Generalized PIO Design for Uncertain Discrete Time Systems

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Abstract: Considering the uncertain discrete time systems of Linear Parameter Varying (LPV) type, in this paper an approach for the synthesis of robust Proporcional+Integral Observers (PIO) is presented. From LPV systems characterized with polytopical uncertainties, the method of design is based on considering a dynamics extended of the typical PIO, in order to transform the design of the matrices of the dynamics of the observer, as a design of the gain of Static Output Feedback (SOF) of a problem of robust control. Under these conditions and from the norms $\mathscr{H}_2/\mathscr{H}_\infty$ described as Linear Matrix Inequalities (LMI), the criteria to obtain the gain in the SOF problem are established; taking into account performance indices in $\mathscr{H}_2$ and $\mathscr{H}_\infty$, under the presence of uncertainties and disturbances. In order to illustrate the results and the performance of the state robust estimation of states, some numerical examples are evaluated.

Key–Words: Discrete-time systems. LPV Systems. Proportional+Integral Observers (PIO). Linear Matrix Inequalities (LMI). $\mathcal{H}_2$-$\mathcal{H}_\infty$ Norms

1 Introduction

The Linear Parameter Variable (LPV) systems are referred as those linear dynamical systems whose representations in state space depends on exogenous non-stationary parameters [19]. The LPV systems are a generalization of LTV systems, establishing an intermediate model between linear and nonlinear dynamics, which can become a representative model for the control of nonlinear processes, allowing the use of all machinery linear control systems linear to the particular case of nonlinear processes [9, 3].

From a practical point of view, LPV system has at least two interesting interpretations [4]: 1) It can be seen as a LTI system with parametric uncertainty. 2) It can be seen as an LTV model, or a model resulting from the linearization of a nonlinear system (SNL) along the trajectories of the parameter α. This last statement is important since the LPV system represents a description of one intermediate system, which enables to design controllers sor observers for nonlinear systems, in a systematic way such as for linear systems. Also, if the nonlinear model is formulated as a linear system parameterized, where parametrization is dependent states, allows a description LPV represent a nonlinear system not locally, taking advantage of the consequences of a global stabilization [3].

On the other hand, the causal observation is the problem of finding estimates for the current values of a set of signals from the present and past values of another set of signals, where both sets of signals are interconnected by the action a dynamic system. The latter is called *observer*, and the procedure is known as an estimate or reconstruction of states. An important feature is that the estimate is asymptotically exact, ie, to converge the actual value of the observed signals as time goes to infinity (asymptotic observability). In the case of LPV systems, this idea is followed: designing a dynamic system that allows the asymptotic estimation of states, under the presence of parametric variations.

In principle, the observer design problem for LPV systems involves an analysis of the observability of such systems. Thus, in [1] the notion of invariant subspaces for LPV systems is presented, introducing the concept of invariant subspaces of parameter variations, which is very important for the design of state observers, characterizing a geometric condition for the observability.

In the same vein, in [20] some characterizations are presented, and necessary and sufficient conditions for the existence of observers for linear functions and variant finite dimensional systems are given. The results are evaluated for affine parameter variant systems and bilinear control systems.

For the design of observers in LPV discrete-time systems, in [7] a synthesis method based on interpolation is discussed. Estimate the stability of the error is evaluated by the existence of a Lyapunov function dependent, affine way, regarding parameters and a rate of asymptotic decay defined. In [12] observation state is used for the synthesis of state feedback controllers in discrete LPV systems. The observer is Luenberger raised type, regardless unknown entries.

On the other hand, in [5] a PIO for estimating state and unknown inputs in discrete systems without uncertainty applies. For the same class of systems, in [10] a PIO for estimating state and unknown inputs and outputs are presented.

In this paper, a method for designing observers for LPV discrete-time systems based on the Proportional+Integral (PI) observers, considering polytopic type parametric variations in the process and sensors, is presented. The stability of the observer system is analyzed by Lyapunov stability of LPV discrete time polytopic systems. The dynamic observer is constructed by a control formulation by static output feedback (SOF), considering performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$; and also, the possibility of reconstructing unknown inputs.

Notation. $\Re$ is the set of real numbers. For a matrix A, A^T denote its transpose. $\mathrm{tr}\,(A)$ defines the trace of the matrix A. $\mathrm{diag}(A, B)$ is a diagonal matrix with entries A and B on its diagonal. In symmetric matrices partitions $\star$ denotes each of the symmetrical blocks. $\mathbb{R}_n$ defines the identity matrix of dimension n.

2 Theoretical Framework

In order to progress in the description of the key tools that will support the results to present, consider the discrete LTI system

$$\begin{aligned} x_{k+1} &= Ax_k + B\omega_k \\ z_k &= Cx_k + D\omega_k \end{aligned} \qquad (1)$$

where $x \in \Re^n$ are the states, $\omega_k \in \Re^m$ are exogenous inputs (noise, disturbance); and $z_k \in \Re^q$ are regulated outputs. The matrices A, B, C and D be of appropriate dimensions.

From Lyapunov stability, it is very well known that (1) system is asymptotically stable if and only if there exists a matrix $P = P^T \in \Re^n \times n$, satisfying the following matrix inequalities:

$$P > 0, \qquad P - A^T PA > 0 \qquad (2)$$

This condition can be described as follows:

Lemma 1 (Quadratic stability) *Be the system* (1). *If* $P = P^T > 0$, *the matrix* $G \in \Re^{n\times n}$ *and* $Q = P^{-1}$, *then the following conditions of asymptotic stability* (1) *are equivalents:*

i) $P > 0$ *and* $P - A^T PA > 0$.

ii) $P > 0$ *and*

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} > 0, \qquad (3)$$

iii) $Q > 0$ *and*

$$\begin{bmatrix} Q & AQ \\ QA^T & Q \end{bmatrix} > 0, \qquad (4)$$

iv) There exist G, *such that* $G + G^T > 0$ *and*

$$\begin{bmatrix} G + G^T - P & GA \\ G^T A^T & P \end{bmatrix} > 0, \qquad (5)$$

Proof: See [16] □

Consider that is desired to place the poles in a particular stable region, such as in a stable region of radius r and center $(\sigma, 0)$, with $|\sigma| < 1$, see the Figure 1.

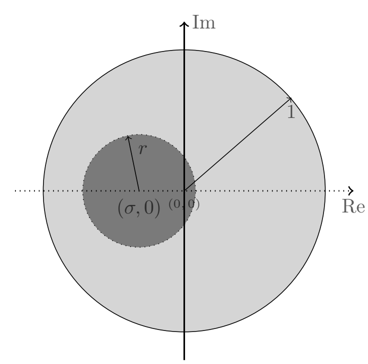

Figure 1: Poles location in circular region.

In this case, the stability condition corresponds [6, 15]:

$$P = P^T > 0, \quad r^2 P - (A - \sigma\mathbb{I}_n)^T P(A - \sigma\mathbb{I}_n) > 0 \qquad (6)$$

It follows by considering the dynamic matrix is given by $\frac{A-\sigma\mathbb{I}_n}{r}$.

Lemma 2 (Quadratic stability and pole placement) *Consider the system* (1). *If there exist* $P = P^T > 0$, *the matrix* $G \in \Re^{n\times n}$ *and* $Q = P^{-1}$, *then the following conditions of asymptotic stability for* (1) *are equivalents* *i)* $P > 0$ *and* $r^2 P - (A - \sigma\mathbb{I}_n)^T P(A - \sigma\mathbb{I}_n) > 0$.

ii) $P > 0$ *and*

$$\begin{bmatrix} rP & A^TP - \sigma P \\ PA - \sigma P & rP \end{bmatrix} > 0, \qquad (7)$$

iii) $Q > 0$ *and*

$$\begin{bmatrix} rQ & AQ - \sigma Q \\ QA^T - \sigma Q & rQ \end{bmatrix} > 0, \qquad (8)$$

iv) There exist G, such that $G + G^T > 0$ *and*

$$\begin{bmatrix} r(G + G^T - P) & GA - \sigma G \\ G^TA^T - \sigma G^T & rP \end{bmatrix} > 0, \qquad (9)$$

Proof: The procedure for the proof of Lema 1 is followed; whereas the dynamic matrix is $\frac{A-\sigma\mathbb{I}_n}{r}$. □

In the same vein, other important results to be taken into account, corresponds to the extended LMIs characterizations of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms for discrete time systems [22, 16]. These standard and extended characterizations of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms as LMIs can be combined with pole placement in particular regions, whereas the dynamic matrix is $\frac{A-\sigma\mathbb{I}_\ltimes}{r}$, which helps to improve the transient response of the dynamics of the state estimator.

In this sense, there are other important results to be taken into account, as they will be used in the development of the proposed technique, corresponds to the extended LMIs characterizations of norms $\mathscr{H}_\infty$ and $\mathscr{H}_2$ of discrete-time systems [22, 16].

Lemma 3 (Extended $\mathscr{H}_2$ performance) *Lets consider the system* (1). *For* $P = P^T > 0$, *next expressions are equivalent*

i) A is stable and $\left\|C(z\mathbb{I} - A)^{-1}B + D\right\|_2 < \mu$.

ii) There exist $P = P^T \in \Re^{n\times n}$ *y* $W = W^T \in \Re^{q\times q}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} P & PA & PB \\ A^TP & P & 0 \\ B^TP & 0 & \mathbb{I} \end{bmatrix} > 0, \quad \begin{bmatrix} W & C & D \\ C^T & P & 0 \\ D^T & 0 & \mathbb{I} \end{bmatrix} > 0, \qquad (10)$$

iii) There exist $P = P^T \in \Re^{n\times n}$, $W = W^T \in \Re^{q\times q}$ *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} P & PA^T & PC^T \\ AP & P & 0 \\ CP & 0 & \mathbb{I} \end{bmatrix} > 0, \quad \begin{bmatrix} W & B^T & D^T \\ B & P & 0 \\ D & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (11)$$

iv) There exist $P = P^T \in \Re^{n\times n}$, $W = W^T \in \Re^{q\times q}$ *y* $G \in \Re^{n\times n}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} G + G^T - P & GA & GB \\ A^TG^T & P & 0 \\ B^TG^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad \begin{bmatrix} W & C & D \\ C^T & P & 0 \\ D^T & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (12)$$

v) There exist $P = P^T \in \Re^{n\times n}$, $W = W^T \in \Re^{q\times q}$ *y* $G \in \Re^{n\times n}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} G + G^T - P & GA^T & GC^T \\ AG^T & P & 0 \\ CG^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad \begin{bmatrix} W & B^T & D^T \\ B & P & 0 \\ D & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (13)$$

Proof: The equivalence of the first three statements has been shown in [18], following the results presented in [8]. The equivalence between *ii)*, *iv)* and *v)* has been shown in [13], from the extensions, derived from the projection lemma presented in [16]. □

It is known that when there are relationships between the dynamic system matrix and the Lyapunov matrix there are obtained conservative results, as is the case of systems with polytopic uncertainties. This situation is overcome, in some degree, uncoupling both arrays and using parameters dependent Lyapunov functions, if applicable.

In the same way as for the $\mathscr{H}_2$ case, there are some results in order to improve performance in $\mathscr{H}_\infty$, as shown below.

Lemma 4 (Relaxed $\mathscr{H}_\infty$ Performance) *Let's consider the system* (1). *With* $P = P^T > 0 \in \Re^{n\times n}$ *and matrix* $G \in \Re^{n\times n}$, *next expressions are equivalent:*

i) A is stable and $\left\|C(z\mathbb{I} - A)^{-1}B + D\right\|_\infty < \gamma$.

ii) There exist P, such that

$$\begin{bmatrix} P & PA & PB & 0 \\ A^TP & P & 0 & C^T \\ B^TP & 0 & \gamma\mathbb{I} & D^T \\ 0 & C & D & \gamma\mathbb{I} \end{bmatrix} > 0. \qquad (14)$$

iii) There existP, such that

$$\begin{bmatrix} P & PA^T & PC^T & 0 \\ AP & P & 0 & B \\ CP & 0 & \gamma\mathbb{I} & D \\ 0 & B^T & D^T & \gamma\mathbb{I} \end{bmatrix} > 0. \qquad (15)$$

iv) There exist P and G such that

$$\begin{bmatrix} G+G^T-P & GA & GB & 0 \\ A^TG^T & P & 0 & C^T \\ B^TG^T & 0 & \gamma\mathbb{I} & D^T \\ 0 & C & D & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (16)$$

v) There exist P and G such that

$$\begin{bmatrix} G+G^T-P & GA^T & GC^T & 0 \\ AG^T & P & 0 & B \\ CG^T & 0 & \gamma\mathbb{I} & D \\ 0 & B^T & D^T & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (17)$$

Proof: Conditions *i)*, *ii)* and *iii)* define the standard characterizations of the norm $\mathscr{H}_\infty$ as LMI and are derived from the *Real Bounded Lemma*. The equivalence between *ii)*, *iv)* and *v)* can be seen in [13], following the results presented in [16]. □.

The standard and extended characterizations of norms $\mathscr{H}_2$ and $\mathscr{H}_\infty$ as LMIs can be combined with poles location in particular regions, only considering that the dynamic matrix corresponds to $\frac{A-\sigma\mathbb{I}_n}{r}$, which is relevant for the observer design, as this helps improving the transient response of the estimator dynamics.

2.1 Robust stability and performance in discrete LPV systems

Consider LPV discrete-time system given by

$$\begin{aligned} x_{k+1} &= A(\alpha(k))x_k + B(\alpha(k))\omega_k \\ z_k &= C(\alpha(k))x_k + D(\alpha(k))\omega_k \end{aligned} \quad (18)$$

The system (18) can be characterized as a polytope, defining

$$\mathcal{P} := \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix} \in \Omega. \quad (19)$$

where Ω is a polytopic set:

$$\Omega := \left\{ \mathcal{P} : \mathcal{P} = \sum_{i=1}^{l} \alpha_i \mathcal{P}_i; \quad \alpha_i \geq 0; \quad \sum_{i=1}^{l} \alpha_i = 1 \right\}; \quad (20)$$

such that any admissible matrix $\mathcal{P}$ of the system can be written as a convex combination of l vertices matrices given, so that

$$\mathcal{P}_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad (21)$$

where A_i, B_i, C_i, D_i, for $i = 1, \ldots, l$, are the vertices of the polytope, and they are known matrices. Thus, this system can be characterized by the convex hull of Ω considering the vertices of the polytope, ie

$$\mathcal{C}_o\,\Omega = \left\{ \begin{pmatrix} A_1 & B_1 \\ C_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & B_l \\ C_l & 0 \end{pmatrix} \right\} \quad (22)$$

for $\alpha_i \geq 0, \ \ i = 1, \ldots, l, \ \ \sum_{i=1}^{l} \alpha_i = 1$.

The stability of polytopic system [11] is given by

$$P_i = P_i^T > 0, \quad P_i - A_i^T P_i A_i > 0, \quad i = 1, \ldots, l \quad (23)$$

Following the results of Lemma 1, the robust stability condition for polytopic system (18) can be summarized as follows:

Lemma 5 (Robust stability) *Consider the system* (18). *If there exist $P_i = P_i^T > 0$, the matrix $G \in \Re^{n\times n}$, $i = 1, \ldots, l$, then the following conditions for robust stability are equivalents*

i) $P_i = P_i^T > 0, \quad P_i - A_i^T P_i A_i > 0,$

ii) There exist G, such that $G + G^T > 0$ and

$$\begin{bmatrix} G+G^T-P_i & GA_i \\ G^TA_i^T & P_i \end{bmatrix} > 0, \quad (24)$$

Proof: The procedure is similar to the proof of Lemma 1 and the considerations presented in [11]. □

Similarly, the robust performance of the polytopic system (18) can be analyzed from the following:

Lemma 6 ($\mathscr{H}_2$ Robust performance) *Consider the system* (18). *For $P_i = P_i^T > 0$, the following statements are equivalent*

i) System (18) *is robustly stable and $\left\|C_i(z\mathbb{I} - A_i)^{-1}B_i + D_i\right\|_2 < \mu$; for $i = 1, \ldots, l$*

ii) There exist $P_i = P_i^T \in \Re^{n\times n}$, $W = W^T \in \Re^{q\times q}$ y $G \in \Re^{n\times n}$, such that: $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{aligned} \begin{bmatrix} G+G^T-P_i & GA_i & GB_i \\ A_i^TG^T & P_i & 0 \\ B_i^TG^T & 0 & \mathbb{I} \end{bmatrix} &> 0 \\ \begin{bmatrix} W & C_i & D_i \\ C_i^T & P_i & 0 \\ D_i^T & 0 & \mathbb{I} \end{bmatrix} &> 0 \end{aligned} \quad (25)$$

Proof: See [16] and [13]. □

Lemma 7 ($\mathscr{H}_\infty$ Robust performance) *Consider the system* (18). *If $P_i = P_i^T > 0 \in \Re^{n\times n}$, $i = 1, \ldots, l$; and the matrix $G \in \Re^{n\times n}$, the following statements are equivalent:*

i) System (18) *is robustly estable and* $\left\|C_i(z\mathbb{I}-A_i)^{-1}B_i+D_i\right\|_\infty<\gamma$.

ii) There exist $P_i=P_i^T>0$ *y* G *such that*

$$\begin{bmatrix} G+G^T-P_i & GA_i & GB_i & 0 \\ A_i^TG^T & P_i & 0 & C_i^T \\ B_i^TG^T & 0 & \gamma\mathbb{I} & D_i^T \\ 0 & C_i & D_i & \gamma\mathbb{I} \end{bmatrix} > 0. \tag{26}$$

Proof: See [16] and [13]. □

Importantly, the robust performance in $\mathscr{H}_2$-$\mathscr{H}_\infty$ for polytopic system (18) can be combined with robust pole placement, considering that the dynamic matrix corresponds to $\frac{A_i-\sigma\mathbb{R}_n}{r}$.

2.1.1 LPV Systems Controlability and Observability

If for LPV system (18), $\alpha=\alpha_0$ constant, this represents a LTI system and study of controllability and observability can be performed under such LTI systems framework. When $\alpha=\delta(k)$ defines a trajectory, it is obtained a LTV system and the conditions of controllability and observability can be analyzed in the context of these systems.

The controllability and observability for LPV systems study has been presented as an extension of the invariant subspaces of LTV systems [2]. This allows to set the following definitions:

Definition 8 *LPV Controlability. Consider the* (18). *It can be said that the pair* $(A(\alpha),B(\alpha))$ *is controllable if there is a compensator, possibly dynamic, state feedback, whose equations are functions of the variable parameter,*

$$\begin{aligned} \zeta_{k+1} &= f(\zeta,x,\alpha) \\ u_k &= g(\zeta,x,\alpha) \end{aligned} \tag{27}$$

such that the resulting closed loop system is quadratically stable.

Similarly the observability is described.

Definition 9 *LPV Observability. Consider the system* (18). *It can be said that the pair par* $(C(\alpha),A(\alpha))$ *is observable if there exists a system (estimator), possibly dynamic, whose equations are function of the variable parameter,*

$$\begin{aligned} \zeta_{k+1} &= f(\zeta,u,y,\alpha) \\ \hat{x}_k &= g(\zeta,y,\alpha) \end{aligned} \tag{28}$$

such that the error estimation dynamic is quadratically stable in the way that: For all $x(t_0)$, $z(t_0)$, $\alpha(\cdot)$, *the condition* $e_k=x_k-\hat{x}_k \rightarrow 0$ *as* $k \rightarrow \infty$ *is assured.*

It's important to emphasize, as is the interest of this work, that controllability and observability conditions will be robust if the dynamic equations (27) and (28) do not depend on the variable parameter α. In addition, the case of linear dependence of (27) and (28) is particularized with respect to x, z, y, and u. Consequently, the interest of the last definition corresponds to describe a procedure for the design of state observers.

3 Main results

In this section, the design procedures of generalized PIO for LPV discrete-time systems are explained.

3.1 A PIO for LPV discrete time systems

A PIO is characterized by the incorporation of an additional integral term estimate of the output estimation error in order to design the observer, which may offer certain degrees of freedom.

In [21] a generalization for PIO design is presented, considering an explicit parametric solution of Sylvester matrix equations for the observer gain. In the following proposal, the design of the generalized PIO gain is obtained by solving a control problem by SOF, considering the closed loop stability and following characterization of the quadratic stability as a feasibility problem LMIs. Accordingly, consider LPV system

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k+Bu_k; \quad x(0)=x_0 \\ y_k &= C(\alpha)x_k \end{aligned} \tag{29}$$

taking the same considerations for the model (18), except that no uncertainty for for the operation of the actuators, since the control matrix B is assumed known and constant. It is recognized that the pair $(C(\alpha),A(\alpha))$ is observable, for all α.

From the PIO model given in [14], consider the following generalized version, whose block diagram is shown in Figure 2:

$$\begin{aligned} \hat{x}_{k+1} &= F\hat{x}_k+K_I\vartheta_k+K_P\left(y_k-\hat{y}_k\right)+Bu_k, \\ \vartheta_{k+1} &= L\vartheta_k+H\left(y_k-\hat{y}_k\right) \\ \hat{y}_k &= J\hat{x}_k \end{aligned} \tag{30}$$

where the matrices F, L, H, J, K_P (proportional gain) and K_I (integral gain) are of appropriate dimensions, which are defined as the observer matrices to be

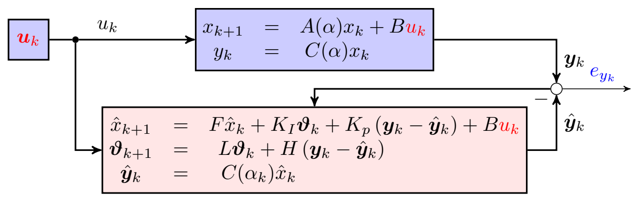

Figure 2: Diagram of blocks for a PIO.

determined, provided that

$$\lim_{k\to\infty}\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} = 0 \tag{31}$$

being $e_k = x_k - \hat{x}_k$ the estimation error. The variable ϑ is related to the "weighted" integral of the output estimation error. The matrix L is a fading effect coefficient for regulating the transient response of the observer. The matrix H is a coefficient of additional integral effect, which improves the stability margin. If $L = 0$, a classical PIO is obtained. If $L \neq 0$, it can be interpreted as a generalized PIO, since the dynamics of the observer is enriched, reaffirming their behavior as PIO, which is important in applications of fault diagnosis based on observers. Also, if steady state $\vartheta_k = 0$, it implies that $y_k - J\hat{x}_k = 0$.

Definition 10 *Dynamic system*(30) *is said to be a generalized PIO of full order for the system* (29)*, if only if, the matrices F, L, H, J, K_P and K_I are such that the expression* (31) *is satisfied.*

Accordingly, the following theorem can be stated:

Theorem 11 *Dynamic system* (30) *is said to be a generalized PIO of full order for the system* (29)*, if only if, the matrices F, L, H, J, K_P and K_I are such that*

1. *$F = A(\alpha)$ and $J = C(\alpha)$.*
2. *The matrix $\mathfrak{A}(\alpha)$:*

$$\mathfrak{A}(\alpha) = \begin{pmatrix} F - K_P C(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix} \tag{32}$$

is stable in the sense of Lyapunov.

Proof: From the dynamic error

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} = (A(\alpha) - K_P C(\alpha))\, x_k - \\ &\quad (F - K_P J)\,\hat{x}_k - K_I\vartheta_k \\ \vartheta_{k+1} &= L\vartheta_k + H\,(C(\alpha)x_k - J\hat{x}_k) \end{aligned}$$

then, if $F = A(\alpha)$ and $J = C(\alpha)$,

$$\begin{aligned} \begin{pmatrix} e_{k+1} \\ \vartheta_{k+1} \end{pmatrix} &= \begin{pmatrix} F - K_P C(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix}\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} \\ &= \mathfrak{A}(\alpha)\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} \end{aligned} \tag{33}$$

The dynamic system (33) must be quadratically stable, so that the condition condition (31) is satisfied, which implies that the matrix $\mathfrak{A}(\alpha)$ be quadratically stable. □

The matrix $\mathfrak{A}(\alpha)$ can be expressed as

$$\begin{aligned} \mathfrak{A}(\alpha) &= \begin{pmatrix} A(\alpha) & 0 \\ 0 & 0 \end{pmatrix} + \\ &\quad \begin{pmatrix} -K_P & -K_I \\ H & L \end{pmatrix}\begin{pmatrix} C(\alpha) & 0 \\ 0 & \mathbb{I}_q \end{pmatrix} \end{aligned} \tag{34}$$

Thus, consider the matrices

$$\begin{aligned} \mathcal{A}_o(\alpha) &= \begin{pmatrix} A(\alpha) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_o = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_q \end{pmatrix}, \\ \mathcal{C}_o(\alpha) &= \begin{pmatrix} C(\alpha) & 0 \\ 0 & \mathbb{I}_q \end{pmatrix}; \end{aligned} \tag{35}$$

and the matrix

$$\mathbb{K} = \begin{pmatrix} -K_P & -K_I \\ H & L \end{pmatrix}. \tag{36}$$

The matrices given in (35), defining the dynamic system

$$\begin{aligned} z_{k+1} &= \mathcal{A}_o(\alpha)z_k + \mathcal{B}_o v_k \\ \eta_k &= \mathcal{C}_o(\alpha)z_k \end{aligned} \tag{37}$$

donde $z_k = \begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix}$. The system (37) is characterized by the polytope

$$\mathcal{P}_i = \begin{pmatrix} \mathcal{A}_{o_i} & \mathcal{B}_o \\ \mathcal{C}_{o_i} & 0 \end{pmatrix} \qquad i = 1, \ldots, l \tag{38}$$

It is easily verifiable that the pair $(\mathcal{A}_o(\alpha), \mathcal{B}_o)$ is controllable. In addition, the system satisfies the conditions for the design of a gain to the problem of SOF [17], given that $\mathcal{B}_o = \mathbb{I}_{n+q}$, the necessary condition is that the pair $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ be observable, which corresponds that the pair $(C(\alpha), A(\alpha))$ be observable. Therefore, the matrix $\mathbb{K}$ corresponding to the feedback gain to the control problem by SOF for the system (37).

Lemma 12 *Consider the system* (37)*, with $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ observable (equivalently, the pair $(C(\alpha), A(\alpha))$ observable). Then, the system admits a SOF control, given by $v(t) = \mathbb{K}\eta(t)$, such that closed loop dynamics is asymptotically stable.*

Proof: In effect, if $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ is observable and the structure of the matrix $\mathcal{B}_o$, the control $v(t) = \mathbb{K}\eta(t)$ stabilizes the closed loop dynamic matrix, given by

$$\mathcal{A}_o(\alpha) + \mathcal{B}_o\mathbb{K}\mathcal{C}_o(\alpha) = \mathfrak{A}(\alpha)$$

□

Consequently, the stabilization problem of the dynamic matrix $\mathfrak{A}(\alpha)$ for the generalized PIO corresponds to design the gain $\mathbb{K}$ into the problem of stabilization by SOF of the system (37). This has the advantage that the design is obtained by direct solution of a problem of stabilizing LPV control systems, which has been extensively studied for those uncertain systems.

3.2 Designing a generalized PIO by SOF

The main result of this work is described by the following theorem:

Theorem 13 *Let be system* (37)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. There exist a gain* $\mathbb{K}$*, for the control by SOF, provided that the matrix* $\mathfrak{A}(\alpha)$ *be stable, if only if, there exist* $P_i = P_i^T > 0$*, for* $i = 1, \ldots, l$*; the matrix* G *and the matrix* Y *such that the following LMI is satisfied*

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0, \qquad (39)$$

where the gain $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \qquad (40)$$

Proof: According to the robust stability established in Lemma 5, $\mathfrak{A}(\alpha)$ will be stable if exist $P_i = P_i^T > 0$ and G are such that

$$\begin{bmatrix} G + G^T - P_i & G\mathfrak{A}(\alpha) \\ \star & P_i \end{bmatrix} > 0$$

By substitution, the matrix inequalities appear, which are linearized by changes of variables $G\mathcal{B}_o = \mathcal{B}_o M$ and $Y = M\mathbb{K}$.

Therefore

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0$$

resulting the LMI defined by (39) and the expression (40) that allows to obtain $\mathbb{K}$. □

Calculating the gain $\mathbb{K}$, the matrices G, H, K_P y K_I of the generalized PIO are obtained. If the location of poles is relevant, then the following result can be applied:

Lemma 14 *Consider the system* (37)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. There exist a gain* $\mathbb{K}$ *for control by SOF, provided that the matrix* $\mathfrak{A}(\alpha)$ *has its poles in the circular region of radius* r *and center* $(\sigma, 0)$*, if only if, there exist* $P_i = P_i^T > 0$ *for* $i = 1, \ldots, l$*, the matrix* G*, and the matrix* Y *such that the following LMI is satisfied*

$$\begin{bmatrix} r(G + G^T - P_i) & G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & rP_i \end{bmatrix} > 0, \qquad (41)$$

where the gain $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \qquad (42)$$

Proof: Proof follows from Theorem 13. □

Given that $F = A(\alpha)$ and $J = C(\alpha)$, for purposes of practical implementation of the generalized PIO, as most observers to LPV systems, consider the matrix $F = A_0$ and $J = C_0$, where A_0 and C_0 are the central matrices of the respective polytopes, which allows to preserver the robustness condition in the design of the generalized PIO.

3.3 Designing a generalized PIO with $\mathscr{H}_2 - \mathscr{H}_\infty$ performance

Consider LPV perturbed discrete time system

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k + B_1(\alpha)\omega_k + Bu_k \\ y_k &= C(\alpha)x_k + D(\alpha)\omega_k, \end{aligned} \qquad (43)$$

where $\omega_k \in \Re^r$ are unknown perturbations. For all parameter α, it is assumed that $(C(\alpha), A(\alpha))$ is observable. In principle, the design of the observer must correspond to minimizing effects of disturbances in the state estimation.

Again, it is considered that the uncertain matrices $A(\alpha), B_1(\alpha), C(\alpha), D(\alpha)$ belong to a convex polytopic set, $\forall\ \alpha_i \geq 0$, $\sum_{i=1}^{l} \alpha_i = 1$, defined by

$$\Omega = \left\{ \sum_{i=1}^{l} \alpha_i \Big(A^{(i)}, B_1^{(i)}, C^{(i)}, D^{(i)} \Big) \right\}. \qquad (44)$$

For the generalized PIO given by (30), if $F = A(\alpha)$ y $J = C(\alpha)$, then

$$\begin{pmatrix} e_{k+1} \\ \vartheta_{k+1} \end{pmatrix} = \begin{pmatrix} F - K_P C(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix} \begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} + \begin{pmatrix} B_1(\alpha) - K_P D(\alpha) \\ HD(\alpha) \end{pmatrix} \omega_k \qquad (45)$$

Considering the matrices defined by (35), the gain matrix given by (36), and the following matrices

$$B_{1_o}(\alpha) = \begin{pmatrix} B_1(\alpha) \\ 0 \end{pmatrix}, \quad D_o(\alpha) = \begin{pmatrix} D(\alpha) \\ 0 \end{pmatrix}, \tag{46}$$

the following dynamic system is derived

$$\begin{aligned} z_{k+1} &= \mathcal{A}_o(\alpha) z_k + \mathcal{B}_{1_o}(\alpha)\omega_k + \mathcal{B}_o v_k, \\ \eta_k &= \mathcal{C}_o(\alpha) z_k + D_o(\alpha)\omega_k, \end{aligned} \tag{47}$$

which admits a SOF control $v_k = \mathbb{K}\eta_k$, such that the closed loop dynamics (45), with the output η_k, satisfying a $\mathscr{H}_2 - \mathscr{H}_\infty$ performance index. The gain $\mathbb{K}$, that defines the generalized PIO, is obtained by solving of a robust optimal control in $\mathscr{H}_2 - \mathscr{H}_\infty$. Consequently, the closed-loop dynamic is

$$\begin{aligned} z_{k+1} &= \left(\mathcal{A}_o(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{C}_o(\alpha)\right) z_k + \\ &\quad \left(\mathcal{B}_{1_o}(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{D}_o(\alpha)\right) \omega_k \\ \eta_k &= \mathcal{C}_o(\alpha) z_k + D_o(\alpha)\omega_k \end{aligned} \tag{48}$$

Such as has been stated, the observer design should consider the minimizing of the effects of the disturbance ω_k in the state estimation. This performance criterion can be imposed by minimizing of the $\mathscr{H}_2$ norm or $\mathscr{H}_i nfty$ norm for the transfer function of the system (48).

3.3.1 Design in $\mathscr{H}_2$

Let be $T_{\omega\eta}(\mathbf{z})$ the transfer function of the disturbance ω to the regulated output η for the system (48). The design problem of a generalized PIO with robust performance in $\mathscr{H}_2$ corresponds to the following:

Theorem 15 *Consider the system* (43) *on the polytope* (44)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (30)*, that is determined by the gain* $\mathbb{K}$ *solving the SOF control problem for the system* (47)*, guaranteeing a suboptimal performance in* $\mathscr{H}_2$ *for* (48)*, ie,* $\left\|T_{\omega\eta}(\mathbf{z})\right\|_2 < \mu$*, from the following optimization problem:*

$$\min_{\substack{P_i, Y, W, G \\ i=1,\ldots,l.}} \operatorname{tr}(W), \qquad \textit{such that}$$

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} & G\mathcal{B}_{1_{o_i}} + \mathcal{B}_o Y \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0,$$

$$\begin{bmatrix} W & \mathcal{C}_{o_i} & \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0, \tag{49}$$

where, $P_i = P_i^T > 0$, $i = 1, \ldots, l$ *and the matrices* G *(with* $G + G^T > 0$*),* W*,* Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{where} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{50}$$

Proof: Assuming that there is a feasible solution to the formulated optimization problem, in accord to Lemma 6, the inequality (25) is linearized by changing variable $Y = \mathcal{B}_o M$, where $G\mathcal{B}_o = \mathcal{B}_o M$; previously the respective substitutions are made, in order to obtain the LMIs (49). □

This formulation reduces the conservatism, which occurs when a fixed Lyapunov matrix $P^T = P$ is used. Thus, it is possible to obtain a Lyapunov function for each vertex of the polytope without forcing a single Lyapunov matrix for the LPV estimation system.

The immediate result that is derived, corresponding to the location of poles in a particular region, together with the minimization of the $\mathscr{H}_2$ norm.

Lemma 16 *Consider the system* (43) *on the polytope* (44)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (30)*, provided that the closed loop dynamic matrix* $\mathfrak{A}(\alpha)$ *has its poles in the circular region of radius* r *and center* $(\sigma, 0)$*, it is determined by the gain* $\mathbb{K}$ *solving the SOF control problem for the system* (47)*, guaranteeing a suboptimal performance in* $\mathscr{H}_2$ *for* (48)*, ie,* $\left\|T_{\omega\eta}(\mathbf{z})\right\|_2 < \mu$*, from the following optimization problem:*

$$\min_{\substack{P_i, Y, W, G \\ i=1,\ldots,l.}} \operatorname{tr}(W), \qquad \textit{such that}$$

$$\begin{bmatrix} r(G + G^T - P_i) & \Upsilon_o & r\left(G\mathcal{B}_{1_{oi}} + \mathcal{B}_o Y \mathcal{D}_{o_i}\right) \\ \star & rP_i & 0 \\ \star & \star & r\mathbb{I} \end{bmatrix} > 0,$$

$$\begin{bmatrix} W & \mathcal{C}_{o_i} & \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0, \tag{51}$$

where $\Upsilon_o = G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_o Y \mathcal{C}_{o_i}$, $P_i = P_i^T > 0$, $i = 1, \ldots, l$ *and the matrices* G*,* W*,* Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{where} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{52}$$

Proof: Proof follows from Theorem 15. □

3.3.2 Design in $\mathscr{H}_\infty$

Theorem 17 *Consider system* (43) *on the polytope* (44)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (30) *is determined by the gain* $\mathbb{K}$*, solving the SOF control problem for the system* (47)*, guaranteeing a suboptimal performance in* $\mathscr{H}_\infty$ *for* (48)*, ie,* $\left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty < \gamma$*, from the following optimization problem:*

$$\min_{\substack{P_i, G, Y, \\ i=1,\ldots,l.}} \left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty, \qquad \textit{such that}$$

$$\begin{bmatrix} G+G^T-P_i & G\mathcal{A}_{o_i}+\mathcal{B}_oY\mathcal{C}_{o_i} & G\mathcal{B}_{1_{o_i}}+\mathcal{B}_oY\mathcal{D}_{o_i} & 0 \\ \star & P_i & 0 & \mathcal{C}_{o_i}^T \\ \star & \star & \gamma\mathbb{I} & \mathcal{D}_{o_i}^T \\ \star & \star & \star & \gamma\mathbb{I} \end{bmatrix} > 0 \quad (53)$$

where $P_i = P_i^T > 0$, $i = 1, \ldots, l$ and the matrices G, Y have appropriate dimensions. Thus, the gain $\mathbb{K}$ is obtained from

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1}G\mathcal{B}_o \quad (54)$$

Proof: Similarly, assuming that there is a feasible solution to the formulated optimization problem, by the change of variable $Y = \mathcal{B}_oM$, with $Q^T\mathcal{B}_o = \mathcal{B}_oM$, to Item *iii)* of the Lemma 7, the respective substitutions are made, in order to obtain the LMI given by (53). □

For $\mathscr{H}_\infty$ robust performance and location of poles in a particular circular region, then:

Lemma 18 *Consider system* (43) *on the polytope* (44)*, with $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ observable. A generalized PIO given by* (30)*, provided that the closed loop dynamic matrix $\mathfrak{A}(\alpha)$ has its poles in the circular region of radius r and center $(\sigma, 0)$, it is determined by the gain $\mathbb{K}$ solving the SOF control problem for the system* (47)*, guaranteeing a suboptimal performance in $\mathscr{H}_\infty$ for* (48)*, ie, $\left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty < \gamma$, from the following optimization problem:*

$$\min_{\substack{P_i, Y, G \\ i=1,\ldots,l.}} \left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty, \qquad \textit{such that}$$

$$\begin{bmatrix} r(G+G^T-P_i) & \Gamma_o & r\left(G\mathcal{B}_{1_{o_i}}+\mathcal{B}_oY\mathcal{D}_{o_i}\right) & 0 \\ \star & rP_i & 0 & r\mathcal{C}_{o_i}^T \\ \star & \star & r\gamma\mathbb{I} & r\mathcal{D}_{o_i}^T \\ \star & \star & \star & r\gamma\mathbb{I} \end{bmatrix} > 0 \quad (55)$$

where $\Gamma_o = G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_oY\mathcal{C}_{o_i}$, $P_i = P_i^T > 0$, $i = 1, \ldots, l$ and the matrices G, Y have appropriate dimensions. Thus, the gain $\mathbb{K}$ is obtained from

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1}G\mathcal{B}_o \quad (56)$$

Proof: The procedure for the proof of Theorem 17 is followed. □

The results in $\mathscr{H}_2$ and $\mathscr{H}\infty$ can be combined to establish mixed performance indices. Importantly, once the gain obtained $\mathbb{K}$ proceed to determine the matrices of the generalized PIO. Similarly, some conditions transient performance in the dynamic of the observer can be imposed, which represents a method of tuning of the PIO design parameters. Thus, a LMIs characterizing the pole location in a particular circular region can be specified, which ensures faster for estimating, important when controllers are designed from the estimated states.

3.4 Generalized PIO design by perturbation reconstruction

Section 3.3 has been dedicated to the design of generalized PIO, imposing performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$ in order to minimize the effects of disturbance signals in the estimation error e_k and in the signal ϑ_k that, as mentioned, is related to the "weighted" integral of the estimation error of the output. At this point, the idea is to consider that ϑ_k represents an approximation of the disturbance ω_k, that is, the signal $\vartheta_k = \hat{\omega}_K$ is defined, then the design of the PIO must ensure that

$$\lim_{k\to\infty} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} = 0 \quad (57)$$

where $\xi_k = \omega_k - \hat{\omega}_k$. This is the more relevant design specification for the synthesis of UIO. Under certain conditions, this design specification can be satisfied by a PIO. In effect, consider the discrete polytopic system (43), with $B_1(\alpha) = B_1$ a constant matrix and $D(\alpha) = 0$, that means the unknown input is in system dynamic. The unknown input may represent actuator or process failures, which which are to rebuild. There, $(C(\alpha), A(\alpha))$ is assumed observable.

A generalized PIO (30) guarantees the condition (57) if some criteria for the choice of design matrices are satisfied, as it is shown below. Under the new features of the dynamics of (43) and if $\vartheta_k = \hat{\omega}_k$, then:

$$\begin{pmatrix} e_{k+1} \\ \hat{\omega}_{k+1} \end{pmatrix} = \begin{pmatrix} A(\alpha) - K_PC(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix}\begin{pmatrix} e_k \\ \hat{\omega}_k \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix}\omega_k \quad (58)$$

In order to satisface (57), ξ_k dynamically should be evaluated, thus

$$\begin{aligned} \xi_{k+1} &= \omega_{k+1} - \hat{\omega}_{k+1} = \omega_{k+1} - \hat{\omega}_{k+1} + \xi_k - \xi_k \\ &= \xi_k - LHC(\alpha)e_k + L\hat{\omega} - \hat{\omega} + \omega_{k+1} - \omega_k \end{aligned} \quad (59)$$

Consider $\varepsilon = \omega_{k+1} - \omega_k$. The first constraint imposed is that ω should be a smooth signal in the sense that $\varepsilon \approx 0$. Combining the dynamic equations of closed loop for errors e and ξ, then

$$\begin{aligned} e_{k+1} &= (A(\alpha) - K_PC(\alpha))e_k - K_I\hat{\omega}_k + B_1\omega_k \\ \xi_{k+1} &= -LHC(\alpha)e_k + \xi_k + L\hat{\omega}_k - \hat{\omega}_k + \varepsilon \end{aligned} \quad (60)$$

From (60), if $K_I = B_1$ and $L = \mathbb{I}$, then the closed loop dynamics will be

$$\begin{pmatrix} e_{k+1} \\ \xi_{k+1} \end{pmatrix} \begin{pmatrix} A(\alpha) - K_P C(\alpha) & B_1 \\ -HC(\alpha) & \mathbb{I} \end{pmatrix} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}, \tag{61}$$

which must be asymptotically stable in order to satisfy the condition (57), through the appropriate choice of design matrices K_P and H.

Following the design procedure, the synthesis of K_P and H of the PIO must be transformed, once established the conditions for the matrices K_I and L, in a control problem by SOF. Thus, in this case, the following matrices are considered

$$\begin{aligned} \mathcal{A}_o(\alpha) &= \begin{pmatrix} A(\alpha) & B_1 \\ 0 & \mathbb{I}_r \end{pmatrix}, \quad \mathcal{B}_o = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_r \end{pmatrix}, \\ \mathcal{C}_o(\alpha) &= \begin{pmatrix} C(\alpha) & 0 \end{pmatrix}; \end{aligned} \tag{62}$$

and the matrix

$$\mathbb{K} = \begin{pmatrix} -K_P \\ -H \end{pmatrix}. \tag{63}$$

The matrices in (62) defining a dynamic system given by (37), where $z_k = \begin{pmatrix} e_k \\ \xi_k \end{pmatrix}$. In addition, matrices defined in (62) characterize a polytope given by (38). If $\mathfrak{A}(\alpha)$ is the closed loop dynamic matrix for (61), the Lemma 12 can be applied, as

$$\mathfrak{A}(\alpha) = \mathcal{A}_o(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{C}_o(\alpha).$$

Theorem 19 *Consider system* (60)*, defining the matrices* $\mathcal{A}_o(\alpha)$*,* $\mathcal{B}_o$*,* $\mathcal{C}_o(\alpha)$*. If*

1. $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *is observable, (necessary condition).*
2. $\varepsilon \approx 0$.
3. $K_I = B_1$ *and* $L = \mathbb{I}_r$*, for the PIO given by* (30).

There exist a gain $\mathbb{K}$ *for SOF control, provided that the matrix* $\mathfrak{A}(\alpha)$ *be stable:* $\lim_{k\to\infty} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} = 0$*, if only if, there exist* $P_i = P_i^T > 0$*, with* $i = 1, \ldots, l$*; the matrix* G *and the matrix* Y *such that following LMI is satisfied*

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0, \tag{64}$$

where $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1} Y, \quad with \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{65}$$

Proof: See the proof of Theorem 13. □

Remark 20 *The necessary condition* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *be observable not only depends on the matrices of the original system, but also to impose* L *and* K_I*, such as these design matrices are defined in* (62)*, establishing difference from the procedure presented in Section 3.1 and Section 3.3. Thus, this design technique proves to be very restrictive.*

From $\mathbb{K}$ the design matrices K_P and H are obtained. For LPV discrete time systems, this procedure is a generalization of the results presented in [5]. Moreover, the Lemma 14 can be applied to ensure that the closed loop poles are located in a particular stable region. In order to show the effectiveness of the technique, some numerical examples and simulations have been made, see Section 4.

4 Numerical Examples

Lets consider the uncertain systems

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & \alpha_1 \\ 0 & 0 & 0.6 + \alpha_2 \end{bmatrix} x_k + \\ & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \omega_k \\ y_k &= \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} x_k \end{aligned}$$

with $-0.8 \leq \alpha_1 \leq 0.8$ y $0 \leq \alpha_2 \leq 0.5$. Consider the design of a PIO following the two presented procedures: robust estimation and reconstruction of unknown input. First, since the system has unstable poles, it is necessary to design a control action that stabilizes the system. Here it can be applied different techniques. For purposes of this example, where the fundamental part is the PIO design, it's considered a state feedback [17]. Thus, the state feedback gain, $u_k = K_c x_k$,

$$K_c = \begin{bmatrix} -0.0736 & -0.2918 & -0.6139 \end{bmatrix}$$

can robustly stabilize the system.

4.1 PIO with $\mathscr{H}_\infty$ performance

The PIO design is based on the lemma 18. The solution factibility has been found for $r = 0.98$ and $\sigma = 0.2$, with $\gamma = 6.3198$. So,

$$G = \begin{bmatrix} 2.8541 & 0.3222 & -0.6461 & 0.1185 \\ 0.3905 & 4.7490 & 0.7329 & 0.1191 \\ 0.1119 & 0.5017 & 7.9695 & 1.0453 \\ -0.0104 & -0.0096 & -0.1292 & 7.5515 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.3508 & 0.7923 \\ 0.9134 & 1.8122 \\ -0.6482 & -0.9088 \\ 0.0120 & 1.5355 \end{bmatrix}$$

Consequently,

$$\mathbb{K} = \left[\begin{array}{c|c} 0.0787 & -0.0302 \\ 0.2005 & 0.0141 \\ -0.0951 & -0.0228 \\ \hline 0.0003 & 0.2006 \end{array}\right],$$

where the PIO design matrices can be found.

Figure 3 shows the magnitudes of the poles distribution for the closed loop PIO, depending on α_1 and α_2. As can be seen, all poles are stable.

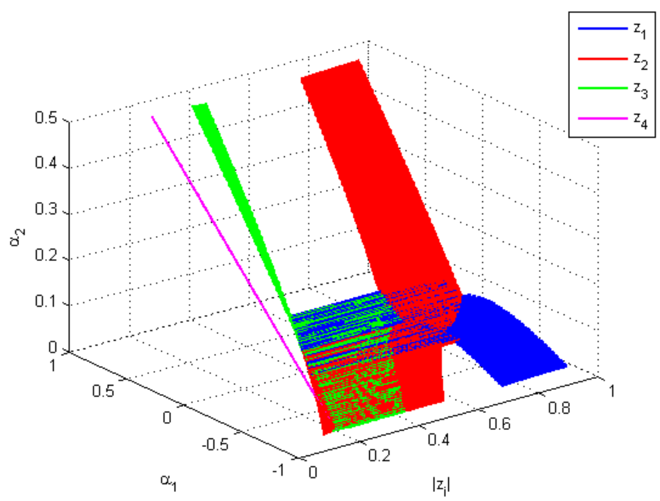

Figure 3: Poles distribution for PIO with $\mathscr{H}\infty$ performance.

Considering variations in the parameters α_1 and α_2, as shown in Figure 4, the corresponding simulations for estimation were performed.

In Figure 5 the temporal behavior of real states x_k are shown, with respect to its estimated $\hat{x}_k$. There, ζ_i, $i = 1, 2, 3$, represent the estimated states. It can be observed a very acceptable tracking estimates. The error is strongly influenced by the disturbance signal ω_k. Indeed, Figure 6 shows the time evolution of error e_k. Here it can be seen some convergence to zero that is altered by ω_k and not by parametric variation. Its important to see that this design procedure seeks to minimize the effects of disturbances (rejection) in the estimation of the states.

Finally, Figure 7 shows the time evolution of another important signal that is ϑ_k, in which the variation around the zero is marked, due to the influence of the perturbation ω_k; this signal is shown in Figure 8.

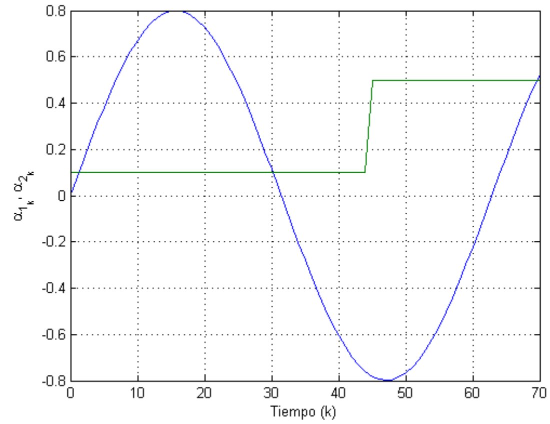

Figure 4: Variations for ρ.

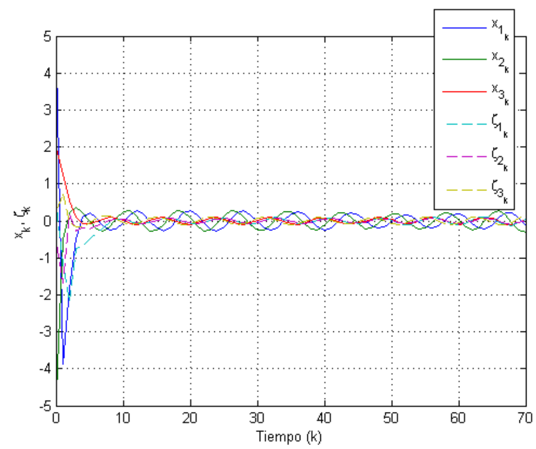

Figure 5: States evolution x_k and $\hat{x}_k$.

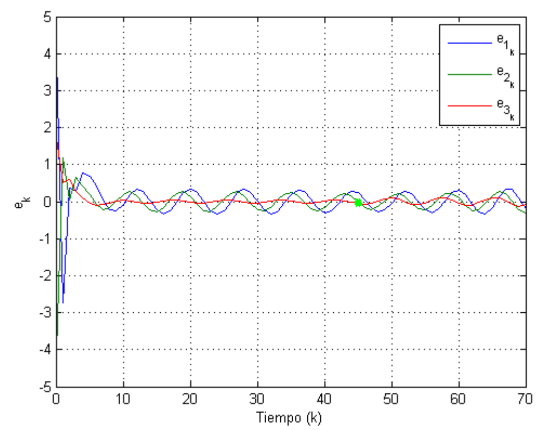

Figure 6: Time behavior of e_k.

4.2 PIO and perturbation reconstruction

Consider the same polytopic system of the previous example. In order to apply Theorem 19 is essential to

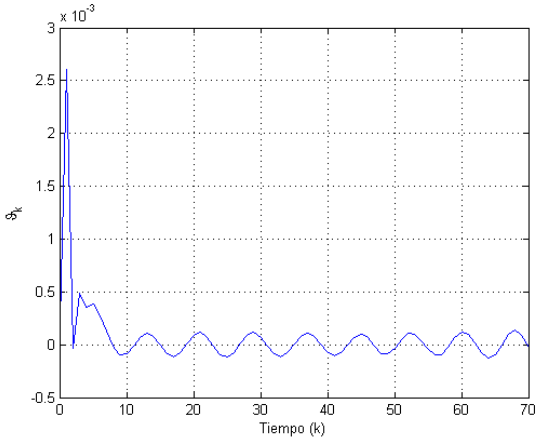

Figure 7: Time behavior of ϑ_k.

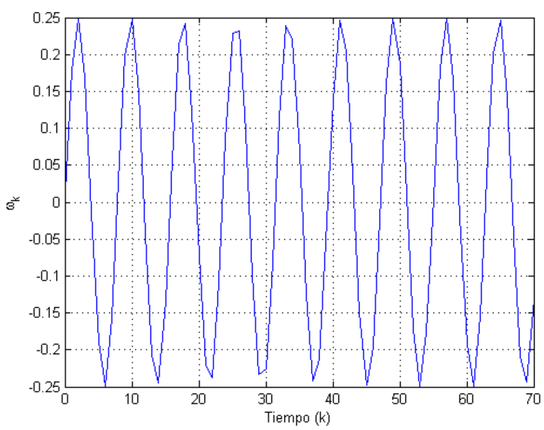

Figure 8: Peturbation signal ω_k.

evaluate the necessary condition, corresponding to the observability of the pair $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$. By inspection, for any value of α_1, if $\alpha_2 = 0.4$ this condition is not satisfied, therefore it can not be calculated the gain $\mathbb{K}$. Hence, by imposing restrictions on the selection of PIO design parameters, as is the case, resulting in a very limited procedure.

For the purpose of verifying the performance as a reconstructor for unknown input, consider the example system but with $B_1 = [1 \;\; 1 \;\; 1]^T = K_I$. Thus, the necessary condition is satisfied. Applying Theorem 19 is obtained

$$\mathbb{K} = \left[\begin{array}{c} -0.4684 \\ 0.0655 \\ -0.3616 \\ \hline -0.1704 \end{array} \right],$$

For the dynamic (62), Figure 9 shows closed loop poles magnitude distribution.

Next it will be shown the comparation between the states and their estimations (Figure 10), and between the perturbation ω_k and its estimation (Figure 11).

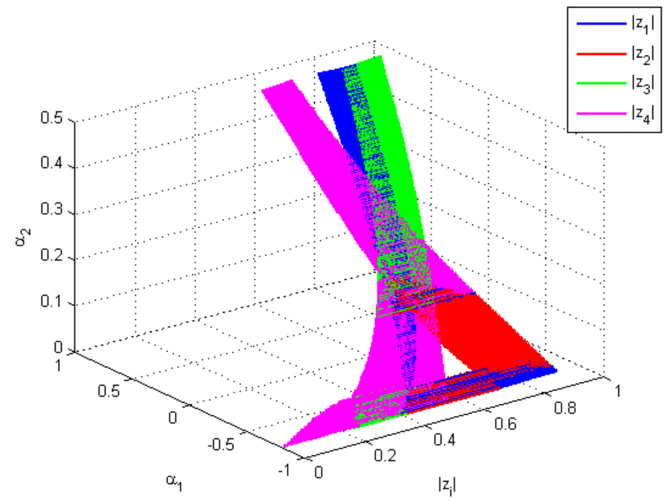

Figure 9: Poles distribution for PIO and perturbation reconstruction.

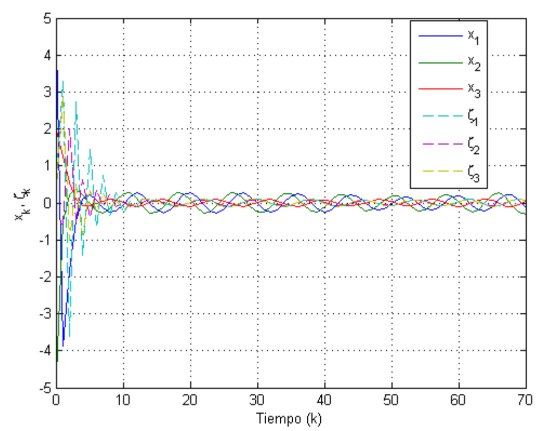

Figure 10: The states x_k and their estimated $\hat{x}_k$.

From Figure 10, it can be seen that the estimated states are similar to the previously evaluated case. From Figure 11, obviating the transient, it can be seen an approximation between the signals with some delay, typical in the dynamically reconstruction process; estimation that, despite design constraints, can be seen with some degree of acceptance. Thus, this approach may be appropriate, for example, in additive fault diagnostic applications for technical processes.

5 Conclusions

A technique for designing of generalized Proportional+Integral observers for linear discrete-time sys-

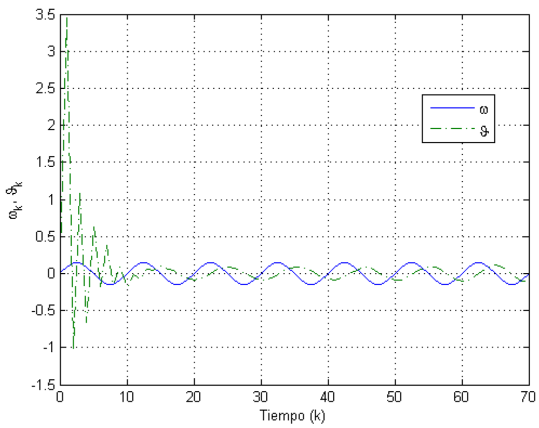

Figure 11: Perturbation ω_k and its estimation ϑ_k.

tems with polytopic uncertainties has been presented. The generalization is to incorporate, in the dynamics of PI observer, a design matrix representing a fading effect coefficient, which allows adjusting the transient response of the observer. Thereafter, the method involves extending the dynamics of PIO, this allows the synthesis of design matrices of the observer, through its transformation into a control problem by static output feedback for LPV systems. Then, considering the extended LMIs characterizations of $\mathscr{H}_2/\mathscr{H}_\infty$ norms and the location of poles in a particular region, the design technique is generalized to LPV systems with disturbances, imposing performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$, specifications for the synthesis of the gain in the control problem by SOF, which results in the matrices of the IOP. The synthesis can be extended to consider multi-objective object specifications. Similarly, exploiting the ability of unknown input reconstruction PI observer, it has design ensuring PIO approximate reconstruction of the disturbance, under appropriate technical considerations. This last method is very restrictive, limiting the selection of parameters PIO, which can result in marginal stability or infeasibility of the solution. Through numerical examples, the effectiveness of the technique has been shown.

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