Stability and stabilization of general 2D delayed systems: an LP approach

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Abstract: This paper studies the problems of stability and stabilization of positive general 2D delayed systems. Necessary and sufficient conditions are proposed for asymptotic stability of positive 2D discrete delayed system. Based on the obtained results, stabilizing controllers for general 2D system with delays are derived. All the obtained results are formulated in term of linear programming conditions, which are computationally tractable. Finally, illustrative examples are given.

Key–Words: Control and synthesis, memory and non-negative memory state feedback, memoryless state feedback, positive 2D delayed systems, linear programing

1 Introduction

In recent years, two-dimensional (2D) state-delayed systems have attracted significant attention of many researchers see for example [1, 2, 3, 4, 5]. Models of 2D delayed systems are derived from the most popular 2D models presented by Roesser [6], Fornasini-Marchesini [7] and Kurek [8]. The interest in 2D delayed systems stems from the fact that time delays correspond to transportation time or computation time, encountered for instance during the processing of visual image which is intrinsically 2D. Moreover, 2D systems for which the states take only non-negative values, when starting from any non-negative boundary condition are called positive 2D systems. These systems emerge naturally in many practical areas such as iteration learning control [9, 10], distributed and parallel computing [11], analysis of iterative algorithms [12, 13], signal filtering [14], digital image processing [15], river pollution and self-purification process [16]. The stability problem has already been studied in [17, 18, 19, 20] for positive 2D systems without delays and in [1, 3, 4, 21, 22] for positive 2D systems with delays. We stress out that the existing results [1, 2, 23, 24, 25] did not consider the problem of designing controllers for general 2D delayed system, such that the 2D closed-loop system is positive and stable. Finally, motivated by all these, it becomes attractive to study 2D state-delayed systems.

In this paper, we present an extension of our results presented in [2]. Then, we focus on stability and stabilization of positive 2D delayed system. The study is carried out by using a change of variable method through augmentation of states. This approach leads to a general 2D system without delays but with higher dimension. Thus, by analyzing the positivity and the stability of such augmented general 2D system, we present checkable conditions for the positivity of general 2D delayed system. Also, we establish delays-dependent and delays-independent necessary and sufficient conditions for the asymptotic stability of positive 2D delayed system. Furthermore, based on these results, we will develop conditions for the existence of stabilizing controllers (memory and memoryless) with some existing results as special case. All the obtained results are formulated in terms of linear programming (LP) conditions. An advantage of our method is the existence of efficient numerical algorithms to solve them.

This paper is organized as follows. In section 2 basic definition and preliminary results concerning positive general 2D system without delays are given. In section 3 checkable conditions for the positivity of general 2D delayed system are given, and also, necessary and sufficient conditions concerning the asymptotic stability of positive general 2D delayed system are presented. Section 4 studies the synthesis problem, then memory, non-negative memory and memoryless controllers are established. Finally, section 5 gives some numerical examples to illustrate the proposed results.

The following notations will be used. $\mathbb{Z}_+$ denotes the set of non-negative integers, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices and $\mathbb{R}_{+}^{n\times m}$ denotes the set of real $n\times m$ matrices with non-negative entries. A^T denotes the transpose of the real matrix A. $\rho(M)$ denotes the spectral radius of a matrix $M\in\mathbb{R}^{n\times n}$ and is defined as: $\rho(M)=\max(|\lambda_1|,\dots,|\lambda_n|)$. $\mathbf{vec}(E)$ denotes the vector formed by the columns of a given matrix E. $\mathbf{diag}(v)$ denotes a diagonal matrix with diagonal components are the elements of v. 1_n is a vector of n elements equal to 1. $0_{n\times p}$ is a matrix of dimension $n\times p$ where all its elements equal to 0 and finally, the $n\times n$ identity matrix will be denoted by I_n.

2 Preliminaries results

Consider the free general 2D system [7, 8] described by the following

$$x_{i+1,j+1}=A_0x_{i,j}+A_1x_{i+1,j}+A_2x_{i,j+1} \tag{1}$$

with $x(i,j)\in\mathbb{R}^n$ is the system's state vector at the point (i,j), A_0, A_1 and A_2 are known square matrices with dimension n.

The boundary conditions for (1) are defined by

$$\begin{cases} x_{i,0}=x_{k0},\ \forall i\in\mathbb{Z}_+, \\ x_{0,j}=x_{0t},\ \forall j\in\mathbb{Z}_+, \end{cases} \tag{2}$$

where x_{k0} and x_{0t} are given sequences of vectors.

Definition 1 *The general 2D system (1) is called a positive general 2D system if all the trajectories generated by (1), with the non-negative boundary conditions (2) remain non-negative.*

Definition 2 *A real matrix M is called a non-negative matrix $\left(M\in\mathbb{R}_{+}^{n\times q}\right)$ if all its elements are non-negative $m_{ij}\geq 0$, $i=1,\dots,n$, $j=1,\dots,q$.*

According to Definition 1, the following result provides checkable conditions for the positivity of the general 2D system (1), see for example [26].

Proposition 1 *The general 2D system (1) is positive if and only if A_0, A_1 and A_2 are non-negative matrices.*

Next, necessary and sufficient conditions concerning the asymptotic stability of the positive 2D system (1) are given.

Definition 3 *[26] A positive 2D system described by (1) is called asymptotically stable if the state evolution corresponding to any set of the non-negative boundary conditions (2) asymptotically tends to zero, i.e.,*

$$\lim_{i,j\to\infty} x_{i,j}=0. \tag{3}$$

Theorem 1 *[24, 2] Assume that the general 2D system (1) is positive. Then the following statements are equivalent*

(i) *The 2D system (1) is asymptotically stable.*

(ii) $\rho(A_0+A_1+A_2)<1$.

(iii) *There exist a vector $\lambda\in\mathbb{R}^n$ such that*

$$\begin{cases} (A_0+A_1+A_2-I_n)\lambda<0, \\ \lambda>0. \end{cases} \tag{4}$$

3 Positive general 2D delayed system

Consider a general 2D delayed system represented by the following 2D state-space system with q delays

$$x_{i+1,j+1}=\sum_{t=0}^{q}\left(A_t^0x_{i-t,j-t}+A_t^1x_{i+1-t,j-t}+A_t^2x_{i-t,j+1-t}\right), \tag{5}$$

where $i,j\in\mathbb{Z}^+$, $x_{i,j}\in\mathbb{R}^n$ is the state vector at the point (i,j), A_t^1, A_t^2, A_t^3, $t=0,1,\dots,q$, are known constant matrices with compatible dimensions and q denote the number of delay terms in each direction.

In what follows, we will use the method of change of variables in order to obtain checkable conditions for the positivity and the asymptotic stability of the 2D delayed system (5). Thus, by using the new variable

$$\tilde{x}_{i,j}=\begin{bmatrix} x_{i,j} \\ x_{i-1,j-1} \\ \vdots \\ x_{i-q,j-q} \end{bmatrix}\in\mathbb{R}^N \text{ where } N=(q+1)n, \tag{6}$$

the 2D system (5) with q delays can be reduced to the following general 2D system without delays

$$\tilde{x}_{i+1,j+1}=\tilde{A}_0\tilde{x}_{i,j}+\tilde{A}_1\tilde{x}_{i+1,j}+\tilde{A}_2\tilde{x}_{i,j+1} \tag{7}$$

with

$$\begin{aligned} \tilde{A}_0&=\begin{bmatrix} A_0^0 & A_1^0 & \dots & A_{q-1}^0 & A_q^0 \\ & & I_{q*n} & & 0_{q*n\times n} \end{bmatrix}, \\ \tilde{A}_1&=\begin{bmatrix} A_0^1 & A_1^1 & \dots & A_{q-1}^1 & A_q^1 \\ & & 0_{q*n\times N*n} & & \end{bmatrix}, \\ \tilde{A}_2&=\begin{bmatrix} A_0^2 & A_1^2 & \dots & A_{q-1}^2 & A_q^2 \\ & & 0_{q*n\times N*n} & & \end{bmatrix}. \end{aligned} \tag{8}$$

Therefore, the dimension of the new general 2D system (7) is $n(q+1)$.

Applying to the general 2D system (7) Proposition 1, we obtain the following result concerning the positivity of the 2D delayed system (5).

Proposition 2 *The general 2D system with q delays (5) is positive if and only if $\tilde{A}_0$, $\tilde{A}_1$ and $\tilde{A}_2$ are non-negative matrices, or equivalently, the matrices A_t^0, A_t^1 and A_t^2 are non-negative $\forall t = 0, 1, \ldots, q$.*

Also, by considering the new general 2D system (7), we are now in place to state the following result concerning the asymptotic stability of the positive 2D delayed system (5). Then, from Theorem 1 applied to the new general 2D system (7) we have the following result.

Theorem 2 *The positive 2D delayed system (5) is asymptotically stable if and only if one of the following equivalent conditions holds.*

(i) *The positive general 2D system (7) is asymptotically stable.*

(ii) $\rho(\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2) < 1$.

(iii) *There exist a vector $\lambda \in \mathbb{R}^{n*(q+1)}$ such that*

$$\begin{cases} (\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 - I_{n*(q+1)})\lambda < 0 \\ \lambda > 0. \end{cases} \quad (9)$$

In the following, from Theorem 2, we present equivalent necessary and sufficient conditions for the asymptotic stability of the positive general 2D delayed system (5).

Lemma 1 *The positive 2D delayed system (5) is asymptotically stable if and only if the following LP problem in the variables λ_0, …, $\lambda_q \in \mathbb{R}^n$ is feasible.*

$$\begin{cases} (A_0^0 + A_0^1 + A_0^2 - I_n)\lambda_0 + \sum_{t=1}^{q} (A_t^0 + A_t^1 + A_t^2)\lambda_t < 0, \\ \lambda_t < \lambda_{t+1},\ t = 0, \ldots, q-1, \\ \lambda_t > 0,\ t = 0, \ldots, q. \end{cases} \quad (10)$$

Proof 1 *To show this, we take into account that the positive general 2D delayed system (5) is asymptotically stable. Then, by using Theorem 2, we have that the positive 2D delayed system (5) is asymptotically stable if and only if there exists $\lambda \in \mathbb{R}^{n*(q+1)}$ such that the LP conditions (9) holds. Now, by using the expression of $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$ given in (8) and by defining $\lambda = \begin{bmatrix} \lambda_0 & \ldots & \lambda_q \end{bmatrix}^T$, with $\lambda_i \in \mathbb{R}^n$, $i = 0, \ldots, q$, we obtain*

$$\begin{cases} \begin{bmatrix} A_0^0 + A_0^1 + A_0^2 - I_n & \ldots & A_q^0 + A_q^1 + A_q^2 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_q \end{bmatrix} < 0, \\ \begin{bmatrix} I_n & -I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & \ldots & 0_{n\times n} & I_n & -I_n \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_q \end{bmatrix} < 0, \\ \lambda_t > 0, t = 0, \ldots, q, \end{cases}$$

the above inequalities are effectively the same inequalities in the LP constraints (10). Finally, the reverse implication can be trivially obtained by the simple matrix manipulation as shown above. Thus, the proof is complete.

Remark 1 *Each of the LP conditions (9) and (10) present a delay dependent constraints for the asymptotic stability of the positive general 2D delayed system (5). These LP conditions present an computational disadvantage in the case of matrices A_t^0, A_t^1, A_t^2, $t = 0, \ldots, q$ of high dimensions, and also in the case of general 2D systems with large time delays.*

Next, we state our main result.

Theorem 3 *The positive 2D system (5) with q delays is asymptotically stable if and only if the following LP problem in the variable $\lambda_0 \in \mathbb{R}^n$ is feasible.*

$$\begin{cases} \left(\sum_{t=0}^{q} (A_t^0 + A_t^1 + A_t^2) - I_n \right) \lambda_0 < 0, \\ \lambda_0 > 0. \end{cases} \quad (11)$$

Proof 2 *By using Lemma 1, we have that the positive 2D delayed system (5) is asymptotically stable if there exists λ_0, …, $\lambda_q \in \mathbb{R}^n$ such that the LP conditions (10) holds. Now, by considering the non-negativity of the matrices A_0^t, A_1^t and A_2^t $\forall t = 0, \ldots, q$ and the second inequality in the LP conditions (10) $\lambda_t < \lambda_{t+1}$, $t = 0, \ldots, q-1$, which is equivalent to $\lambda_0 < \lambda_t$, $\forall t = 1, \ldots, q$, then, we give $(A_t^0 + A_t^1 + A_t^2)\lambda_0 < (A_t^0 + A_t^1 + A_t^2)\lambda_t$, $\forall t = 1, \ldots, q$, which leads to*

$$\sum_{t=1}^{q} (A_t^0 + A_t^1 + A_t^2)\lambda_0 < \sum_{t=1}^{q} (A_t^0 + A_t^1 + A_t^2)\lambda_t$$

Thus, by adding $(A_0^0 + A_0^1 + A_0^2 - I_n)\lambda_0$ to both sides of the above inequality, we have

$$\left(\sum_{t=0}^{q} (A_t^0 + A_t^1 + A_t^2) - I_n \right) \lambda_0 < 0.$$

The reverse implication can be trivially obtained. Thus, the proof is complete.

Remark 2 *Theorem 3 provide that the stability conditions (11) of the positive general 2D delayed system (5) are delays-independent.*

For $A_t^0 = 0$, $\forall t = 0, \ldots, q$ and from Theorem 3 we have the following Corollary.

Corollary 1 *The positive 2D Fomasini-Marchesini state-space model with q delays is asymptotically stable if and only if the following LP problem in the variable $\lambda \in \mathbb{R}^n$ is feasible.*

$$\begin{cases} \left(\sum_{t=0}^{q} (A_t^1 + A_t^2) - I_n\right) \lambda < 0, \\ \lambda > 0. \end{cases} \tag{12}$$

4 Stabilization of general 2D delayed system

Consider the following forced general 2D linear system with q delays described by

$$\begin{aligned} x_{i+1,j+1} = & \sum_{t=0}^{q} \left(A_t^0 x_{i-t,j-t} + A_t^1 x_{i+1-t,j-t} + A_t^2 x_{i-t,j+1-t}\right) \\ & + B^0 u_{i,j} + B^1 u_{i+1,j} + B^2 u_{i,j+1}, \end{aligned} \tag{13}$$

where $i, j \in \mathbb{Z}^+$, $x_{i,j} \in \mathbb{R}^n$ and $u_{i,j} \in \mathbb{R}^m$ are the state vector and the control input at the point (i, j) respectively. The matrices A_t^0, A_t^1, A_t^2, $t = 0, 1, \ldots, q$, B^0, B^1 and B^2 are known matrices.

4.1 Memory state feedback controller

The problem addressed in this section is that of designing a feedback controller of the form

$$u_{i,j} = \sum_{t=0}^{q} K_t x_{i-t,j-t}. \tag{14}$$

for which 2D closed-loop system is positive and stable.

Applying the control (14) to the forced general 2D delayed system (13) yields the 2D closed-loop system

$$\begin{aligned} x_{i+1,j+1} = \sum_{t=0}^{q} \big(& (A_t^0 + B^0 K_t) x_{i-t,j-t} + (A_t^1 + B^1 K_t) x_{i+1-t,j-t} \\ & + (A_t^2 + B^2 K_t) x_{i-t,j+1-t} \big). \end{aligned} \tag{15}$$

In the following, necessary and sufficient conditions are developed for the positivity and the asymptotic stability of the 2D closed-loop system (15).

Theorem 4 *The 2D closed-loop system (15) is positive and asymptotically stable if and only if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, Z_0, …, $Z_q \in \mathbb{R}^{m\times n}$ is feasible.*

$$\begin{cases} \left(\sum_{t=0}^{q} (A_t^0 + A_t^1 + A_t^2) - I_n\right) \lambda + (B^0 + B^1 + B^2) \left(\sum_{t=0}^{q} Z_t\right) 1_n < 0, \\ A_t^0 diag(\lambda) + B^0 Z_t \geq 0, t = 0, \ldots, q, \\ A_t^1 diag(\lambda) + B^1 Z_t \geq 0, t = 0, \ldots, q, \\ A_t^2 diag(\lambda) + B^2 Z_t \geq 0, t = 0, \ldots, q, \\ \lambda > 0. \end{cases} \tag{16}$$

Moreover, the gain matrices K_0, … and K_q satisfying (15) can be computed as follows:

$$K_t = Z_t diag(\lambda)^{-1}, t = 0, \ldots, q. \tag{17}$$

where λ, Z_0, … and Z_q are any feasible solution to the above LP problem (16).

Proof 3 *Assume that the 2D closed-loop system (15) is positive and asymptotically stable, we want to show that the conditions (16) holds. By using Theorem 3, the positive 2D closed-loop system (15) is asymptotically stable if and only if there exist $\lambda > 0$ such that $(\sum_{t=0}^{q} (A_t^0 + B^0 K_t + A_t^1 + B^1 K_t + A_t^2 + B^2 K_t) - I_n)\lambda < 0$. Now, we define $K_t = Z_t diag(\lambda)^{-1}$, $t = 0$, …, q. Thus, the above inequalities are effectively the first and the fifth inequalities in the LP constraints (16). The others three inequalities in the LP constraints (16), are obtained as follows. By using Proposition 2 the 2D closed-loop system (15) is positive if and only if the matrices $A_t^0 + B^0 K_t$, $A_t^1 + B^1 K_t$, $A_t^2 + B^2 K_t$ are non-negative $\forall t = 0$, …, q, or equivalently, if and only if $(A_t^0 + B^0 K_t) diag(\lambda_t) \geq 0$, $(A_t^1 + B^1 K_t) diag(\lambda_t) \geq 0$, $(A_t^2 + B^2 K_t) diag(\lambda_t) \geq 0$, $\forall t = 0$, …, q. Then, by recalling $K_t = Z_t diag(\lambda)^{-1}$, $t = 0$, …, q, we see that the above inequalities are equivalent to the others three inequalities in the LP constraints (16). The reverse implication can be trivially obtained by a simple matrix manipulation as shown above.*

4.2 Constrained controller

The problem addressed in the following is designing a nonnegative memory state feedback controller of the form

$$u_{i,j} = \sum_{t=0}^{q} K_t x_{i-t,j-t} \geq 0, \tag{18}$$

for which the 2D closed-loop system (15) is positive and asymptotically stable.

Now, we state our result developed for the existence of a non-negative controller ensuring the positivity and the asymptotic stability of the 2D closed-loop system (15).

Theorem 5 *There exist a non-negative controller of the form (18) such that the 2D closed-loop system (15) is positive and asymptotically stable if and only if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $Z_0 \in$*

$\mathbb{R}^{m\times n}$, ..., Z_q *is feasible.*

$$\begin{cases} \left(\sum_{t=0}^{q}(A_t^0+A_t^1+A_t^2)-I_n\right)\lambda+(B^0+B^1+B^2)\left(\sum_{t=0}^{q}Z_t\right)1_n<0, \\ A_t^0 diag(\lambda)+B^0Z_t\geq 0,\, t=0,\dots,q, \\ A_t^1 diag(\lambda)+B^1Z_t\geq 0,\, t=0,\dots,q, \\ A_t^2 diag(\lambda)+B^2Z_t\geq 0,\, t=0,\dots,q, \\ Z_t\geq 0,\, t=0,\dots,q, \\ \lambda>0. \end{cases} \tag{19}$$

Moreover, the gain matrices $K_0,\dots$ and K_q satisfying (15) can be computed as follows:

$$K_t = Z_t diag(\lambda)^{-1},\, t=0,\dots,q. \tag{20}$$

where λ, Z_0, ... and Z_q are any feasible solution to the above LP problem (19).

Proof 4 *Note that the control law (18) is non-negative if and only if K_0, ... and K_q are non-negative matrices. Then, by using the change of variables $K_t = Z_t diag(\lambda_t)^{-1}$, $t=1,\dots,q$, we have necessarily that Z_0, ... and Z_q are non-negative matrices. To complete this proof, we can follow the same line of arguments as in the Proof of Theorem 4.*

4.3 Memoryless controller

In the case when we do not have access to the delayed states or when the delays are unknown, a memoryless state feedback controller of the form

$$u_{i,j} = Kx_{i,j} \tag{21}$$

can be designed for system (13). In this case, the matrices A_t^0, A_t^1 and A_t^2 $\forall t=1,\dots,q$ must be non-negative.

The following result can be derived from Theorem 4 by taking $K=K_0$ and $K_t=0$, $\forall i=1,\dots,q$

Theorem 6 *Assume that the matrices A_t^0, A_t^1 and A_t^2 are non-negative $\forall t=1,\dots,q$. Then the general 2D delayed system (13) with the memoryless controller (21) is positive and asymptotically if and only if the following LP problem in the variables $\lambda\in\mathbb{R}^n$ and $Z\in\mathbb{R}^{m\times n}$ is feasible.*

$$\begin{cases} \left(\sum_{t=0}^{q}(A_t^0+A_t^1+A_t^2)-I_n\right)\lambda+(B^0+B^1+B^2)Z1_n<0, \\ A_0^0 diag(\lambda)+B^0Z\geq 0, \\ A_0^1 diag(\lambda)+B^1Z\geq 0, \\ A_0^2 diag(\lambda)+B^2Z\geq 0, \\ \lambda>0. \end{cases} \tag{22}$$

Moreover, the gain matrix K can be computed as follows:

$$K = Zdiag(\lambda)^{-1}, \tag{23}$$

where λ and Z are any feasible solution to the above LP problem (22).

Remark 3 *By taking $A_k^0 = 0_{n\times n}$ and $B^0 = 0_{n\times m}$, the system described by (13) is the well-known Fornasini-Marchesini Second model [7] and it can be stabilizing by using Theorem 4 for a memory state feedback control, Theorem 5 for a non-negative memory state feedback control or Theorem 6 for a memoryless state feedback control.*

4.4 Standard LP Formulation

Previously, we have seen that the provided stabilization results are formulated as linear matrix constraints. Specifically, the LP problems (16), (19) and (22) are not in the standard form because they involve matrix variables. We would like to show that these LP's can be re-expressed in the well-known standard form, which involves vector constraints with a single unknown vector variable. This can be done by using the Kronecker product and **vec** operation.

In what follows, we propose a standard LP form for the problem (22)

$$\begin{bmatrix} \sum_{t=0}^{q}(A_t^0+A_t^1+A_t^2)-I_n & 1_n^T\otimes(B^0+B^1+B^2) \\ -I_n & 0_{n\times(n*m)} \end{bmatrix} w<0,$$

$$\begin{bmatrix} -\sum_{t=0}^{q} v_iv_i^T\otimes A_0^0 v_i & -I_n\otimes B^0 \\ -\sum_{t=0}^{q} v_iv_i^T\otimes A_0^1 v_i & -I_n\otimes B^1 \\ -\sum_{t=0}^{q} v_iv_i^T\otimes A_0^2 v_i & -I_n\otimes B^2 \end{bmatrix} w\leq 0, \tag{24}$$

where the new vector variable w is defined as

$$w = \begin{bmatrix} \lambda & vec(Z) \end{bmatrix}^T \tag{25}$$

Finally, we would like to mention that these LP programs can be solved, for example, by using `linprog` function in Matlab, or by using Yalmip software.

5 Numerical Examples

The following examples are given for illustrating the proposed control synthesis design.

5.1 Example 1: stability

Using Theorem 3, we want to check the asymptotic stability of a positive general 2D system of the form (5) with $q=2$ delays and the following matrices

$$A_0^0 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix},\ A_0^1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix},$$

$$A_0^2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.2 \end{bmatrix}, \; A_1^0 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix},$$

$$A_1^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \; A_1^2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$A_2^0 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}, \; A_2^1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2^2 = \begin{bmatrix} 0 & 0 \\ 0.02 & 0.1 \end{bmatrix}.$$

By using Theorem 3, or equivalently, by solving the LP problem (11), one feasible solution gives

$$\lambda = \begin{bmatrix} 110.5354 \\ 22.4027 \end{bmatrix} > 0.$$

Then, the LP conditions (11) of Theorem 3 are satisfied and the above positive general 2D system with $q = 2$ delays is asymptotically stable.

5.2 Example 2: memory controller

Consider a 2D general model with $q = 2$ delays described by (13) with the following matrices:

$$A_0^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}, \; A_0^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{bmatrix},$$

$$A_0^2 = \begin{bmatrix} 0.15 & 0.3 \\ 0.25 & 0.3 \end{bmatrix}, \; A_1^0 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$

$$A_1^1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \; A_1^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2^0 = \begin{bmatrix} 0 & 0.03 \\ 0 & 0 \end{bmatrix}, \; A_2^1 = \begin{bmatrix} 0.14 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \; B^0 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix},$$

$$B^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}, \; B^2 = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}.$$

Based on Theorem 4, we have designed a memory controller to stabilize the 2D system (13) described with the above matrices. Thus, by solving the LP problem (16) we obtained the following feasible solution

$$\lambda = \begin{bmatrix} 222.6013 \\ 322.8234 \end{bmatrix}, \; Z_1 = \begin{bmatrix} 2.7759 & 4.4153 \\ 1.6741 & 1.0953 \end{bmatrix},$$

$$Z_0 = \begin{bmatrix} 14.5757 & -115.1878 \\ -76.5841 & -90.8803 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} -4.3315 & -2.8975 \\ 10.3211 & 6.4989 \end{bmatrix}.$$

Then, the following gain matrices can be calculated

$$K_0 = \begin{bmatrix} 0.0655 & -0.3568 \\ -0.3440 & -0.2815 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.0125 & 0.0137 \\ 0.0075 & 0.0034 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.0195 & -0.0090 \\ 0.0464 & 0.0201 \end{bmatrix},$$

which guarantee the positivity and the asymptotic stability of the 2D closed-loop system.

6 Conclusion

Necessary and sufficient conditions are proposed for the asymptotic stability of positive general 2D delayed system. Also, conditions for the existence of stabilizing controllers for general 2D delayed system are stated. All the obtained results have been developed in terms of linear programming conditions, which can be expressed in standard LP form. The obtained results can be extended to the Roesser model with delays. Several examples are provided in order to illustrate the proposed results.

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