SISO Problems of H_2-Optimal Synthesis with Allocation of Control Actions

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Abstract: - This paper is devoted to H_2-optimization of linear time invariant (LTI) systems with scalar controls and external disturbances. It is supposed that the intensities of control actions are not given initially for all channels of control. In conformity, a joined problem of H_2-optimal synthesis is posed including allocation of control actions. To solve this problem, a specific spectral approach in frequency domain is developed based on polynomial factorization. Some theoretical details are discussed and numerical algorithm is proposed for practical implementation of this approach, which allows to increase a computational efficiency of synthesis. Applicability and effectiveness of the proposed approach are illustrated by numerical example.

Key-Words: - control allocation, optimization, H_2-control, feedback, functional.

1 Introduction

Various problems of feedback control analytical synthesis for LTI systems often arise nowadays both in theoretical researches and in theirs practical applications. It is quite suitable to use optimization approach for these problems solution to provide high effectiveness of the controllers to be designed. Note that numerical methods of optimization could be excellently realized using modern computer technologies. This makes possible to automatize time-consuming processes of control systems design and simulation. It is very significant to reduce computational consumption for controllers design in case of real time onboard adaptive tuning for autonomous moving plants.

Nevertheless, application of well-known methods of optimal synthesis can be ineffective because of limited processing powers of onboard computers. Thereby design algorithms should be developed to minimize computational complexity of their practical implementation.

Problems of mean-squire optimization [1-8], taking into account presence of external random actions to the plant, occupy a special position in the H-theory of analytical synthesis. These problems are typical for movement control of marine ships, affected by the sea surface waves [9-10]. Mathematical model of such disturbance is usually described as time-invariant Gaussian random process with prescribed spectral power density. Let us note that a spectrum of the sea disturbance always has dominating frequencies, determined by mean value of sea surface wave period.

The main problem, which is discussed in this paper, is the special SISO-problem statement with simultaneous search of transfer function of the optimal controller and the vector, characterizing control action allocation for the channels of control. The specific spectral approach [5-8] to H_2-optimal control synthesis is used as a background that simplifies both a research and calculation.

2 The Problem of H_2-Synthesis

Let us consider a LTI plant with the mathematical model of the state space form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{h}d(t), \\ y &= \mathbf{c}\mathbf{x} + b_y u, \end{aligned} \tag{1}$$

where $\mathbf{x} \in E^n$ is the state space vector, y, u and d are the scalar values: y is the measured variable, u is the control and d represents an external disturbance. All components of the matrices $\mathbf{A}$, $\mathbf{h}$, $\mathbf{c}$ are given constants, the pair $\{\mathbf{A},\mathbf{c}\}$ is observable. Vector $\mathbf{b}$ and number b_y are not specified a priori.

External disturbance $d(t)$ is considered as the random stationary process with zero mathematical expectation and with the following spectral density

$$S_d(s) \equiv S_d(\omega)\big|_{\omega=-js} \equiv S_1(s)S_1(-s),$$
$$S_1(s) \equiv N(s)/T(s).$$

where $N(s)$ and $T(s)$ are Hurwitz polynomials.

Let us introduce mean–squire functional, which is given on motions of the plant (1):

$$I = \lim_{T\to\infty}\frac{1}{T}\int_0^T [y^2(t) + k^2 v^2(t)]dt = \langle y^2\rangle + k^2\langle v^2\rangle\,, \quad (2)$$

where k is initially given constant, and

$$\begin{aligned} &\nu = B(s)u,\\ &B(s) = A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{b} + A(s)b_y\,,\\ &A(s) = \det(\mathbf{I}s - \mathbf{A}). \end{aligned} \quad (3)$$

Let us accept that controller to be designed has tf-model

$$u = W(s)y\,, \quad (4)$$

where $W(s)$ is transfer function of the controller, $W(s) = W_1(s)/W_2(s)$ and $W_1(s)$, $W_2(s)$ are polynomials at that. Let us consider a set Ω_0 of rational fractions with any degrees, and, in addition, let introduce the subset $\Omega^* \subset \Omega_0$:

$$\Omega^* = \left\{\begin{matrix} W\in\Omega_0 : \mathrm{Re}\,\delta_i(W) < 0, \\ \Delta_3(\delta_i(W)) = 0, i = \overline{1, n_3} \end{matrix}\right\}, \quad (5)$$

where

$$\begin{aligned} &\Delta_3(s) = A(s)W_2(s) - B(s)W_1(s)\,,\\ &n_3 = \deg\Delta_3(s)\,. \end{aligned} \quad (6)$$

As a result, Ω^* is a set of controllers (4), stabilizing closed-loop system with characteristic polynomial $\Delta_3(s)$, having degree n_3.

Note that the functional (2) depends on the transfer function $W(s)$ and the vector $\boldsymbol{\beta} = \left(\mathbf{b}^{\mathrm{T}} \,\vdots\, b_y\right)^{\mathrm{T}} \in E^{n+1}$ for the closed-loop system (1), (4). This allows to pose the following minimization mean-square (or H_2 [6]) problem with requirement of the closed-loop system stability:

$$I = I(W, \boldsymbol{\beta}) \to \min_{W\in\Omega^*,\ \boldsymbol{\beta}\in E^{n+1}} \quad (7)$$

Let us notice, that unlike the classical theory of the mean-square optimization, problem (7) requires to compute not only transfer function $W(s)$ of the controller, but also the vector $\boldsymbol{\beta}$, characterizing allocation of the control action effectiveness with respect to the state space equations or the channels of control.

3 Transition to Equivalent problem

To simplify the solution of the problem (7), let us provide its transformation to certain equivalent form. First of all, let us transform the model (1) of the plant to the input-output representation

$$A(s)y = B(s)u + P(s)d\,, \quad (8)$$

$$P(s) = A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{h}\,. \quad (9)$$

Now we stress that the freedom with allocation of the control action *u* for the model (8) is determined by the fact that polynomial $B(s)$, having a degree

$$m = \deg B(s) \le n\,, \quad (10)$$

is not fixed a priori. In accordance with (3), the value of functional (2) is uniquely defined by selection both this polynomial and the transfer function $W(s)$ of the controller (4). Then the problem (7) can be transformed to the following equivalent form

$$I = I(W, B) \to \min_{W\in\Omega^*,\ B(p)\in\Omega_b}\,, \quad (11)$$

where Ω_b is the set of polynomials $B(s)$, satisfying (10).

Now we shall consider the problem (11) for the closed-loop system (8), (4) with the functional (2), which is equivalent to the initial one.

Let us introduce the following additional designations:

$$\begin{aligned} &\{W_0, B_0\} = \arg\min_{W\in\Omega^*,\ B(p)\in\Omega_b} I(W, B)\,,\\ &I_0 = I(W_0, B_0),\ W_0(s) = W_{01}(s)/W_{02}(s)\,. \end{aligned} \quad (12)$$

Let us also introduce an auxiliary closed-loop control system, involving the plant with the model

$$A(s)y = v + P(s)d\,, \quad (13)$$

and the controller

$$v = V(s)y\,, \quad (14)$$

where $\nu \in E^1$ is a new scalar control action. All the rest variables and parameters of the system (13), (14) are the same as for the system (4), (8).

Now let us consider the functional (2), which is given on the motions of the closed-loop system (13), (14), and note that its value depends only on the transfer function $V(s)$ in (14). This case allows us to pose the following mean-square synthesis problem

$$I = I(V) \to \min_{V\in\Omega^*}\,; \quad (15)$$

for its solution we shall use the new notations

$$V_0 = \arg\min_{V\in\Omega^*} I(V),\ I_{0v} = I(V_0)\ ,\\ V_0(s) = V_{01}(s)/V_{02}(s)\ . \tag{16}$$

The following statement is valid with respect to the problems (11) and (15):

Theorem 1: The problem (11) of the functional (2) minimization for the closed-loop system (8), (4) is equivalent to the problem (15) of the same functional minimization for the closed-loop system (13), (14) in the sense that

$$V_0(s) \equiv B_0(s)W_0(s)\ ,\ I_{0v} = I_0\ . \tag{17}$$

Proof: Let us suppose that the solution (12) of the problem (11) has already been obtained by any way. The characteristic polynomial

$$\Delta_0(s) = A(s)W_{02}(s) - B_0(s)W_{01}(s)\ , \tag{18}$$

of the closed-loop connection (8), (4) is Hurwitz, and the functional (2) attains the minimum value

$$I_0 = \int_0^\infty \left[\, |W_{02}(j\omega)|^2 + k^2 |B_0(j\omega)W_{01}(j\omega)|^2 \right]\times \\ \times \left|\frac{P(j\omega)}{\Delta_0(j\omega)}\right|^2 S_d(\omega)d\omega. \tag{19}$$

Let us form an auxiliary controller

$$v = V^*(s)y\ , \tag{20}$$

where $V^*(s) \equiv B_0(s)W_0(s)$, and implement this one to close the plant (13). Characteristic polynomial of the closed-loop system (13), (20) obviously coincides with (18), whence it follows, that $V^*(s) \in \Omega^*$. Also the controller (20) provides the value (19) of the functional (2), which means that $I^* = I(V^*) = I_0$. Let us show that $I^* = I_{0v}$ using proof by contradiction.

Really, let us suppose that there exists such a controller $v = \tilde{V}(s)y$ with the transfer function $\tilde{V} = \tilde{V}_1/\tilde{V}_2$ that $\tilde{V}(s) \in \Omega^*$ for the plant (13), and $\tilde{I} = I(\tilde{V}) < I^* = I_0$. At that, let us represent $\tilde{V}_1(s)$ in the form $\tilde{V}_1(s) \equiv \tilde{B}(s)\overline{V}_1(s)$, where polynomial $\tilde{B}(s)$ satisfies the condition $\deg \tilde{B}(s) \le n$ and consider the controller

$$u = \tilde{W}(s)y\ ,\ \tilde{W}(s) = \overline{V}_1(s)/\tilde{V}_2(s)\ . \tag{21}$$

Therefore, we have $I(\tilde{W},\tilde{B}) < I_0$ for the closed-loop system (8), (21), i.e. the pair $\{\tilde{W},\tilde{B}\}$ provides better result than the pair $\{W_0, B_0\}$, but this is impossible. Hence it is proven that $I^* = I_{0v} = I_0$, $V^*(s) \equiv V_0{}^*(s)$. Consequently, the solution (12) of the problem (11) coincides with the solution of the problem (15).

On the contrary, let us suppose that the solution (16) of the problem (15) is computed. Then let us make by any way the presentation

$$V_{01}(s) \equiv \overline{B}(s)\overline{V}_{01}(s)\ , \tag{22}$$

so that $\deg \overline{B}(s) \le n$, and construct the controller

$$u = \overline{W}(s)\ ,\ \overline{W}(s) = \overline{V}_{01}(s)/V_{02}(s)\ . \tag{23}$$

Similarly to the previous case, we can show, that the pair $\{\overline{W}, \overline{B}\}$ is optimal solution of the problem (11), providing the minimal value $I_0 = I_{0v}$ of the closed-loop system (8), (23). Hence, the proof is completed. ■

4 Spectral Approach to Synthesis

So than the problem (11) of the optimal control synthesis may be reduced to the equivalent standard one (15), as it follows from the theorem 1.

This can be interpreted as the partial case of H_2 optimal synthesis problem, which can be solved by using a lot of various methods: for example, such as "2-Riccati " approach or LMI technique. However, there are serious troubles, remarked in the papers [5-8], related with the singularity of this problem in the context of H-theory.

As is shown in the paper [6], overcoming of these difficulties can be provided by the implementation of the special spectral approach, which is presented in detailes in the papers [5-8]. In accordance with this approach, the unique solution of the problem (15) is determined by the following transfer function of the optimal controller (14):

$$V_0(s) = \frac{[A(s)T(s)R(s) + P_1(s)N(s)]/G(-s)}{\left[T(s)R(s) - k^2 A(-s)P_1(s)N(s)\right]/G(-s)}\ . \tag{24}$$

Hurwitz polynomials $G(s)$ and $P_1(s)$ are the results of the factorizations

$$k^2 A(s)A(-s) + 1 \equiv G(s)G(-s)\ ,\\ P(s)P(-s) \equiv P_1(s)P_1(-s)\ , \tag{25}$$

and $R(s)$ is the following auxiliary polynomial:

$$R(s)=\sum_{i=1}^{n}\frac{G(-s)}{g_i-s}\frac{S_1(g_i)P_1(g_i)}{A(g_i)G'(-g_i)}. \qquad (26)$$

Here $G'(-g_i)=dG(-s)/ds|_{s=g_i}$, g_i $(i=\overline{1,n})$ are the roots of the polynomial $G(-s)$ (for simplicity it is assumed that all the roots are distinct). Divisions to the polynomial $G(-s)$ in (24) are realized totally (without a remainder). Spectral presentation (24) of the solution allows formulating the following statements:

Theorem 2: The transfer function $V_0(s)$ (16) of the optimal controller (14) always can be presented as

$$V_0(s)\equiv B_0(s)W_0(s), \qquad (27)$$

where $B_0(s)$ is a polynomial of degree n, and $W_0(s)=W_{01}(s)/W_{02}(s)$ is a proper rational fraction, i.e.

$$\deg B_0(s)=n,\ \deg W_{01}(s)\le \deg W_{02}(s). \qquad (28)$$

Proof: Let us represent the transfer function (24) as

$$V_0(s)\equiv\frac{V_{01}(s)}{V_{02}(s)}\equiv$$
$$\equiv\frac{\upsilon_\mu s^\mu+\upsilon_{\mu-1}s^{\mu-1}+\ldots+\upsilon_1 s+\upsilon_0}{\delta_\nu s^\nu+\delta_{\nu-1}s^{\nu-1}+\ldots+\delta_1 s+\delta_0}, \qquad (29)$$

and use notations $p=\deg N(s)$, $q=\deg T(s)$, $r=\deg P(s)$ to express degrees of the polynomials $V_{01}(s)$, $V_{02}(s)$. Taking into account that $\deg G(s)=n$ and $\deg R(s)=n-1$, we derive from the formula (24):

$$\mu=\max\{n+q+n-1,\ p+r\}-n,$$
$$\nu=\max\{n+q-1,\ p+n+r\}-n.$$

So far as $p<q$ and $r<n$, we have

$$\mu=n+q-1,\ \nu=\max\{q-1,\ p+r\}-n. \qquad (30)$$

Let us remark that decreasing of the polynomial $V_{01}(s)$ degree in comparison with (30) is impossible; but as for polynomial $V_{02}(s)$, it is possible if and only if $p+r=q-1$. Nevertheless, this is only a partial situation for the concrete value k, and we suppose that this is not the case.

We can state on the base of (30) that the numerator V_{01} in (29) always can be presented as a product of two polynomials:

$$V_{01}(s)\equiv B_0(s)W_{01}(s),\ \deg B_0(s)=n,$$
$$\deg W_{01}(s)=q-1. \qquad (31)$$

At the same time, it is true that $\deg V_{02}(s)=q-1$ or $\deg V_{02}(s)=p+r>q-1$. Hence, accepting $W_0(s)=W_{01}(s)/W_{02}(s)$, where $W_{02}(s)\equiv V_{02}(s)$, we arrive to the statement of theorem. ■

Theorem 2 determines a solution of the initial problem (7) for the plant (1) and for the controller (4).

Theorem 3: Let us suppose that the optimal transfer function $V_0(s)$ (16) for the problem (15) has been computed, taking into account its presentation (27). Then the solution

$$\{W^*,\boldsymbol{\beta}^*\}=\arg\min_{W\in\Omega^*,\ \boldsymbol{\beta}\in E^{n+1}} I(W,\boldsymbol{\beta}), \qquad (32)$$

of the problem (7) is determined by the transfer function $W^*(s)\equiv W_0(s)$ and, also, by the vector $\boldsymbol{\beta}^*=\left(\mathbf{b}^{*\mathrm{T}}\ \vdots\ b_y^*\right)^{\mathrm{T}}$, where $\mathbf{b}^*$ is a solution of the linear algebraic system

$$\mathbf{C}_s\mathbf{b}=\mathbf{b}_{s0}-b_y^*\mathbf{a}_s. \qquad (33)$$

Here the matrix

$$\mathbf{C}_s=\begin{pmatrix} c_{1(n-1)} & c_{2(n-1)} & \cdots & c_{n(n-1)} \\ c_{1(n-2)} & c_{2(n-2)} & \cdots & c_{n(n-2)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{11} & c_{21} & \cdots & c_{n1} \\ c_{10} & c_{20} & \cdots & c_{n0} \end{pmatrix}$$

consists of coefficients of the polynomials

$$C_i(s)=c_{i(n-1)}s^{n-1}+c_{i(n-2)}s^{n-2}+\ldots$$
$$\ldots+c_{i1}s+c_{i0},\ i=\overline{1,n},$$

and the vectors $\mathbf{b}_{s0}=\left(b_{0(n-1)}\ b_{0(n-2)}\ \ldots\ b_{01}\ b_{00}\right)^{\mathrm{T}}$, $\mathbf{a}_s=(a_{n-1}\ a_{n-2}\ \ldots\ a_1\ a_0)^{\mathrm{T}}$ are determined by the coefficients of the polynomials

$$B_0(s)=b_{0n}s^n+b_{0(n-1)}s^{n-1}+b_{0(n-2)}s^{n-2}+\ldots$$
$$\ldots+b_{01}s+b_{00},$$
$$A(s)=s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0,$$

respectively, and $b_y^*=b_{0n}$.

Proof: the controller (4) with the transfer function $W\equiv W_0$ characterizes a solution of the problem (11) for the plant (8) simultaneously with the polynomial $B_0(s)$, in accordance to the Theorems 1, 2. Nevertheless, as it was mentioned above, the problem (11) is equivalent to the problem (7), i.e. its solution is the controller (3) simultaneously with the

vector
$\boldsymbol{\beta} = \left(\mathbf{b}^{\mathrm{T}} \;\vdots\; b_y\right)^{\mathrm{T}} \in E^{n+1}$, satisfying the identity

$$A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{b} + A(s)b_y \equiv B_0(s)\,. \qquad (34)$$

Introducing the following notation:

$$\mathbf{C}(s) \equiv A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1} \equiv$$
$$\equiv (C_1(s) \quad C_2(s) \quad \ldots \quad C_n(s)),$$

let us rewrite the identity (34) as

$$(C_1(s) \quad C_2(s) \quad \ldots \quad C_n(s))\mathbf{b} \equiv B_0(s) - A(s)b_y\,.$$

By equating of the coefficients for the same degrees of s in the last identity, we obtain the equality $b_y = b_y^* = b_{0n}$ and also we obtain the linear algebraic system (33). Let us remark that a condition of the pair $\{\mathbf{A}, \mathbf{c}\}$ observability provides a relative primality of the polynomials $C_i(s),\ i = \overline{1,n}$ that guarantees nonsingularity of the matrix $\mathbf{C}_s$, i.e. there exists the unique solution $\mathbf{b} = \mathbf{b}^*$ of the system (33). ■

5 Algorithm and numerical example

Spectral approach to mean-squire synthesis, presented by the theorems referred above, is the base for constructing of a computational algorithm, which allows to find a solution of control allocation problem.

Let us accept the matrices $\mathbf{A}$, $\mathbf{h}$, $\mathbf{c}$ of the plant (1), weighting coefficient k, and polynomials $N(s)$, $T(s)$ as initial data. It is necessary to carry out the following computations to solve the problem (7).

1. Compute the polynomials

$A(s) = \det(\mathbf{I}s - \mathbf{A})$, $P(s) = A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{h}$,
$C_i(s),\ i = \overline{1,n}$, where
$(C_1(s) \quad C_2(s) \quad \ldots \quad C_n(s)) \equiv A(s)\mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}$.

2. Execute factorizations of two polynomials

$$\begin{aligned} &k^2 A(s)A(-s) + 1 \equiv G(s)G(-s)\,, \\ &P(s)P(-s) \equiv P_1(s)P_1(-s)\,, \end{aligned} \qquad (35)$$

i.e. compute Hurwitz polynomials $G(s)$ and $P_1(s)$.

3. Compute the auxiliary polynomial:

$$R(s) = \sum_{i=1}^{n} \frac{G(-s)}{g_i - s} \frac{N(g_i)P_1(g_i)}{A(g_i)T(g_i)G'(-g_i)}\,, \qquad (36)$$

where $G'(-g_i) = dG(-s)/ds\,|_{s=g_i}$, g_i $(i = \overline{1,n})$ are distinct roots of the polynomial $G(-s)$.

4. Construct the auxiliary transfer function

$$V_0(s) = V_{01}(s) / V_{02}(s)\,, \qquad (37)$$

$$V_{01}(s) = [A(s)T(s)R(s) + P_1(s)N(s)] / G(-s)\,,$$
$$V_{02}(s) = \left[T(s)R(s) - k^2 A(-s)P_1(s)N(s)\right] / G(-s)\,,$$

5. Select any n roots ξ_i of the polynomial $V_{01}(s)$, take the rest roots ζ_j $\left(j = \overline{1, \mu - n}\right)$ and generate the polynomials

$$\begin{aligned} B_0(s) &= \prod_{i=1}^{n} (s - \xi_i)\,, \\ W_{01}(s) &= \prod_{j=1}^{\mu - n} \left(s - \zeta_j\right). \end{aligned} \qquad (38)$$

Let remark for example that if the value k is large (i.e. controller has to be designed in the economical mode of action) then the pair of complex conjugate roots ξ_i close to $\pm\beta j$ must be roots of the polynomial $W_{10}(s)$ to provide demanded frequency properties of the closed-loop connection.

6. Construct two vectors

$$\mathbf{b}_{s0} = \left(b_{0(n-1)} \quad b_{0(n-2)} \quad \ldots \quad b_{01} \quad b_{00}\right)^{\mathrm{T}},$$
$$\mathbf{a}_s = \left(a_{n-1} \quad a_{n-2} \quad \ldots \quad a_1 \quad a_0\right)^{\mathrm{T}},$$

with the coefficients of two polynomials

$$B_0(s) = b_{0n}s^n + b_{0(n-1)}s^{n-1} + b_{0(n-2)}s^{n-2} +$$
$$\ldots + b_{01}s + b_{00}\,,$$
$$A(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0\,,$$

respectively. Construct the matrix

$$\mathbf{C}_s = \begin{pmatrix} c_{1(n-1)} & c_{2(n-1)} & \ldots & c_{n(n-1)} \\ c_{1(n-2)} & c_{2(n-2)} & \ldots & c_{n(n-2)} \\ \ldots & \ldots & \ldots & \ldots \\ c_{11} & c_{21} & \ldots & c_{n1} \\ c_{10} & c_{20} & \ldots & c_{n0} \end{pmatrix},$$

consisting of the coefficients of the polynomials

$$C_i(s) = c_{i(n-1)}s^{n-1} + c_{i(n-2)}s^{n-2} + \ldots$$
$$\ldots + c_{i1}s + c_{i0}\,,\ i = \overline{1,n}.$$

7. Set $b_y = b_{0n}$ and compute a solution $\mathbf{b} = \mathbf{b}^*$ of the linear algebraic system

$$\mathbf{C}_s \mathbf{b} = \mathbf{b}_{s0} - b_y^* \mathbf{a}_s . \qquad (39)$$

8. Accept the transfer function of the optimal controller (3) as $W_0(s) = W_{01}(s)/B_0(s)$, where $W_{02}(s) \equiv V_{02}(s)$, the optimal vector $\mathbf{b} = \mathbf{b}^*$, and the optimal value $b_y = b_y^*$.

Let us consider the practical example, demonstrating applicability and effectiveness of the proposed algorithm. A marine ship, moving on the plane with the constant longitudinal speed, can be presented by the model (1), having the following matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ 0 \end{pmatrix},\ \mathbf{c} = (0 \quad 0 \quad 1)\,. \quad (40)$$

Vector $\mathbf{x} \in E^3$ consists of the following components: drift angle, angular velocity and yaw angle. Let a speed of motion is $V = 10$ m/s: then the components of the matrices (40) are: $a_{11} = -0.0936$, $a_{12} = 0.634$, $a_{21} = 0.0480$, $a_{22} = -0.717$, $h_1 = 1.51 \cdot 10^{-4}$, $h_2 = 0.00902$. External disturbance $d(t)$ is scalar, presented as random time invariant wave process with power spectral density $S_d(s) \equiv S_1(s) S_1(-s)$, $S_1(s) \equiv N(s)/T(s)$, where

$$\begin{aligned} N(s) &= 2\sqrt{D_d \alpha(\alpha^2 + \beta^2)}, \\ T(s) &= s^2 + 2\alpha s + \alpha^2 + \beta^2. \end{aligned} \qquad (41)$$

Here $\beta = 0.455$ is a mean oscillation frequency, $\alpha = 0.21\beta$ is blurriness of the spectrum, and $D_d = 1.52 \cdot 10^{-4}$ is its dispersion. Let us consequently execute all the steps of proposed algorithm with value $k = 10$, and receive:

1. Initial polynomials:

$$\begin{aligned} &A(s) = s^3 + 0.811 s^2 + 0.0367 s, \\ &P(s) = 0.00903 s + 0.000852, \\ &C_1(s) = 0.0480,\ C_2(s) = s + 0.0936, \\ &C_3(s) = s^2 + 0.811 s + 0.0367. \end{aligned}$$

2. Results of the factorization of two polynomials:

$$\begin{aligned} &G(s) = 10.0 s^3 + 12.6 s^2 + 5.04 s + 1, \\ &P_1(s) = 0.00903 s + 0.000852. \end{aligned}$$

3. The auxiliary polynomial:

$$R(s) = -0.00194\, s^2 + 0.00206\, s - 0.000455\,.$$

4. The auxiliary transfer function:

$$V_0(s) = V_{01}(s)/V_{02}(s)\,,$$

$$\begin{aligned} V_{01}(s) = (&0.195 s^4 + 0.234 s^3 + 0.115 s^2 + \ldots \\ &\ldots + 0.0436 s + 0.00302) \cdot 10^{-3}\,, \\ V_{02}(s) = &-(0.126 s + 0.0984) \cdot 10^{-3}\,. \end{aligned}$$

5. The roots of the polynomial $V_{01}(s)$ are $\xi_1 = -0.780$, $\xi_{2,3} = -0.168 \pm 0.453 j$, $\xi_4 = -0.085$. We use three of them to construct B_0:

$$\begin{aligned} B_0(s) &= \prod_{i=1}^{3} (s - \xi_i) = \\ &= s^3 + 1.12 s^2 + 0.496 s + 0.182\,, \\ W_{01}(s) &= (0.194 s + 0.0166) \cdot 10^{-3}. \end{aligned}$$

6. The auxiliary vectors and matrix:

$$\mathbf{b}_{s0} = (1.12 \;\; 0.496 \;\; 0.182)^{\mathrm{T}}\,,$$
$$\mathbf{a}_s = (0.811 \;\; 0.0367 \;\; 0)^{\mathrm{T}}\,,$$

$$\mathbf{C}_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0.811 \\ 0.0480 & 0.0936 & 0.0367 \end{pmatrix}.$$

7. Solution $\mathbf{b} = \mathbf{b}^*$ of the system (39)

$$\mathbf{b} = \mathbf{b}^* = (3.15 \;\; 0.211 \;\; 0.306)^{\mathrm{T}}\,.$$

8. Transfer function of the optimal controller

$$W_0(s) = -\frac{0.194 s + 0.0166}{0.126 s + 0.0984}\,,$$

with the optimal parameters $\mathbf{b} = \mathbf{b}^*$, $b_y = b_y^* = 1$.

Figures 1, 2 represent step and frequency responses of the closed-loop system, computed for the optimal controller, designed above.

Figure 3 demonstrates dynamics of the closed-loop system under the action of the mentioned random wave disturbance and additional unit impulse function, effecting on the system from the 750-th to 950-th s. It can be seen that response to rectangular impulse can be isolated from the time-invariant oscillations that allows us to state impulse disturbance for detection of wrecks. The noticed feature occurs because of the frequency response value $A_y(0) = 1.6$ at zero frequency significantly surpasses the value $A_y(0.455) = 0.92$ at the central frequency of the external disturbance.

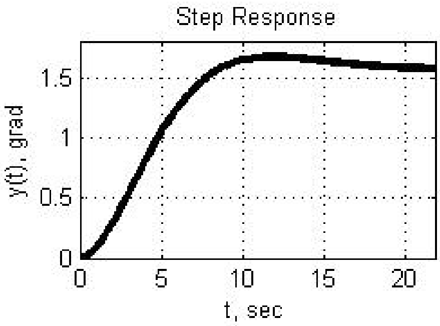

Fig. 1. Step response of the closed-loop system.

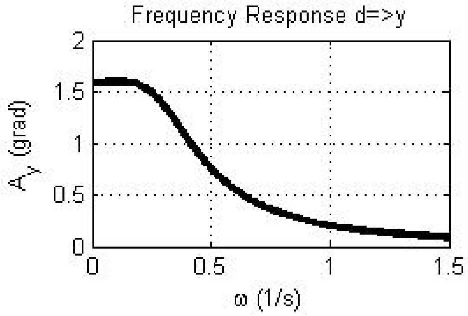

Fig. 2. Frequency response of the closed-loop system.

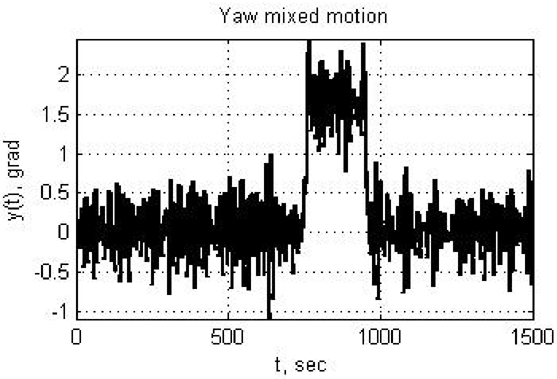

Fig. 3. Joined response of the closed-loop system to the wave and impulse disturbance.

It will be desirable to attempt to increase the coefficient $k_r = k_r(k) = A_y(0)/A_y(0.455)$ of elevation by the choice of the factor k in the functional (2). To realize this attempt, let us consider the plots of the functions $A_y(0.455, k)$ (magnitude at the central frequency of the spectrum), $k_r(k)$, and $T_p(k)$ (peak time of the step response), which are presented on the Figure 4.

These plots demonstrate that increasing of the parameter k results to increasing of the coefficient $k_r(\lambda)$, but the peak time $T_p(\lambda)$ increases too, that is undesirable. In addition, the large value of k results to decreasing a degree of stability of the closed-loop system.

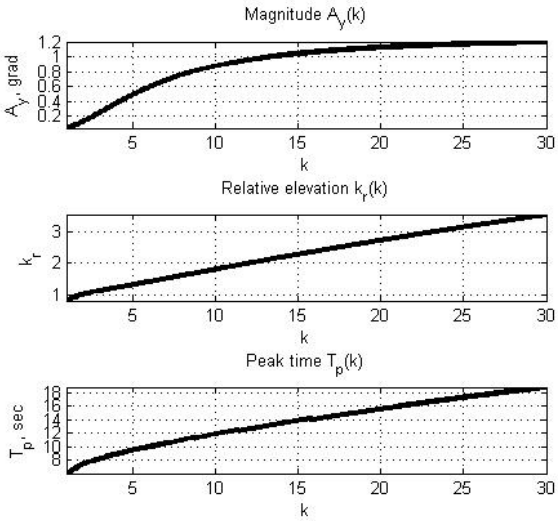

Fig.4. Characteristics of the closed-loop system with various parameter k.

6 Conclusion

The main object of this paper is to propose novel approach to mean-square problem with allocation of the control action for SISO LTI plants. The optimal controller is presented in the specific spectral form (24). Application of the proposed approach allows to construct a simple algorithm of controller synthesis, which can be used, for example, for motion control system design in the laboratory conditions. In addition, it could be applied onboard in real-time regime for detection of the situations close to wrecking. Applicability and efficiency of the proposed designed scheme is illustrated with the help of the practical example for the moving plant control. The object of the future research is solution of the mean-square problem with allocation of the control action for MIMO (multi-input and multi-output) plants, maybe taking into account control delays and robust features.

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