Analizing Exact Controllability of ℓ-Order Linear Systems

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Abstract: In recent years there has been growing interest in the descriptive analysis of ℓ-order time invariant linear dynamical system $x^\ell = A_{\ell-1}x^{\ell-1} + \ldots + A_0x^0$ where A_i are square complex matrices and x^i denotes the i-th derivative of x. We are interested to mesure the minimum number of controls B that are needed in order to make the system $x^\ell = A_{\ell-1}x^{\ell-1} + \ldots + A_0x^0 + Bu$ controllable.

Key–Words: ℓ-order time invariant linear dynamical system, Controllability, Exact controllability.

1 Introduction

It is well known the interest generated in many research communities about the study of control of high order linear systems. These kinds of systems have found applications in different fields as in vibration and structural analysis, spacecraft control and robotics control among others ([1], [6], [10], [13], [17]).

In [10], Guang-Ren Duan presents a modelling of the motion of there-axis dynamic flight based on high order linear systems

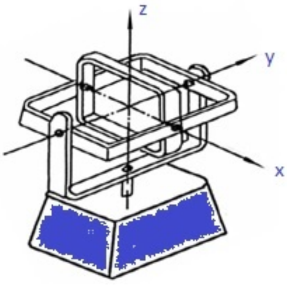

Figure 1: The there-axis dynamic flight motion simulator

The analysis of controllability of high order linear systems can be realized extending the knowledge about the topic on first order linear dynamical systems

The concept of controllability is fundamental in dynamic systems and it is studied under different approaches (see [3], [6], [7], [19], [20], [21], for example).

One of the approaches is the exact controllability concept that following [21] is based on the maximum multiplicity to identify the minimum set of inputs required to achieve full control of the system. In other words, exact controllability can be formulated as follows: Given a system, we are allowed to act on the solution using a suitable control with minimal inputs.

We are interested in to analyze exact controllability for high order linear systems. Other possible approaches to controllability are structural controllability that study the possibility that to adding a set of weights in the non-zero terms of the matrices defining the system such that the corresponding new system is controllable in a classical sense. This notion of controllability is crucial in control network dynamic systems (see [19] for more information). More recent results over the structural controllability can be found on [16]. Another important aspect of control is the notion of output controllability that describes the ability of an external data to move the output from any initial condition to any final in a finite time. Some results about can be found in [8].

Taking into account that there are plenty of engineering problems, economic or biological problems that they are modelled by means high order singular linear systems, also called generalized high order linear systems where it is crucial to control them, we also introduce in the paper, the study of exact controllability for this kind of systems.

2 Linearization

We consider the ℓ-order time-invariant linear systems

$$x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)}, \qquad (1)$$

($x^{(i)}$ denotes the ith-derivative).

One of methods to study ℓ-order linear systems is linearizing the system (see [11] for example), obtaining a linear system in the following manner. Given a

ℓ-order linear system

$$x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)}$$

and calling

$$X = \begin{pmatrix} x^{(0)} \\ x^{(1)} \\ \vdots \\ x^{(\ell-1)} \end{pmatrix},$$

we have

$$X^{(1)} = \mathbf{A}X \tag{2}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & I_n & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_n \\ A_0 & A_1 & \ldots & A_{\ell-1} \end{pmatrix} \in M_{\ell n}(\mathbb{C}), \tag{3}$$

and

$$X^{(1)} = \begin{pmatrix} x^{(1)} \\ x^{(1)} \\ \vdots \\ x^{(\ell)} \end{pmatrix},$$

In the same manner we can linearize the ℓ-order linear system

$$x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)} + Bu \tag{4}$$

we have

$$X^{(1)} = \mathbf{A}X + \mathbf{B}u \tag{5}$$

with

$$\mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B \end{pmatrix} \in M_{\ell n \times m}(\mathbb{C}) \tag{6}$$

Remark 1 *The solution of the high order linear system*

$$x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)} + Bu$$

can be easily obtained considering its equivalent linear form $X^{(1)} = \mathbf{A}X + \mathbf{B}u$:

$$x(t) = \Pi e^{t\mathbf{A}}x(0) + \Pi \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}u(s)ds$$

where $\Pi = \begin{pmatrix} I & 0 & \ldots & 0 \end{pmatrix}$.

2.1 Relation between eigenstructure of high order linear system an linearized one

Let $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)}$ be a high order linear equation, we can associate it the following polynomial matrix

$$P(s) = I_n s^\ell - A_{\ell-1}s^{\ell-1} - \ldots - A_1 s - A_0,$$

and we have the following result

Proposition 2

$$\det P(s) = \det(sI_{\ell n} - \mathbf{A})$$

Proof:

Making block column elementary transformations we have

$$\det \begin{pmatrix} sI_n & -I_n & \ldots & 0 \\ 0 & sI_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -I_n \\ -A_0 & -A_1 & \ldots & sI_n - A_{\ell-1} \end{pmatrix} =$$

$$\det \begin{pmatrix} 0 & -I_n & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -I_n \\ -A_0 - sA_1 - \ldots - s^{\ell-1}A_{\ell-1} + s^\ell I_n & 0 & \ldots & 0 \end{pmatrix}$$

□

Then, the eigenvalues of the matrix $\mathbf{A}$ are the zeros of the $n\ell$-degree scalar polynomial $\det P(s)$ that are called eigenvalues of $P(s)$.

Let s_0 be an eigenvalue of polynomial matrix $P(s)$. Then, there exists a vector $v_0 \neq 0$, such that $P(s_0)(v_0) = 0$. This vector is called an eigenvector of $P(s)$ corresponding to the eigenvalue s_0.

The eigenvectors v_0 of $P(s)$ of eigenvalue s_0 are related to the eigenvectors V of $\mathbf{A}$ of eigenvalue s_0.

Proposition 3 *Let V be an eigenvector of $\mathbf{A}$ of eigenvalue $s_0 \in \mathbb{C}$ and consider where $\Pi = \begin{pmatrix} I_n & 0 & \ldots & 0 \end{pmatrix}$. Then,* $\Pi V \in \operatorname{Ker} P(s_0)$.

Proof:

Calling $V = (v_1, \ldots v_\ell) \in \mathbb{C}^{n\ell}$, it suffices to observe that

$$\begin{aligned} sv_1 &= v_2 \\ sv_2 &= v_3 \\ &\vdots \\ sv_{\ell-1} &= v_\ell \\ -A_0v_1 - A_1v_2 - \ldots + sv_\ell - A_{\ell-1}v_\ell &= 0 \end{aligned}$$

and that $\Pi V = v_1$. □

Corollary 4 *Suppose $A_1 = \ldots = A_{\ell-1} = 0$. Then, v_1 is an eigenvector of A_0 of eigenvalue $\lambda = s_0^\ell$.*

Proof:
$v_1 \in \text{Ker}\,(s_0^\ell I - A_0) = \text{Ker}\,(\lambda I - A_0)$ with $\lambda = s_0^\ell$. □

Corollary 5 *Suppose $A_0 = A_1 = \ldots = A_{\ell-1}$. Then, v_1 is an eigenvector of A_0 of eigenvalue $\lambda = \frac{s_0^\ell}{1+s_0+\ldots+s_0^{\ell-1}}$.*

Proof:
It is easy to see that

$$A_0((1 + s_0 + \ldots + s_0^{\ell-1})v_1) = s_0^\ell v_1$$

Then,

$$A(v_1) = \frac{s_0^\ell}{1 + s_0 + \ldots + s_0^{\ell-1}} v_1.$$

□

Remark 6 *If $1 + s_0 + \ldots + s_0^{\ell-1} = 0$ then, $v_1 = 0$ and $V = 0$.*

We will call a Jordan chain of length ℓ for $P(s)$ corresponding to complex number s_0 the sequence of n-dimensional vectors $v_0, \ldots, v_{\ell-1}$, such that

$$\sum_{k=0}^{i} \frac{1}{k!} P^{(k)}(s_0) v_{i-k} = 0, \quad i = 0, \ldots, \ell - 1 \tag{7}$$

where $P^{(k)}$ denotes the k-derivative of $P(s)$ for the variable s. If s_0 is an eigenvalue, there exists a Jordan chain of length at least 1 made up of the eigenvector.

3 Controllability and stability

The concept of controllability of dynamical systems has captivated great attention for some years. It concerns to the ability of a system to transfer the state vector from one concrete vector value to another in finite time.

Given a state space representation of a linear dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{8}$$

that for simplicity, from now on we will write as the pair of matrices (A, B).

Definition 7 *A system is called controllable (see [3]) if, for any $t_1 > 0$, $x(0) \in R^n$ and $w \in \mathbb{R}^n$, there exists a control input $u(t)$ such that $x(t_1) = w$.*

It is well known that there are many possible control matrices B in the system 8 that satisfy the controllability condition:
Controllability condition: the pair (A, B) is controllable if and only if,

$$\text{rank}\begin{pmatrix} sI - A & B \end{pmatrix} = n. \tag{9}$$

(See [12] for more details).

In our particular setup, controllability character is expressed in the following manner

Proposition 8

$$\text{rank}\begin{pmatrix} s\mathbf{I} - \mathbf{A} & \mathbf{B} \end{pmatrix} = \ell n. \tag{10}$$

But, it can be computed directly from the initial equation 1 in the following manner.

Theorem 9 ([6]) *The ℓ-order linear system 4 is controllable if and only if,*

$$\text{rank}\begin{pmatrix} s^\ell I - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} = n. \tag{11}$$

for all $s \in \mathbb{C}$.

Example:

1.- Case $A_{\ell-1} = A_1 = 0$. The linearized system is controllable if and only if the pair of matrices (A_0, B) is controllable.

2.- Case $A_0 = 0$. It is not difficult to prove that the linearized system is controllable if and only if $n \geq m$ and the matrix B has full rank.

3.- In the case of $A_0 = A_1 = \ldots = A_{\ell-1}$ the controllability character of the system 4 is the same than $\dot{x} = A_0 x + Bu$. Because of:

$$\begin{array}{l}\text{rank}\begin{pmatrix} s^\ell I_n - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} = \\ \text{rank}\begin{pmatrix} s^\ell I_n - s^{\ell-1}A_0 - \ldots - sA_0 - A_0 & B \end{pmatrix} = \\ \text{rank}\begin{pmatrix} s^\ell I_n - (s^{\ell-1} - \ldots - s - 1)A_0 & B \end{pmatrix}\end{array}$$

Taking into account that $s^\ell - 1 = (s-1)(s^{\ell-1} - \ldots - s - 1))$, the roots s_i of $(s^{\ell-1} - \ldots - s - 1))$ are the $\ell - 1$ roots of unit different of one ($s_i \neq 1$).

Then,

for each s_i

$$\begin{array}{l}\text{rank}\begin{pmatrix} s^\ell I_n - (s^{\ell-1} - \ldots - s - 1)A_0 & B \end{pmatrix} = \\ \text{rank}\begin{pmatrix} s_i^\ell I & B \end{pmatrix} = n,\end{array}$$

and for all $s \neq s_i$, we have

$$\begin{array}{l}\text{rank}\begin{pmatrix}s^\ell I_n - (s^{\ell-1} - \ldots - s - 1)A_0 & B\end{pmatrix} = \\ \text{rank}\begin{pmatrix}\frac{s^\ell}{(s^{\ell-1}-\ldots-s-1)}I_n - A_0 & B\end{pmatrix} = \\ \text{rank}\begin{pmatrix}\lambda I - A & B\end{pmatrix}.\end{array}$$

Another qualitative property of the linear systems and then for high order linear systems is the stability. This concept is strongly linked to the notion of equilibrium state. For time-invariant linear systems $x^{(1)} = Ax^{(0)}$ the stability can be analyzed directly from the matrix A as we show with the following well known results.

Proposition 10 ([3]) *An equilibrium state of $x^{(1)} = Ax^{(0)}$ is stable in the sense of Lyapunov, if and only if all eigenvalues of A have negative or zero real part and those with zero real parts are distinct roots of the minimal polynomial of A.*

Proposition 11 ([3]) *The zero state of $x^{(1)} = Ax^{(0)}$ is asymptotically stable if and only if all eigenvalues of A have negative real part.*

For the case of $x^{(1)} = Ax^{(0)} + Bu$ the stability is determined for the poles of transfer function $G(s) = (sI - A)^{-1}B$, and we have the following corollary.

It is well known the powerful of feedback concept, which is used extensively in engineering systems. The principle of feedback is to try to correct actions on the difference between desired and actual performance. For example the feedback can be used to stabilize $x^{(1)} = (A + BF)x^{(0)}$; that is to say, to make its zero solution asymptotically stable.

Corollary 12 *The zero state of $x^{(1)} = Ax^{(0)} + Bu$ is asymptotically stabilizable if and only if*

$$\begin{array}{l}\text{rank}\begin{pmatrix}sI - A & B\end{pmatrix} = n, \\ \forall s \in \mathbb{C}^+ = \{s \in \mathbb{C} \mid Re\,(s) \geq 0\},\ s \text{ finite}\end{array}.$$

In our particular setup one use feedback $u = -F_{\ell-1}x^{(\ell-1)} - \ldots - F_0x^{(0)}$, with $F_i \in M_{m\times n}(\mathbb{C})$ of the state vector x to move the eigenvalues of the closed-loop system $x^{(\ell)} = (A_{\ell-1} - BF_{\ell-1})x^{(\ell-1)} + \ldots + (A_0 - BF_0)x^{(0)}$ to desired locations.

Then, asymptotic stability character can be expressed in the following manner

Proposition 13

$$\begin{array}{l}\text{rank}\begin{pmatrix}s\mathbf{I} - \mathbf{A} & \mathbf{B}\end{pmatrix} = \ell n, \\ \forall s \in \mathbb{C}^+ = \{s \in \mathbb{C} \mid Re\,(s) \geq 0\},\ s \text{ finite}\end{array} \tag{12}$$

But, it can be computed directly from the initial equation 1 in the following manner.

Theorem 14 *The ℓ-order linear system 4 is asymptotically stabilizable if and only if,*

$$\text{rank}\begin{pmatrix}s^\ell I - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B\end{pmatrix} = n. \tag{13}$$

for all $s \in \mathbb{C}$.

In is interesting to observe that.

Proposition 15 *The ℓ-order controllability character is invariant under feedback operation.*

Proof:
It suffices to observe that:

$$\begin{pmatrix} sI_n & -I_n & \ldots & 0 & 0 \\ 0 & sI_n & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & I_n & 0 \\ -A_0 + BF_0 & -A_1 + BF_1 & \ldots & sI_n - A_{\ell-1} + BF_{\ell-1} & B \end{pmatrix} =$$

$$\begin{pmatrix} sI_n & -I_n & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & I_n & 0 \\ -A_0 & -A_1 & \ldots & sI_n - A_{\ell-1} & B \end{pmatrix} \begin{pmatrix} I & & 0 & 0 \\ & \ddots & & \\ 0 & & I & 0 \\ F_0 & \ldots & F_{\ell-1} & I \end{pmatrix} \tag{14}$$

□

4 Exact controllability

Between different aspects in which we can study the controllability, we have the Exact controllability that serves to measure the minimum set of controls that are needed to steer the whole system toward any desired state, in particular measure the minimum set of controls to stabilize the system. Then, the goal of this paper is to find the set of all possible matrices B, having the minimum number of columns corresponding to the minimum number $n_B(A)$ of independent controllers required to control the whole network.

Definition 16 *Let A be a matrix representing the homogeneous linear dynamical system $\dot{x} = Ax$. The exact controllability $n_B(A)$ is the minimum of the rank of all possible matrices B making the system 8 controllable.*

$$\begin{aligned} n_B(A) &= \min\{\text{rank}\, B, \forall B \in M_{n\times i},\ 1 \leq i \leq n, \\ &\quad (A, B) \text{ controllable}\}. \end{aligned}$$

If confusion is not possible we will write simply n_B.

Example:

1) Obviously, if $A = 0$, $n_B = n$.

2) If

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

with $\lambda_i \neq \lambda_j$ for all $i \neq j$, then $n_B = 1$. It suffices to take

$$B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Notice that, the controllability matrix $\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ is a VanderMoonde matrix.

3) Not every matrix B having n_B columns is valid to make the system controllable. For example if

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

the system (A, B) is not controllable, (rank $\begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$, or equivalently rank $\begin{pmatrix} A - \lambda I & B \end{pmatrix} = 2$ for $\lambda = 2, 3$.

It is straightforward that n_B is invariant under similarity, that is to say: for any invertible matrix S we have $n_B(A) = n_B(S^{-1}AS)$. As a consequence, if necessary we can consider A in its canonical Jordan form.

Extending the similarity equivalence to $\ell+1$-ples of matrices in the following manner.

Definition 17 *The ℓ-ple of matrices $(A_{(\ell-1)}, \dots, A_0, B)$ and $({A'}_{(\ell-1)}, \dots, {A'}_0, B')$ are similar if and only if there exists an invertible matrix S such that*

$$A'_\ell = S^{-1}A_{(\ell-1)}S, \dots, {A'}_0 = S^{-1}A_0S, B' = S^{-1}B$$

The exact controllability character is also preserved by this equivalence relation.

Proposition 18 *The exact controllability $n_{\mathbf{B}}(\mathbf{A})$ is invariant under ℓ-pla similarity.*

Proof:

$$\begin{pmatrix} sI & -I & \dots & 0 & 0 \\ 0 & sI & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -{A'}_0 & -{A'}_1 & \dots & sI - {A'}_{\ell-1} & B \end{pmatrix} = \begin{pmatrix} S^{-1} & & \\ & \ddots & \\ & & S^{-1} \end{pmatrix} \cdot \begin{pmatrix} sI & -I & \dots & 0 & 0 \\ 0 & sI & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -A_0 & -A_1 & \dots & sI - A_{\ell-1} & B \end{pmatrix} \cdot \begin{pmatrix} S & & & \\ & \ddots & & \\ & & S & \\ & & & I \end{pmatrix}$$

□

So, if it is necessary we can consider some matrix A_i is its canonical reduced form.

Proposition 19 ([21])

$$n_B = \max_i \{\mu(\lambda_i)\}$$

where $\mu(\lambda_i) = \dim \operatorname{Ker}(A - \lambda_i I)$.

Theorem 20

$$n_{\mathbf{B}} = \max_i \{\mu(\lambda_i)\}$$

where $\mu(\lambda_i) = \dim \operatorname{Ker}(\lambda_i I - \mathbf{A})$.

Corollary 21

$$n_{\mathbf{B}} = \max_i \{\nu(s_i)\}$$

where

$$\nu(s_i) = \dim \operatorname{Ker}(s^\ell I - s^{\ell-1}A_{\ell-1} - \dots - sA_1 - A_0)$$

for all $s_i \in C$ *such that*

$$\det(s^\ell I - s^{\ell-1}A_{\ell-1} - \dots - sA_1 - A_0) = 0.$$

Particular cases

1. If $A_0 = 0$ then $n_{\mathbf{B}}(\mathbf{A}) = n$, because of, for $s = 0$, rank $\begin{pmatrix} s^\ell I - s^{\ell-1}A_{\ell-1} - \dots - sA_1 & B \end{pmatrix} =$ rank B

2. If $A_{\ell-1} = \ldots = A_1 = 0$ and $A_0 \neq 0$ corresponding to the system $x^{\ell} = A_0 x$. Then $n_{\mathbf{B}} = n_B$

3. For the case of $A_0 = A_1 = \ldots = A_{\ell-1}$ in the system 1, the exact controllability can be measured computing the exact controllability of $\dot{x} = A_0 x$

Example:
Let $x^{(2)} = A_0 x$ be a homogenous system with

$$A_0 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

We have that $\nu(s_i) = 2$ for $s_i = 2, 3$, (and 0 for all $s \neq 2, 3$), then $n_{\mathbf{B}} = 2$.

In fact, rank $\begin{pmatrix} s^2 I - A_0 & B \end{pmatrix} = 8$, for all $s \in \mathbb{C}$ with

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but rank $\begin{pmatrix} s^2 I - A_0 & B \end{pmatrix} = 7$, for all $s \in \mathbb{C}$ and for any matrix

$$B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix}$$

Example:
Let $x^{(2)} = Ax^{(1)} + Ax^{(0)}$ be an homogeneous second order linear system, with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

In this case, it suffices to compute the exact controllability of the system $x^{(1)} = Ax^{(0)}$.

All possible matrices B with a minimal number of columns making the system $x^{(1)} = Ax^{(0)} + Bu$ controllable are

$$\begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

for all α, β with $\alpha \cdot \beta \neq 0$.

Following corollary 21, the problem to explicit the possible matrices B having minimal number of columns making the system controllable is linked to the eigenstructure of the polynomial matrix $P(s)$.

Example:
Let $x^{(\ell)} = Ax^{(0)}$ be a homogeneous ℓ-order linear system and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix A with $\lambda_i \neq \lambda_j$ for all $i \neq j$. Then, $n_{\mathbf{B}} = 1$ and all possible matrices B are $[\alpha_1 v_1 + \ldots + \alpha_n v_n]$ where $\alpha_i \neq 0$ and v_i are the eigenvectors corresponding to the eigenvalues λ_i for all $i = 1, \ldots, n$.

5 Exact controllability of ℓ-order singular linear systems

The study of generalized linear systems has experimented a great deal of interest in recent years. High order generalized systems applied to power systems, were studied by Campbell and Rose [2]. These systems were also used in conjunction with the analysis and modelling of flexible beams (see [17], [18]). Antoniou in [1], presents an algorithm for computation the transfer functions for generalized second-order singular systems

We consider the space of matrix-vectors $(E, A_{(\ell-1)}, \ldots, A_0)$ where $E, A_{(\ell-1)}, \ldots, A_0 \in M_n(\mathbb{C})$, corresponding to a ℓ-order time-invariant singular linear systems

$$Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)}, \tag{15}$$

As in the case of standard ℓ order linear systems, we are interested in exact controllability of these kind of systems. The procedure that we are going to use is the same as the one used for the standard case.

The ℓ-order singular linear system

$$Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0 x^{(0)} + Bu \tag{16}$$

can be linearized to the system

$$\mathbf{E}X^{(1)} = \mathbf{A}X + \mathbf{B}u \tag{17}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are as matrices in 3 and

6 respectively. The matrix

$$\mathbf{E} = \begin{pmatrix} I_n & & & \\ & \ddots & & \\ & & I_n & \\ & & & E \end{pmatrix}$$

5.1 Controllability of ℓ-order singular linear systems

Let $Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ be a ℓ-order singular linear system,

Proposition 22 ([9]) *The system 17 is controllable if and only if:*

$$\left.\begin{array}{l} \text{rank}\begin{pmatrix}\mathbf{E} & \mathbf{B}\end{pmatrix} = n\ell \\ \text{rank}\begin{pmatrix}s\mathbf{E} - \mathbf{A} & \mathbf{B}\end{pmatrix} = n\ell,\ \forall s \in \mathbb{C} \end{array}\right\}, \tag{18}$$

Remark 23 *Notice that if $E = I_n$ then $\mathbf{E} = I_{n\ell}$ and the controllability character coincides with the standard case.*

Remark 24 *The controllability of high order singular linear systems is defined as the controllability of the associated linearized system. (For information about controllability of singular linear systems, see [4], [5], [9] for example).*

Controllability can be computed directly, from ℓ-order singular linear system:

Theorem 25 *The system 17 is controllable if and only if:*

$$\left.\begin{array}{l} \text{rank}\begin{pmatrix}E & B\end{pmatrix} = n \\ \text{rank}\begin{pmatrix}s^\ell E - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B\end{pmatrix} = n, \\ \quad \forall s \in \mathbb{C} \end{array}\right\}, \tag{19}$$

Definition 26 *Let $\mathbf{E}X^{(1)} = \mathbf{A}X$ be a homogeneous singular linear system. The exact controllability $n_B(\mathbf{E}, \mathbf{A})$ is the minimum of the rank of all possible matrices B making the system controllable.*

$$\begin{array}{l} n_B(\mathbf{E}, \mathbf{A}) = \\ \min\{\text{rank}\, B, \forall B \in M_{n\times i}\ 1 \leq i \leq n, \\ \qquad\qquad (\mathbf{E}, \mathbf{A}, \mathbf{B}) \text{ controllable}\}. \end{array} \tag{20}$$

For high order singular systems it is obvious that $n_B(\mathbf{E}, \mathbf{A}) \geq n - \text{rank}\, E = n_E$.

Theorem 27 *The exact controllability n_B is*

$$n_B = \max\{n_E, \mu(\lambda_i)\}$$

where $\mu(\lambda_i) = \dim \text{Ker}\,(\lambda_i\mathbf{E} - \mathbf{A})$ and λ_i (for each i) is the eigenvalue of pencil $s\mathbf{E} - \mathbf{A}$.

Example:

Let $Ex^{(2)} = A_0x$ be a homogeneous second order singular system with

$$E = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad A = \begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}.$$

We have:

$$\begin{array}{l} n_E = 2 - 1 = 1 \\ \det(s\mathbf{E} - \mathbf{A}) = -s^2 + 2 = -(s - \sqrt{2})(s + \sqrt{2} \\ \dim \text{Ker}\,(\sqrt{2}\mathbf{E} - \mathbf{A}) = 1 \\ \dim \text{Ker}\,(-\sqrt{2}\mathbf{E} - \mathbf{A}) = 1 \end{array}$$

Then,

$$n_B(\mathbf{E}, \mathbf{A}) = 1$$

For example, we can consider

$$B = \begin{pmatrix}1\\ -1\end{pmatrix}$$

Observe that not any matrix B serves to obtain a controllable system.

For example, if one consider

$$B = \begin{pmatrix}1\\ 1\end{pmatrix}$$

the system $Ex^{(2)} = A_0x + Bu$ is not controllable.

6 Conclusions

In this paper, the concept of controllability of a linear systems in the sense that they can be steered from any initial state to any final state in finite time, has been revised as well as its generalization to those of higher order.

Among other points of view where the controllability can be treated, the exact controllability based on the maximum multiplicity for high order linear systems is analyzed.

A generalization to singular high order linear systems is introduced.

I each section, simple examples have been put to make the work more comprehensible.

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