in Section 4 to illustrate our results. And conclusions are finally drawn in Section 5.

Notations: We use standard notations throughout this paper. $\mathbb{R}^n$ and $\mathbb{C}$ denote the n-dimensional Euclidean space and the sets of complex numbers, respectively. I_n represents the identity matrix of dimension n, and $\boldsymbol{0}_n$ represents the zero matrix of dimension n. $diag\{\cdot\}$ and $det(\cdot)$ represent the diagonal matrix and determinant of a matrix, respectively. $Re(\cdot)$ and $Im(\cdot)$ denote real and imaginary part, respectively. For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). $\bar{a}$ represents the conjugate complex number of a. The superscript T denotes the transpose. $\Lambda(A)$ denotes all the eigenvalues of matrix A.

2 Problem formulations and preliminaries

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a weighted directed graph of order N with the set of nodes $\mathscr{V}$=$\{v_1, v_2, \cdots, v_N\}$, set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and the nonsymmetric weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with real adjacency elements a_{ij}. The node indexes belong to a finite index set $\ell = \{1, 2, \cdots, N\}$. An edge of $\mathscr{G}$ is denoted by $e_{ij} = (v_j, v_i)$. The adjacency elements associated with the edges of the graph are nonzero, i.e., $e_{ij} \in \mathscr{E}$ if and only if $a_{ij} \neq 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \ell$. In this paper, the graph is always supposed to be simple (without self-loops and multiple edges), i.e., $a_{ii} = 0$ for all $i \in \ell$. The in-degree and out-degree of the node i can be defined as follows:

$$deg_{in}(i) = \sum_{j=1, j\neq i}^{N} a_{ij},$$
$$deg_{out}(i) = \sum_{j=1, j\neq i}^{N} a_{ji}.$$

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with the weighted adjacency matrix A is defined as follows:

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \sum_{k=1, k\neq i}^{N} a_{ik}, & j = i. \end{cases}$$

The communication topology graph $\mathscr{G}$ is said to contain a directed spanning tree if there exists at least one node which has a directed path to all the other nodes.

2.1 Problem formulations

In this section, some basic knowledge on graph theory, problem formulations, some definitions and lemmas are given as the preliminaries of this paper.

2.2 Graph theory

Suppose that the network has N followers and m leaders, $m \geq 2$, where all the followers are divided into m groups and each group has a leader. Introduce a function $\sigma : \ell := \{1, 2, \cdots, N\} \to \{1, 2, \ldots, m\}$, where $\sigma(i) = j$ denotes the i-th agent belongs to the j-th group. For simplicity, in the sequel, we use σ_i instead of $\sigma(i)$. Let ϕ_j denote the set of all the nodes in the j-th group with $\{1, 2, \ldots, N\} = \phi_1 \cup \phi_2 \cup \cdots \cup \phi_m$ and $\phi_p \cap \phi_q = \emptyset$ for $p \neq q$.

The dynamic of each follower in the first-order MASs is described as follows:

$$\dot{x}_i(t) = u_i(t), i \in \ell := \{1, 2, \cdots, N\}, \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the position state of the i-th agent, and $u_i \in \mathbb{R}^n$ is the control input.

The dynamics of the active virtual leader in every group for multi-agent system (1) is described as follows:

$$\dot{y}_{\sigma_i}(t) = a_{\sigma_i}(t), i \in \ell, \sigma_i \in \{1, 2, \ldots, m\}, \quad (2)$$

where $y_{\sigma_i} \in \mathbb{R}^n$ is the position state of the σ_i-th virtual leader. $a_{\sigma_i} \in \mathbb{R}^n$, and if $a_{\sigma_i} \neq 0$, then the σ_i-th virtual leader is dynamic, otherwise the σ_i-th virtual leader is static.

In multi-agent systems, time delays are usually inevitable due to the possible slow process of interactions among agents, which may cause that the agents can not get instant information from each other. Consensus algorithms without considering time delays may lead to decreased system performance. Therefore, this paper will adopt the control protocol with time delays as follows:

$$\begin{aligned} u_i(t) = a_{\sigma_i}(t) + \alpha \sum_{j=1 j\neq i}^{N} a_{ij}[x_j(t-\tau) \\ - x_i(t-\tau)] + \alpha \sum_{j=1}^{N} l_{ij} y_{\sigma_j}(t-\tau) \\ - \alpha d_i[x_i(t-\tau) - y_{\sigma_i}(t-\tau)], \end{aligned} \quad (3)$$

where $\alpha, d_i, a_{ij}, l_{ij}$ are defined as above and time delay $\tau > 0$ is a constant.

Remark 1. *Our protocol (3) not only takes full account of the effect of time delay but also assumes that the virtual leader in protocol (3) can be static or dynamic, which generalizes the control protocol in [15] to a more practical case.*

Define $e_i(t) = x_i(t) - y_{\sigma_i}(t)$, $e(t) := [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$, $H := L + D$, then we get

$$\dot{e}(t) = -\alpha H e(t-\tau), \quad (4)$$

where $D = diag\{d_1, d_2, \ldots, d_N\}$ with the pinned coefficient $d_i \geq 0$ is the pinned matrix.

Remark 2. *Summarizing the above analysis, the group consensus tracking of system (1) can be equivalently converted into the asymptotical stability of the error system (4).*

The dynamics of each follower in the second-order MASs is described by

$$\begin{cases} \dot{x}_i(t) = v_i(t) + \delta_i^1(t), \\ \dot{v}_i(t) = u_i(t) + \delta_i^2(t), i \in \ell := \{1, 2, \cdots, N\}, \end{cases} \quad (5)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are defined as above, $v_i \in \mathbb{R}^n$ is the velocity of the i-th agent. $\delta_i^j(t) (j = 1, 2)$ are the bounded disturbances of system (5), which satisfies $|\delta_i^j| \leq \tilde{\delta} < +\infty$ for all $j = 1, 2; i \in \ell$.

The dynamics of the active virtual leader in each group of multi-agent system (5) is described as follows:

$$\begin{cases} \dot{y}_{\sigma_i}(t) = w_{\sigma_i}(t), \\ \dot{w}_{\sigma_i}(t) = a_{\sigma_i}(t), i \in \ell, \sigma_i \in \{1, 2, \ldots, m\}, \end{cases} \quad (6)$$

where $y_{\sigma_i} \in \mathbb{R}^n$ and $a_{\sigma_i} \in \mathbb{R}^n$ are defined as above, $w_{\sigma_i} \in \mathbb{R}^n$ is the velocity the σ_i-th virtual leader. Since it is difficult to obtain the velocity value w_{σ_i} of the leader in real time in some practical engineering system, it is necessary to design a distributed observer to estimate it. To be more specific, denote by $\hat{v}_i$ an estimate of w_{σ_i} by agent i. Then, in order to guarantee that the follower-agent i can track the active leader in each group, the following observer-based pinning control protocol is given as follows:

$$\begin{aligned} \dot{\hat{v}}_i(t) = a_{\sigma_i}(t) + \frac{\alpha}{\beta}[\sum_{j=1 j\neq i}^{N} a_{ij}(x_j(t-\tau) \\ - x_i(t-\tau)) - d_i(x_i(t-\tau) - y_{\sigma_i}(t-\tau)) \\ + \sum_{j=1}^{N} l_{ij} y_{\sigma_j}(t-\tau)], \end{aligned} \quad (7)$$

$$\begin{aligned} u_i(t) = a_{\sigma_i}(t) - \beta(v_i(t) - \hat{v}_i(t)) \\ + \alpha \sum_{j=1 j\neq i}^{N} a_{ij}[x_j(t-\tau) - x_i(t-\tau)] \\ + \beta \sum_{j=1 j\neq i}^{N} a_{ij}[v_j(t-\tau) - v_i(t-\tau)] \\ + \alpha \sum_{j=1}^{N} l_{ij} y_{\sigma_j}(t-\tau) + \beta \sum_{j=1}^{N} l_{ij} w_{\sigma_j}(t-\tau) \\ - \alpha d_i[x_i(t-\tau) - y_{\sigma_i}(t-\tau)] \\ - \beta d_i[v_i(t-\tau) - w_{\sigma_i}(t-\tau)], \end{aligned} \quad (8)$$

where $\alpha, \beta, a_{ij}, l_{ij}$ are defined as above and time delay $\tau > 0$ is a constant.

Define $\xi(t) := [x^{*T}(t), v^{*T}(t), \hat{v}^{*T}(t)]^T$. Then, system (1) under observer-based control protocol (7)(8) can be further rewritten as

$$\dot{\xi}(t) = W_1 \xi(t) + W_2 \xi(t-\tau) + \delta(t), \quad (9)$$

where $W_1 := \begin{bmatrix} \mathbf{0}_N & I_N & \mathbf{0}_N \\ \mathbf{0}_N & -\beta I_N & \beta I_N \\ \mathbf{0}_N & \mathbf{0}_N & \mathbf{0}_N \end{bmatrix}$, $W_2 := \begin{bmatrix} \mathbf{0}_N & \mathbf{0}_N & \mathbf{0}_N \\ -\alpha H & -\beta H & \mathbf{0}_N \\ -\frac{\alpha}{\beta} H & \mathbf{0}_N & \mathbf{0}_N \end{bmatrix}$, matrix H and $\delta(t)$ are defined as above.

Remark 3. *Summarizing the above analysis, the bounded group consensus tracking of system (5) under the observer-based pinning control protocol (7) (8) can be equivalently converted into the boundness of the tracking error system (9), respectively. This provides us an efficient method to study the bounded group consensus tracking of system (5).*

2.3 Important lemmas

Some basic lemmas are given in this subsection.

Lemma 1. *[19] For a nonsingular matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*
(i) matrix A is positive stable;
(ii) all eigenvalues of A have positive real part, i.e., $Re(\lambda_i(A)) > 0$ for all $i = 1, 2, \ldots, n$.

Lemma 2. *[3] For the communication topology graph $\bar{\mathscr{G}}$ with N followers and a virtual leader, the matrix $H = L + B$ is positive stable if and only if the communication topology graph $\bar{\mathscr{G}}$ contains a directed spanning tree, where L is Laplacian matrix, $B = diag\{b_1, b_2, \ldots, b_N\}$ is the leader adjacency matrix associated with topology graph $\bar{\mathscr{G}}$, where $b_i > 0$ if node 0 has access to the node i (that is, there exists a directed path from the virtual leader to agent i, $i \in \ell$), and otherwise $b_i = 0$.*

Lemma 3. *[18] Let $Y = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where $A_{ij} \in \mathbb{R}^{n\times n}, i,j = 1,2,3$. If matrix $A_{ij}, i,j = 1,2,3$ commute pairwise, then $det(Y) = det(A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{32}A_{23})$.*

Lemma 4. *[5] Consider the following systems:*

$$\begin{cases} \dot{x} = f(x_t), t > 0, \\ x(\theta) = \psi(\theta), \theta \in [-r, 0], \end{cases} \tag{10}$$

where $x_t(\theta) = x(t+\theta)$, $\forall\, \theta \in [-r, 0]$ and $f(0) = 0$. Let $\mathbb{C}([-r, 0], \mathbb{R}^n)$ be a Banach space of continuous functions defined on an interval [-r,0], taking values in $\mathbb{R}^n$ with the topology of uniform convergence, and with a norm $\|\psi\|_c = \max_{\theta\in[-r,0]} \|\psi(\theta)\|$.

Let ϕ_1, ϕ_2 and ϕ_3 be continuous, nonnegative, nondecreasing functions with $\phi_1(s) > 0$, $\phi_2(s) > 0$, $\phi_3(s) > 0$ for $s > 0$, $\phi_1(0) = \phi_2(0) = 0$. For system (10), suppose that the function f: $\mathbb{C}([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ takes bounded subsets of $\mathbb{C}([-r, 0], \mathbb{R}^n)$ in bounded subsets of $\mathbb{R}^n$. If there exist a continuous function V: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a continuous nondecreasing function $\phi(s)$ with $\phi(s) > s$ for $s > 0$ such that
(i) $\phi_1(\|x\|) \le V(t, x) \le \phi_2(\|x\|)$,
(ii) $\dot{V}(t, x(t)) \le -\phi_3(\|x\|)$, if $V(t+\theta, x(t+\theta)) < \phi(V(t, x(t)))$, $\theta \in [-r, 0]$,
then the solution $x = 0$ is uniformly asymptotically stable.

Lemma 5. *[4] Given a complex coefficient third-order polynomial as follows: $f(s) = s^3 + a_1 s^2 + a_2 s + a_3$, where $a_1, a_2, a_3 \in \mathbb{C}$, all the roots of polynomial $f(s)$ are in the open left half plane, i.e., $f(s)$ is Hurwitz stable if and only if:*

(i) $Re(a_1) > 0$;
(ii) $Re(a_1)Re(a_1\bar{a}_2 - a_3) - Im^2(a_2) > 0$;
(iii) $[Re(a_1)Re(a_2\bar{a}_3) - Re^2(a_3)][Re(a_1) \times Re(a_1\bar{a}_2 - a_3) - Im^2(a_2)] - [Re(a_1) \times Im(\bar{a}_1 a_3) + Re(a_3)Im(a_2)]^2 > 0$.

Lemma 6. *[17] Suppose that a symmetric matrix is partitioned as $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, where S_{11}, S_{22} are square matrices. Then the following three conditions are equivalent:*

(i) $S > 0 (< 0)$;
(ii) $S_{11} > 0 (< 0)$ and $S_{22} - S_{12}^T S_{11}^{-1} S_{12} > 0 (< 0)$;
(iii) $S_{22} > 0 (< 0)$ and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T > 0 (< 0)$.

For the directed topology graph $\bar{\mathscr{G}}$ with N followers and a virtual leader, Lemma 2 established the relationship between the communication topology graph and the matrix $L + B$. Naturally, we can't help asking the following problem: **for the multi-agent system with m groups and m leaders in this paper, is it possible for us to establish similar relationship between the matrix $L + D$ and a new communication topology graph?**

Our answer to this question is positive. A lemma similar to Lemma 2 is established as follows:

Lemma 7. *The following three conditions are equivalent:*
(a) The matrix $H = L + D$ in system (4)/(9) is positive stable.
(b) There exists a new refactored communication topology graph $\bar{\mathscr{G}}$ containing a directed spanning tree.
(c) The nodes with zero in-degree in the follower's communication topology graph should be chosen as

the pinned nodes, whose pinned coefficient can choose arbitrary nonzero constants.

Proof. (I) We first prove the equivalence between (a) and (b). The specific refactoring method is outlined as follows:

Step 1: Determine what kind of nodes should be pinned.

Step 2: Add virtual leader (denoted by node 0) in the communication topology graph of the followers, and guarantee that node 0 has access to those pinned nodes.

Then, we obtain a new topology graph $\bar{\mathscr{G}}$.

If the i-th agent is pinned, then the corresponding pinned coefficient $d_i > 0$. Here, we may choose $d_i = 1$. For the case $d_i \neq 1$, similar to the proof of Lemma 2, we can still prove this lemma.

It should be further noted that the leader adjacency matrix B of $\bar{\mathscr{G}}$ is equal to the the pinned matrix D of system (1). By Lemma 2, Lemma 7 obviously holds.

(II) Next, we prove the equivalence between (b) and (c).

$(c) \Rightarrow (b)$. According to the refactored method, there must exists a directed path from the node 0 to the pinned nodes. For a given unpinned node i, in order to prove that (b) holds. We need to prove that node 0 also has a directed path to the unpinned nodes. Here, we prove it by contradiction.

Suppose that node 0 has no directed path to a unpinned node, then node 0 will have no directed path to its parent node. Since node 0 has no directed path to its parent node, node 0 must have no directed path to the parent's parent node. And so forth, we can find a node with zero-degree, but there does not exist a directed path from node 0 to it, which is in contradiction with our pinned strategy.

$(b) \Rightarrow (c)$. If a node with zero in-degree is not pinned, according to our refactored mehod, there does not exist a directed path from node 0 to the node, which means that the communication topology graph $\bar{\mathscr{G}}$ does not contain a directed spanning tree. This is in contradiction with (b). Hence, the nodes with zero in-degree must be pinned. Furthermore, from the proof in (I), we know that the pinned coefficient can be arbitrary constants for the pinned nodes. Hence, (c) holds.

Summarizing the above analysis, this lemma holds. □

Based on lemma 7, our pinning strategy is given as follows: **choose the nodes with zero in-degree as the pinned nodes, whose pinned coefficient can be arbitrary nonzero constants.**

Remark 4. *Lemma 7 provides us a new method to determine the stability of $L + D$ by checking whether the refactored communication graph $\bar{\mathscr{G}}$ contains a directed spanning tree, which can be further guaranteed by choosing the pinning strategy. Lemma 7 can also be regarded as an generation of Lemma 2 above.*

Lemma 8. *Suppose that the nodes with zero in-degree are chosen as the pinned nodes. Then, all eigenvalues of matrix $W = W_1 + W_2$ have negative real part, i.e., matrix W is Hurwitz stable if and only if system parameters α, β satisfy*

(I) If $Im(\mu_i) = 0$,

$$\begin{cases} \alpha > 0, \\ \beta > \max\{\frac{1}{\mu_i+1}\}. \end{cases} \tag{11}$$

(II) If $Im(\mu_i) \neq 0$,

$$\begin{cases} \beta > \max\{\frac{Re(\mu_i)}{|\mu_i|^2+Re(\mu_i)}\}, \\ \max\{0, \frac{\mathcal{Y}-\sqrt{\mathcal{Y}^2-4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1)}}{2|\mu_i|^2 Im^2(\mu_i)}\} < \alpha < \\ \min\{\frac{\mathcal{Y}+\sqrt{\mathcal{Y}^2-4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1)}}{2|\mu_i|^2 Im^2(\mu_i)}, \\ \frac{(Re(\mu_i)+1)(\beta^2|\mu_i|^2+\beta^2 Re(\mu_i)-\beta Re(\mu_i))}{Im^2(\mu_i)}\}, \\ \mathcal{Y}^2 - 4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1) > 0, \\ 0 < \mathcal{Y} < \sqrt{\mathcal{Y}^2 - 4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1)} \\ +2|\mu_i|^2(Re(\mu_i)+1)(\beta^2|\mu_i|^2+\beta^2 Re(\mu_i) \\ -\beta Re(\mu_i)), \end{cases} \tag{12}$$

where α, β are defined above, $\mathcal{Y} = \beta|\mu_i|^2(Re(\mu_i) + 1)(\beta|\mu_i|^2 + \beta Re(\mu_i) - 2Re(\mu_i)) - \beta Re(\mu_i) Im^2(\mu_i) + Re^3(\mu_i)$, μ_i is the eigenvalue of matrix H.

Proof. Assume μ_i be the i-th eigenvalue of matrix H. By Lemma 7, $Re(\mu_i) > 0$, $\mu_i \in \Lambda(H)$, $i \in \ell$. Note that the characteristic polynomial of matrix W is given as follows:

$$\begin{aligned} det(sI_{3N} - W) &= det \begin{bmatrix} sI_N & -I_N & \mathbf{0}_N \\ \alpha H & sI_N + \beta I_N + \beta H & -\beta I_N \\ \frac{\alpha}{\beta} H & \mathbf{0}_N & sI_N \end{bmatrix} \\ &= det\{s^3 I_N + (\beta H + \beta I_N)s^2 + \alpha H s + \alpha H\} \\ &= \prod_{i=1}^{N} \{s^3 + \beta(\mu_i + 1)s^2 + \alpha\mu_i s + \alpha\mu_i\} \\ &:= \prod_{i=1}^{N} g_i(s). \end{aligned}$$

Therefore, matrix W is Hurwitz stable if and only if all the roots of polynomial $g_i(s) = 0$ lie in the left half open plane. Furthermore, $g_i(s)$ can be rewritten as

$$\begin{aligned} g_i(s) = s^3 &+ \beta[Re(\mu_i) + 1 + iIm(\mu_i)]s^2 \\ &+ \alpha[Re(\mu_i) + iIm(\mu_i)]s + \alpha[Re(\mu_i) + iIm(\mu_i)]. \end{aligned}$$

($\mathcal{A}$) If $Im(\mu_i) = 0$, by Hurwitz criteria of the real coefficient polynomial, $g_i(s)$ is Hurwitz stable if and only if

$$\begin{cases} \beta(\mu_i + 1) > 0, \\ \alpha\mu_i > 0, \\ \beta\alpha\mu_i(\mu_i + 1) > \alpha\mu_i, \end{cases}$$

which is equivalent to $\alpha > 0, \beta > \frac{1}{\mu_i+1}$. Thus, (11) holds.

($\mathcal{B}$) If $Im(\mu_i) \neq 0$, by Lemma 5, the complex coefficient third-order polynomial $g_i(s)$ is Hurwitz stable if and only if the following inequalities hold:

(1) $\beta[Re(\mu_i) + 1] > 0,$

(2) $[Re(\mu_i) + 1][\beta^2|\mu_i|^2 + \beta^2 Re(\mu_i) - \beta Re(\mu_i)] - \alpha Im^2(\mu_i) > 0,$

(3) $\alpha\beta^2|\mu_i|^2[|\mu_i|^2 + Re(\mu_i)][Re(\mu_i) + 1] - \alpha\beta Re(\mu_i)[2|\mu_i|^2 + 2|\mu_i|^2 Re(\mu_i) + Im^2(\mu_i)] - \alpha^2|\mu_i|^2 Im^2(\mu_i) + \alpha Re^3(\mu_i) - \beta^3 Im^2(\mu_i)[Re(\mu_i) + 1] > 0.$

The above inequalities can be rewritten as

$$\begin{cases} (i) \quad (Re(\mu_i) + 1)(\beta^2|\mu_i|^2 + \beta^2 Re(\mu_i) \\ \qquad -\beta Re(\mu_i)) - \alpha Im^2(\mu_i) > 0, \\ (ii) \quad |\mu_i|^2 Im^2(\mu_i)\alpha^2 - [\beta|\mu_i|^2(Re(\mu_i) + 1)(\beta|\mu_i|^2 \\ \qquad +\beta Re(\mu_i) - 2Re(\mu_i)) - \beta Re(\mu_i) Im^2(\mu_i) \\ \qquad +Re^3(\mu_i)]\alpha + \beta^3 Im^2(\mu_i)(Re(\mu_i) + 1) < 0. \end{cases}$$

We can further get

$$\begin{cases} (i) \quad \alpha < \frac{[Re(\mu_i)+1][\beta^2|\mu_i|^2+\beta^2 Re(\mu_i)-\beta Re(\mu_i)]}{Im^2(\mu_i)}, \\ (ii) \quad \mathcal{Y} > 0, \\ \qquad \Delta = \mathcal{Y}^2 - 4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i) + 1) > 0, \\ \qquad \frac{\mathcal{Y}-\sqrt{\mathcal{Y}^2-4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1)}}{2|\mu_i|^2 Im^2(\mu_i)} < \alpha < \\ \qquad \frac{\mathcal{Y}+\sqrt{\mathcal{Y}^2-4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i)+1)}}{2|\mu_i|^2 Im^2(\mu_i)}, \\ \qquad \mathcal{Y} - \sqrt{\mathcal{Y}^2 - 4\beta^3|\mu_i|^2 Im^4(\mu_i)(Re(\mu_i) + 1)} < \\ \qquad 2|\mu_i|^2(Re(\mu_i) + 1)(\beta^2|\mu_i|^2 + \beta^2 Re(\mu_i) - \beta Re(\mu_i)), \\ \qquad (Re(\mu_i) + 1)(\beta^2|\mu_i|^2 + \beta^2 Re(\mu_i) - \beta Re(\mu_i)) > 0, \end{cases}$$

where $\mathcal{Y} = \beta|\mu_i|^2(Re(\mu_i) + 1)(\beta|\mu_i|^2 + \beta Re(\mu_i) - 2Re(\mu_i)) - \beta Re(\mu_i) Im^2(\mu_i) + Re^3(\mu_i)$.

According to the above inequalities, obviously, (12) holds.

Analyzing the above discussions, $Re(\lambda_i(W)) < 0$, for all $\lambda_i \in \Lambda(W)$, $i \in \ell$, if and only if polynomial $g(s)$ is Hurwitz stable, which can be further equivalent to system parameters α, β satisfying the inequalities (11) and (12). □

3 Main results

In this section, we aim to use Lyapunov function to find the upper bound τ^* of time delays τ such that system (1)/ (5) can achieve group consensus tracking asymptotically for all $\tau \in [0, \tau^*)$.

3.1 group consensus tracking of first-order MASs with time delays

In this subsection, we mainly apply Lyapunov-Krasovskii function method to solve the tracking issue of system and give the upper bound of delay τ by employing the LMI toolbox in MATLAB.

Theorem 1. *Suppose that the nodes with zero in-degree are chosen as the pinned nodes. Then, system (1) under protocol (3) can achieve group consensus tracking asymptotically, if there exist symmetric positive definite matrices P, Q, $R \in \mathbb{R}^{N\times N}$ such that*

$$\begin{bmatrix} -PH - H^T P + Q & PH & \mathbf{0}_{N\times N} \\ H^T P & -\frac{R}{\tau} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & -Q + \tau H^T R H \end{bmatrix} < 0. \quad (13)$$

Proof. Define a Lyapunov-Krasovskii function for system (4) as follows:

$$V(t) = e^T(t)Pe(t) + \int_{t-\tau}^{t} e^T(s)Qe(s)\mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)R\dot{e}(s)\mathrm{d}s\mathrm{d}\theta, \quad (14)$$

where $P, Q, R \in \mathbb{R}^{N\times N}$ are symmetric positive definite matrices.

Taking the derivative of $V(t)$ along the trajectory of (4) yields

$$\dot{V}(t) = -2e^T(t)PHe(t-\tau) + e^T(t)Qe(t) - e^T(t-\tau)Qe(t-\tau) - \int_{t-\tau}^{t} \dot{e}^T(\theta)R\dot{e}(\theta)\mathrm{d}\theta + \tau e^T(t-\tau)H^T RHe(t-\tau).$$

Due to the fact that $e(t-\tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(s)\mathrm{d}s$, and for any x, $y \in \mathbb{R}^n$ and any symmetric positive definite matrix $\bar{R}$, we get

$$2x^T y \le x^T \bar{R}^{-1} x + y^T \bar{R} y.$$

Then we have

$$\begin{aligned} -2e^T(t)PHe(t-\tau) &= -2e^T(t)PHe(t) + 2e^T(t)PH\int_{t-\tau}^{t} \dot{e}(s)\mathrm{d}s \\ &\le -2e^T(t)PHe(t) + \tau e^T(t)PHR^{-1}H^T Pe(t) + \int_{t-\tau}^{t} \dot{e}^T(s)R\dot{e}(s)\mathrm{d}s. \end{aligned}$$

Consequently,

$$\begin{aligned}\dot{V}(t) &\leq -2e^T(t)PHe(t) + \tau e^T(t)PHR^{-1}H^TPe(t)\\ &\quad + e^T(t)Qe(t) - e^T(t-\tau)Qe(t-\tau)\\ &\quad + \tau e^T(t-\tau)H^TRHe(t-\tau)\\ &= e^T(t)(-PH - H^TP + \tau PHR^{-1}H^TP + Q)e(t)\\ &\quad + e^T(t-\tau)(-Q + \tau H^TRH)e(t-\tau) < 0,\end{aligned}$$

that is,

$$[e^T(t), e^T(t-\tau)] \begin{bmatrix} \mathcal{J} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathcal{K} \end{bmatrix} [e(t), e(t-\tau)]^T < 0.$$

where $\mathcal{J} = -PH - H^TP + \tau PHR^{-1}H^TP + Q$, $\mathcal{K} = -Q + \tau H^TRH$.

Group consensus tracking for multi-agent system (1) can be achieved if the following matrix inequality holds

$$\begin{bmatrix} \mathcal{J} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathcal{K} \end{bmatrix} < 0. \tag{15}$$

Then, by *Schur complement formula*, (15) holds if and only if (13) holds. This completes the proof. □

Next, we will consider the following problem: **how to seek the optimal solution of the time-delay τ in the matrix inequality (13)?** In the sequel, we try to use optimization method to give the upper bound τ^*.

By *Schur complement formula*, matrix inequality (13) holds if and only if

$$\begin{bmatrix} -PH - H^TP + Q & PH & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ H^TP & -\frac{R}{\tau} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & -Q & H^TR \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & RH & -\frac{R}{\tau} \end{bmatrix} < 0,$$

which can be converted into the following form

$$\begin{bmatrix} -PH - H^TP + Q & PH & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ H^TP & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & -Q & H^TR \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & RH & \mathbf{0}_{N\times N} \end{bmatrix} < h \begin{bmatrix} \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & R & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & R \end{bmatrix}, \tag{16}$$

where $h = \dfrac{1}{\tau}$, matrix $R > 0$. Introduce an intermediate variable $X \in \mathbb{R}^{N\times N}$, by (16), it is easy to get

$$Y := \begin{bmatrix} -PH - H^TP + Q & PH & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ H^TP & -X & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & -Q & H^TR \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & RH & -X \end{bmatrix} < 0,$$

where $X < hR$, $R > 0$.

Then the minimum value of h (i.e., the maximum value of time delays τ) can be computed by solving the following optimization problem:

$$\begin{aligned} &\min \quad h \\ &s.t. \begin{cases} 0 < P \\ 0 < Q \\ 0 < R \\ Y < 0 \\ X < hR, \end{cases} \end{aligned} \tag{17}$$

which can easily be solved by employing the LMI Toolbox in MATLAB.

3.2 Bounded group consensus tracking of second-order MASs with time delays

In this subsection, we aim to use Lyapunov function method to establish the bounded tracking control criteria of time-delay system (9).

Theorem 2. *Suppose that the nodes with zero in-degree are chosen as the pinned nodes and control gains α, β satisfy inequality (11) and (12), then system (5) with observer-based control protocol (7) and (8) can achieve the bounded group consensus tracking for $\tau < \tau^* := \dfrac{\bar{\mu}}{2\epsilon}$, specifically,*

$$\lim_{t\to+\infty} \|\xi(t)\| \leq M_{\tilde{\delta}} := \lambda_+\tilde{\delta}\sqrt{\frac{2\tau}{\lambda_-(\bar{\mu} - 2\tau\epsilon)}} < +\infty, \tag{18}$$

where $M_{\tilde{\delta}}$ depends on $\tilde{\delta}$.

Proof. Define a Lyapunov function $V(\xi(t)) = \xi^T(t)P\xi(t)$ with symmetric positive definite matrix P.

Noting that

$$\begin{aligned}\xi(t-\tau) &= \xi(t) - \int_{-\tau}^{0} \dot{\xi}(t+s)\mathrm{d}s\\ &= \xi(t) - W_1\int_{-\tau}^{0} \xi(t+s)\mathrm{d}s\\ &\quad - W_2\int_{-\tau}^{0} \xi(t+s-\tau)\mathrm{d}s\\ &\quad - \int_{-\tau}^{0} \delta(t+s)\mathrm{d}s,\end{aligned}$$

system (9) can be rewritten as

$$\begin{aligned}\dot{\xi}(t) &= (W_1+W_2)\xi(t) - W_2W_1\int_{-\tau}^{0}\xi(t+s)\mathrm{d}s \\ &\quad - W_2^2\int_{-\tau}^{0}\xi(t+s-\tau)\mathrm{d}s - W_2\int_{-\tau}^{0}\delta(t+s)\mathrm{d}s \\ &= W\xi(t) - W_2\int_{-\tau}^{0}\delta(t+s)\mathrm{d}s - W_3\int_{-\tau}^{0}\xi(t+s)\mathrm{d}s \\ &\quad - W_4\int_{-\tau}^{0}\xi(t+s-\tau)\mathrm{d}s,\end{aligned}$$

where $W := W_1+W_2$, $W_3 := W_2W_1$, $W_4 := W_2^2$.

The derivative of $V(\xi(t))$ is given by

$$\begin{aligned}\dot{V}(\xi(t)) &= \xi^T(t)(PW+W^TP)\xi(t) \\ &\quad - 2\xi^T(t)PW_3\int_{-\tau}^{0}\xi(t+s)\mathrm{d}s \\ &\quad + 2\xi^T(t)PW_4\int_{-\tau}^{0}\xi(t+s-\tau)\mathrm{d}s \\ &\quad - 2\xi^T(t)PW_2\int_{-\tau}^{0}\delta(t+s)\mathrm{d}s.\end{aligned} \tag{19}$$

Due to the fact that for any x, $y\in\mathbb{R}^n$ and any symmetric positive definite matrix $\bar{R}$, we get

$$2x^Ty \le x^T\bar{R}^{-1}x + y^T\bar{R}y.$$

Then, it follows that

$$\begin{aligned}& - 2\xi^T(t)PW_3\int_{-\tau}^{0}\xi(t+s)\mathrm{d}s \\ &\le \tau\xi^T(t)PW_3P^{-1}W_3^TP\xi(t) \\ &\quad + \int_{-\tau}^{0}\xi^T(t+s)P\xi(t+s)\mathrm{d}s.\end{aligned}$$

Similarly,

$$\begin{aligned}& - 2\xi^T(t)PW_4\int_{-\tau}^{0}\xi(t+s-\tau)\mathrm{d}s \\ &\le \tau\xi^T(t)PW_4P^{-1}W_4^TP\xi(t) \\ &\quad + \int_{-\tau}^{0}\xi^T(t+s-\tau)P\xi(t+s-\tau)\mathrm{d}s, \\ & - 2\xi^T(t)PW_2\int_{-\tau}^{0}\delta(t+s)\mathrm{d}s \\ &\le \tau\xi^T(t)PW_2P^{-1}W_2^TP\xi(t) \\ &\quad + \int_{-\tau}^{0}\delta^T(t+s)P\delta(t+s)\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{V}(\xi(t)) &\le \xi^T(t)(PW+W^TP)\xi(t) + \tau\xi^T(t)\times \\ &\quad (PW_2P^{-1}W_2^TP + PW_3P^{-1}W_3^TP \\ &\quad + PW_4P^{-1}W_4^TP)\xi(t) \\ &\quad + \int_{-\tau}^{0}\xi^T(t+s)P\xi(t+s)\mathrm{d}s \\ &\quad + \int_{-\tau}^{0}\xi^T(t+s-\tau)P\xi(t+s-\tau)\mathrm{d}s \\ &\quad + \int_{-\tau}^{0}\delta^T(t+s)P\delta(t+s)\mathrm{d}s \\ &\le -\xi^T(t)Q\xi(t) + \tau\xi^T(t)(PW_2P^{-1}W_2^TP \\ &\quad + PW_3P^{-1}W_3^TP + PW_4P^{-1}W_4^TP)\xi(t) \\ &\quad + \int_{-\tau}^{0}V(t+s)\mathrm{d}s + \int_{-\tau}^{0}V(t+s-\tau)\mathrm{d}s \\ &\quad + \tau\tilde{\delta}^2\lambda_+,\end{aligned}$$

where λ_+ is the largest eigenvalue of matrix P.

Choose $\phi(s) = qs$, $q > 1$. By Lemma 4, there must exist a constant $q > 1$ such that $V(t+\eta) \le qV(t)$, $\eta\in[-\tau,0]$. Thus, we can further get

$$\begin{aligned}\dot{V}(\xi(t)) &\le -\xi^T(t)Q\xi(t) + \tau\xi^T(t)(PW_2P^{-1}\times \\ &\quad W_2^TP + PW_3P^{-1}W_3^TP + PW_4\times \\ &\quad P^{-1}W_4^TP + qP + q^2P)\xi(t) + \tau\tilde{\delta}^2\lambda_+.\end{aligned} \tag{20}$$

Let $\epsilon = \|PW_2P^{-1}W_2^TP\| + \|PW_3P^{-1}W_3^TP\| + \|PW_4P^{-1}W_4^TP\| + q\|P\| + q^2\|P\|$ and $\bar{\mu}$ denote the smallest eigenvalue of matrix Q, then from (20), we get

$$\begin{aligned}\dot{V}(\xi(t)) &\le (-\bar{\mu}+\tau\epsilon)\xi^T(t)\xi(t) + \tau\tilde{\delta}^2\lambda_+ \\ &\le -\frac{\bar{\mu}-2\tau\epsilon}{2\lambda_+}V(\xi(t)) + \tau\tilde{\delta}^2\lambda_+ \\ &\le -\gamma V(\xi(t)) + \tau\tilde{\delta}^2\lambda_+.\end{aligned}$$

Furthermore,

$$\begin{aligned}V(\xi(t)) &\le V(\xi(0))e^{-\gamma t} \\ &\quad + \frac{\tau\tilde{\delta}^2\lambda_+}{\gamma}(1-e^{-\gamma t}),\end{aligned} \tag{21}$$

where $\gamma := \dfrac{\bar{\mu}-2\tau\epsilon}{2\lambda_+}$.

Then,

$$\begin{aligned}\|\xi(t)\|^2 &\le \frac{V(\xi(0))}{\lambda_-}e^{-\gamma t} \\ &\quad + \frac{2\tau\tilde{\delta}^2\lambda_+^2}{\lambda_-(\bar{\mu}-2\tau\epsilon)}(1-e^{-\gamma t}),\end{aligned} \tag{22}$$

where λ_- is the smallest eigenvalue of matrix P.

If we choose $\tau < \dfrac{\bar{\mu}}{2\epsilon}$, then $\gamma > 0$. From (22), it is easy to prove that this theorem holds. Moreover, for the special case $\tilde{\delta} = 0$, system (5) can achieve group consensus tracking.

□

4 Numerical examples

In this section, we will give two examples to show the effectiveness of our theoretical results.

Example 1. *(Graph refactoring) Consider a multi-agent system with 8 followers and 2 leaders. All the agents are divided into two groups, where node 1, 2, 3, 4 and leader* 0_1 *form the first group and the rest nodes belong to the second group. Fig. 1 and Fig. 2 below show the communication topology graph among the followers and all the agents, respectively. According to the proof of Lemma 7, the refactored graph* $\bar{\mathscr{G}}$ *is given in Fig. 3 below. By Lemma 7, since* $\bar{\mathscr{G}}$ *contains a directed spanning tree, the matrix* $L + D$ *is positive stable.*

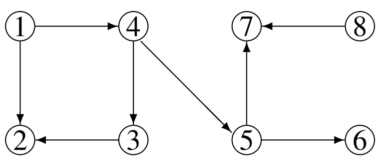

Fig. 1. The topology graph $\mathscr{G}_f$ between followers.

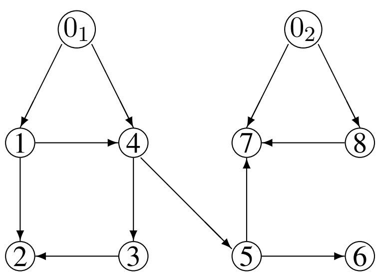

Fig. 2. The topology graph $\mathscr{G}_a$.

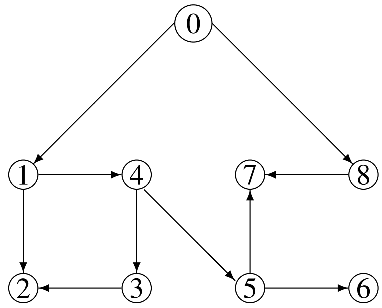

Fig. 3. A new refactored topology graph $\bar{\mathscr{G}}$.

Remark 5. *As shown in Fig. 2 and Fig. 3, we can easily get that the communication topology graph* $\mathscr{G}_a$ *and* $\bar{\mathscr{G}}$ *contain topology* $\mathscr{G}_f$*, in other words, the topology* $\mathscr{G}_f$ *is invariable in the process of information exchange.*

On the other hand, the corresponding Laplacian matrix of $\mathscr{G}_f$ and the leader adjacency matrix B of $\bar{\mathscr{G}}$ as well as the pinned matrix D are easily obtained as follows:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = D = diag\{1, 0, 0, 0, 0, 0, 0, 1\}.$$

Since the refactored graph contains a directed spanning tree, $L+D$ must be positvie stable. In fact, by simple calculation, matrix $H = L+D$ has eigenvalues $\mu_1 = \mu_2 = 2$, $\mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = 1$. This illustrates our Lemma 7.

Example 2. *Consider a network with eight agents, and the directed topology is given in Fig. 4, where the first sub-network consists of agent 1, 2, 3, 4 and leader* 0_1*, and the second sub-network contains the rest agents. See Fig. 4 for the communication topology graph* $\mathscr{G}$ *and the corresponding refactored topology graph* $\bar{\mathscr{G}}$*, respectively. It is clear that the communication topology graph* $\bar{\mathscr{G}}$ *contains a directed spanning tree.*

The corresponding Laplacian matrix and pinned matrix are, respectively, given as

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$D = diag\{1, 0, 0, 0, 0, 0, 0, 1\}$.

Numerical computation shows that matrix $H = L + D$ has eigenvalues $\mu_1 = \mu_2 = 1$, $\mu_3 = 2$, $\mu_4 = 0.8616$, $\mu_5 = 3.0981 + 0.9734i$, $\mu_6 = 3.0981 - 0.9734i$, $\mu_7 = 1.9711 + 0.6380i$, $\mu_8 = 1.9711 - 0.6380i$. From Lemma 1, obviously, the matrix H is positive stable.

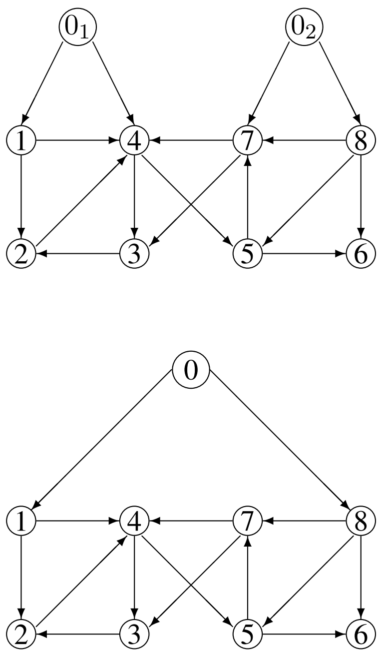

Fig. 4. The communication topology graph $\mathscr{G}$ and $\bar{\mathscr{G}}$.

From (17), it is easy to get the upper bound $\tau^* = 0.2937$ by employing LMI Toolbox in MATLAB. Therefore, we tend to use Hopf bifurcation theory to give the upper bound τ^*.

Case 1. First-order case

Suppose the position of two dynamic virtual leaders as $x_{\sigma_i}(t) = 0.5t + 4$, $1 \le i \le 4$; $x_{\sigma_i}(t) = 0.8t + 7$, $5 \le i \le 8$ and $x_{\sigma_i}(t) = \sin(t) + \cos(t)$, $1 \le i \le 4$; $x_{\sigma_i}(t) = \sin(t) - \cos(t)$, $5 \le i \le 8$, respectively. Fig. 5 shows that the agents in the same group can follow the dynamic virtual leader when $\tau = 0.38$.

Case 2. Second-order case

By Theorems 2, the group consensus tracking with/without disturbances can be achieved when $\tau <$

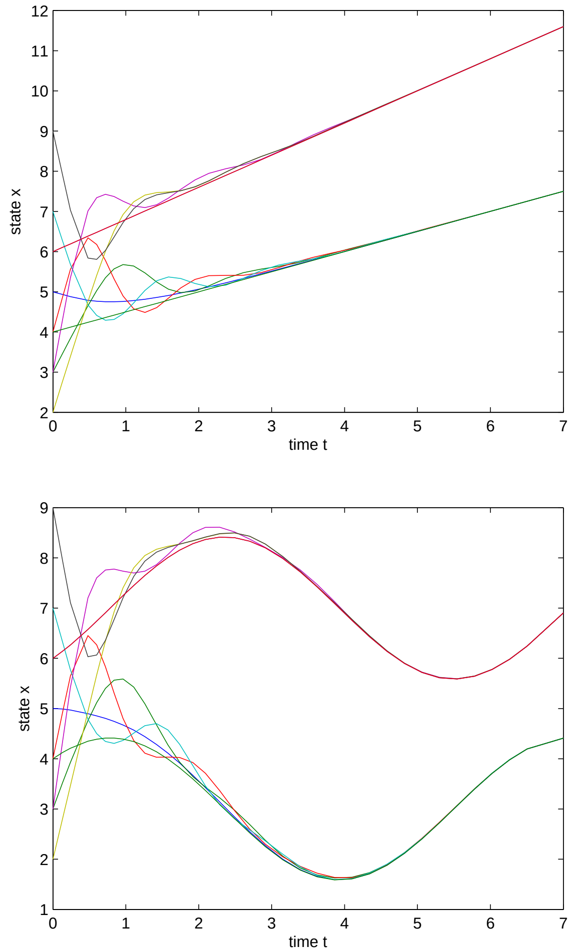

Fig. 5. The state trajectories of all the agents with $\tau = 0.38$.

0.2703, respectively. See Fig. 6 and Fig. 7 for details.

5 Conclusions

In this paper, group consensus tracking issue of continuous-time first-order and second-order multi-agent systems with active virtual leaders and time delays under a pinning control protocol has been addressed,respectively. For the first-order case, we have proposed some necessary and/or sufficient group consensus tracking conditions by using Lyapunov-Krasovskii function. For the second-order case, Lyapunov function method is applied to establish the corresponding group consensus tracking conditions. Moreover, the relationship between the communication topology graph and matrix is found by using the method of graph refactoring (see Lemma 7 and Example 1 for details). Finally, an example is provided to verify the effectiveness of our theoretical results.

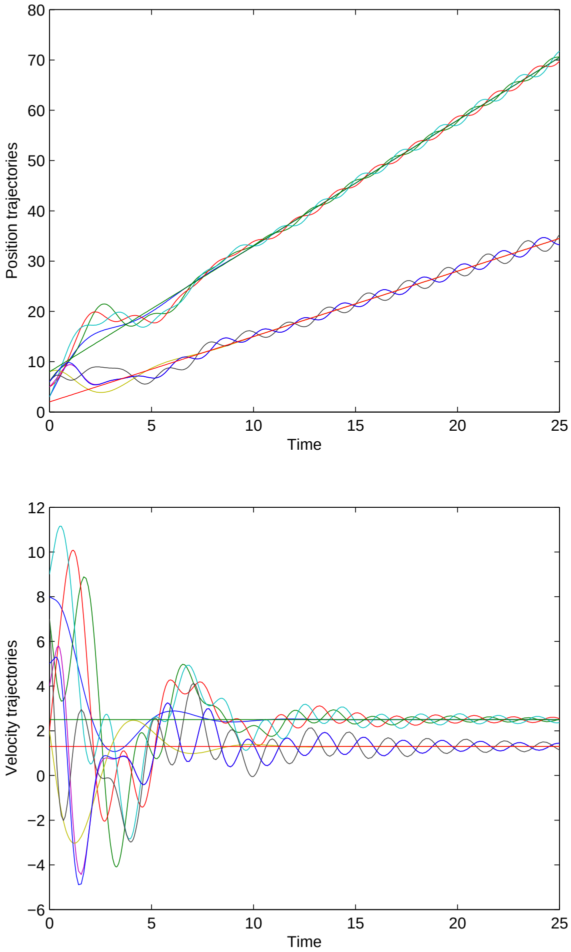

Fig. 6. The state trajectories with disturbances.

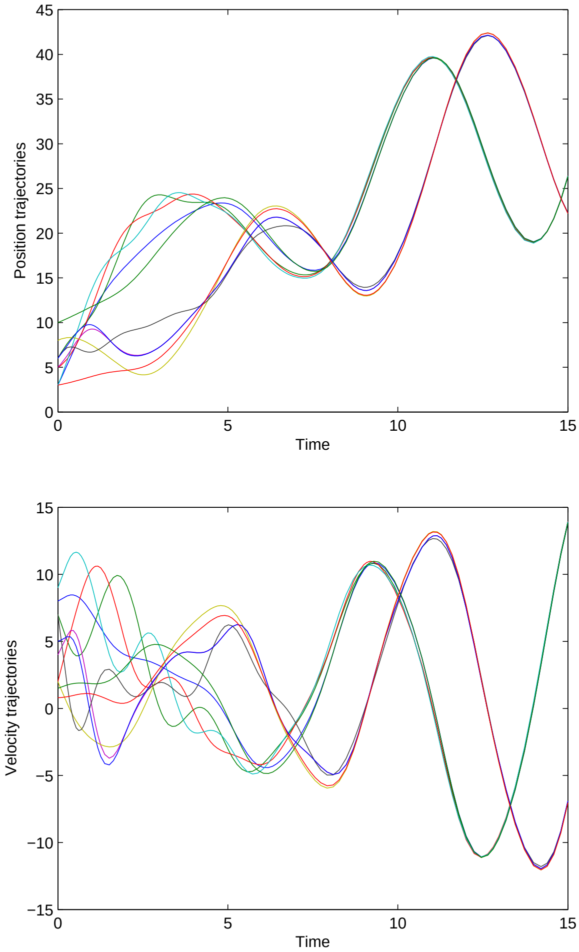

Fig. 7. The state trajectories with $\tau = 0.26$ in a noise-free environment .

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