Some properties and application of $B_k(n,p)$

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Abstract: By the transition probability flow graphs methods, we considered the distribution $G_k(p)$, and derived its probability generating function. Based on $G_k(p)$, we came to the probability distribution of $NB_k(r,p)$ and showed the correlation between them. Following $NB_k(r,p)$, we got the distribution of $B_k(n,p)$, and then we discussed its some properties including the mean and convergence. Finally, we employed the properties to study a reliability question.

Key–Words: distribution of order k; transition probability flow graphs; mean; probability generating function; reliability

1 Introduction

The present paper discusses the probability and statistics properties of the binomial distribution of order k, an important distribution in Bernoulli trials that denoted by $B_k(n,p)$. To the best of our knowledge, the probability generating function of $B_k(n,p)$ has not yet appeared in literature since it is too complicated [11]. Therefore, its probability distribution and mean are difficult to be obtained. By the transition probability flow graphs methods applied widely in the sampling inspection field, we discuss the geometric and the negative binomial distributions of order k [1, 4, 10], and obtain their probability distributions. Following the distributions, we derive the binomial distribution of order k [6, 10, 13], obtain its mean and other properties.

The prototype of transition probability flow graphs methods is the signal flow graphs theory applied to systems engineering widely, which was originated by Mason [9]. Koyama [8] firstly introduced it into the study of sampling inspection. It gradually became a perfect theory by Fan's work [2,3]. Now we provide a brief description for the methods.

Based on decomposing the Markov chain formed by the variation of a nonnegative integer-valued random variable, ascertaining the states and routes, and setting probability functions to the routes, we can arrive at a flow graph of the process being similar to the transition probability graph of the chain. By the series-parallel operation rules, we can get the probability generating function of the random variable from the flow graph.

Let τ be a nonnegative integer-valued random variable with probability space $(\Omega, \mathcal{F}, P)$, set $B_n = \{\tau = n\}$, then for any a fixed $B \in \mathcal{F}$, the transition probability function of τ is defined by $G_\tau(x;B) = \sum_{n=0}^{\infty} P(BB_n)x^n, |x| \leq 1$. Particularly, $B = \Omega$ yields the probability generating function of τ as $G_\tau(x) = \sum_{n=0}^{\infty} P(\tau = n)x^n, |x| \leq 1$.

Consider a Markov chain that takes on countable number of possible values. The transition process from state s_1 into s_2 denoted by $r : s_1 \Rightarrow s_2$ is called a route, and its transition time named step is a random variable. By the Markovian property that the steps of $s_1 \Rightarrow s_2$ and $s_2 \Rightarrow s_3$ are independent. The transition probability function of the route r is defined by $G_r(x) = \sum_{n=0}^{\infty} P_r(n)x^n, |x| \leq 1$, where $P_r(n)$ is the $n-$step transition probability of r. The route from s_1 into s_3 by way of s_2 denoted by $r_1 \cdot r_2$ is called a series route if the routes $r_1 : s_1 \Rightarrow s_2$ and $r_2 : s_2 \Rightarrow s_3$ are independent. The route denoted by $r_1 + r_2$ is called a parallel route of $r_1 : s_1 \Rightarrow s_2$ and $r_2 : s_1 \Rightarrow s_2$ if they are mutually exclusive. The conclusion of Lemma 1 is obvious.

Lemma 1 *[2, 12] Let $G_{r_1}(x)$ and $G_{r_2}(x)$ be respectively the transition probability functions of the routes r_1 and r_2, then $G_{r_1 \cdot r_2}(x) = G_{r_1}(x) \cdot G_{r_2}(x)$ and $G_{r_1+r_2}(x) = G_{r_1}(x) + G_{r_2}(x)$.*

The route $l : s_1 \Rightarrow s_2$ is called a straight route if no state is repeated in it. The route $o : s_1 \Rightarrow s_1$ is called a loop route on state s_1 if s_1 can be repeated infinitely, and all the repeated routes are independent identically distributed. For the straight route and loop route, we have

Lemma 2 *[3, 14] Let $G_l(x)$ be the transition probability functions of the straight routes $l : s_1 \Rightarrow s_2$, $G_{o_1}(x)$ and $G_{o_2}(x)$ be respectively the transition probability functions of one repetition of the loop routes on state s_1, then*

$$G_{l\cdot \boldsymbol{o}_1}(x) = G_l(x)/[1 - G_{o_1}(x)],$$

$$G_{l\cdot(\boldsymbol{o}_1+\boldsymbol{o}_2)}(x) = G_l(x)/[1 - G_{o_1}(x) - G_{o_2}(x)].$$

2 The probability distribution of the binomial distribution of order k

In the present section, following the transition probability flow graphs methods, we derive the probability generating functions of $G_k(p)$ and $NB_k(r, p)$, further, we get the probability distribution of the binomial distribution of order k. Let $X_{(k)}$ denote the number of trials until the occurrence of success run with length k in Bernoulli trials with success probability p. We denote its probability distribution by $G_k(p)$, and call it the geometric distribution of order k with parameter vector p.

Theorem 3 *The probability generating function of $X_{(k)}$ distributed as $G_k(p)$ is given by*

$$G_{X_{(k)}}(x) = \frac{p^k x^k - p^{k+1}x^{k+1}}{1 - x + qp^k x^{k+1}}, \tag{1}$$

where $q = 1 - p$.

Proof. Let τ_n be the state that the trial process will be in at the end of the first n trials, where the number of trials n is also named transition time. Then $\{\tau_n, n = 1, 2, \cdots\}$ is a Markov process with state space $S = \{0, 1, \cdots, k\}$, where 0 is the beginning state and k is the ending state. The event $\{\tau_n = s,\ s \in S\}$ denotes the occurrence of s consecutive success at the end of the foregoing n trials. Hence, $\{\tau_n = k\} = \{X_{(k)} = n\}$. From the beginning to the ending, we can obtain the transition probability flow graphs of $\{\tau_n, n = 1, 2, \cdots\}$ as shown in Fig. 1.

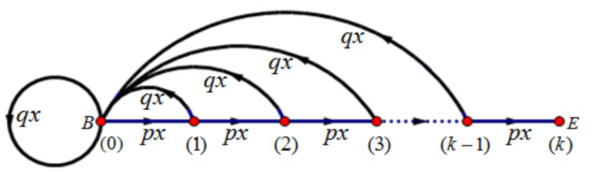

Fig. 1: The transition probability flow graphs of $G_k(p)$

There are k parallel loop routes at the beginning state B with respectively the transition probability functions qx, $qx(px), qx(px)^2, \cdots, qx(px)^{k-1}$, and k series straight routes from B to E with the same transition probability function px. Then we get the transition probability functions of the loop route and the straight route from B to E denoted respectively by $O(x)$ and $L(x)$ as follows

$$O(x) = \sum_{i=0}^{k-1}(qx)(px)^i,\ L(x) = p^k x^k.$$

By Fig. 1 and Lemma 2, we get the probability generating function of $X_{(k)}$ (i.e. the transition time of the route $B \to E$)

$$G_{X_{(k)}}(x) = \frac{p^k x^k}{1 - \sum_{i=0}^{k-1}(qx)(px)^i} = \frac{p^k x^k - p^{k+1}x^{k+1}}{1 - x + qp^k x^{k+1}}.$$

This completes the proof. □

Next, we consider the probability distribution of the negative binomial distribution of order k. Let $X_{(k,r)}$ be a random variable denoting the number of trials until the rth occurrence of the success run with length k in Bernoulli trials with success probability p. We denote the probability distribution of $X_{(k,r)}$ by $NB_k(r, p)$ and call it the negative binomial distribution of order k with parameter vector (r, p).

Theorem 4 *The probability generating function of $X_{(k,r)}$ distributed as $NB_k(r, p)$ is presented as follows*

$$G_{X_{(k,r)}}(x) = \left(\frac{p^k x^k - p^{k+1}x^{k+1}}{1 - x + qp^k x^{k+1}}\right)^r. \tag{2}$$

Proof. Let $X^1_{(k)}$ be the number of trials until the occurrence of the 1^{th} success run, and $X^s_{(k)}$ be the number of trials from the ending of the $(s-1)^{\text{th}}$ success run to the occurrence of the s^{th} one respectively, $s = 2, \cdots, r$. It is obvious that $X^1_{(k)}, X^2_{(k)}, \cdots, X^r_{(k)}$ are independent and identically distributed random variables each having the probability generating function as formula (1). Hence we get the generating function of $X_{(k,r)} = X^1_{(k)} + X^2_{(k)} + \cdots + X^r_{(k)}$ as follows

$$G_{X_{(k,r)}}(x) = \prod_{s=1}^{r} G_{X^s_{(k)}}(x) = \left(G_{X^1_{(k)}}(x)\right)^r$$

$$= \left(\frac{p^k x^k - p^{k+1}x^{k+1}}{1 - x + qp^k x^{k+1}}\right)^r.$$

Theorem 4 has been proven. □

Theorem 5 *If the random variable $X_{(k,r)}$ is distributed as $NB_k(r,p)$, then*

$$P(X_{(k,r)}=n)=\sum_{\substack{n_1,n_2,\cdots,n_k\ni\\ n_1+2n_2+\cdots kn_k=n-kr}}$$

$$\binom{n_1+n_2+\cdots+n_k+r-1}{n_1,n_2,\cdots,n_k,r-1}\left(\frac{q}{p}\right)^{n_1+n_2+\cdots+n_k}p^n,$$

where $\quad n=kr,kr+1,kr+2,\cdots.$

Proof. Following formula (2), we get

$$G_{X_{(k,r)}}(x)=\left(\frac{p^kx^k}{1-qx(1+px+\cdots+p^{k-1}x^{k-1})}\right)^r$$

$$=\left(\frac{p^kx^k}{1-\frac{q}{p}(px+\cdots+p^kx^k)}\right)^r$$

$$=p^{kr}x^{kr}\left(1-\frac{q}{p}(px+\cdots+p^kx^k)\right)^{-r}$$

$$=p^{kr}x^{kr}\sum_{n=0}^{\infty}\binom{r+n-1}{r-1}\left(\frac{q}{p}\right)^n(px+\cdots+p^kx^k)^n$$

$$=p^{kr}x^{kr}\sum_{n=0}^{\infty}\binom{r+n-1}{r-1}\left(\frac{q}{p}\right)^n\times$$

$$\sum_{\substack{n_1,\cdots,n_k\ni\\ n_1+\cdots+n_k=n}}\binom{n}{n_1,\cdots,n_k}(px)^{n_1+2n_2+\cdots+kn_k}$$

$$=\sum_{n=0}^{\infty}\sum_{\substack{n_1,\cdots,n_k\ni\\ n_1+\cdots+n_k=n}}\binom{r+n-1}{r-1}\binom{n}{n_1,\cdots,n_k}\times$$

$$\left(\frac{q}{p}\right)^n(px)^{n_1+2n_2+\cdots+kn_k+kr}$$

$$=\sum_{n_1,n_2,\cdots,n_k}\binom{n_1+n_2+\cdots+n_k+r-1}{n_1,n_2,\cdots,n_k,r-1}\times$$

$$\left(\frac{q}{p}\right)^{n_1+n_2+\cdots+n_k}(px)^{n_1+2n_2+\cdots+kn_k+kr},$$

$$=\sum_{n=kr}^{\infty}\sum_{\substack{n_1,\cdots,n_k\ni\\ \sum_{i=1}^k in_i=n-kr}}\binom{n_1+\cdots+n_k+r-1}{n_1,\cdots,n_k,r-1}\times$$

$$\left(\frac{q}{p}\right)^{n_1+n_2+\cdots+n_k}p^nx^n.$$

So, from the above we can easy come to the probability distribution of $X_{(k,r)}$. □

Remark 6 $NB_k(1,p)=G_k(p)$.

Let $N_n^{(k)}$ be the number of success run of length k in n Bernoulli trials with success probability p. The probability distribution of $N_n^{(k)}$ denoted by $B_k(n,p)$ is called the binomial distribution of order k with parameter vector (n,p). Note that when $k=1$, $B_1(n,p)$ is the usual binomial distribution $B(n,p)$.

Theorem 7 *The probability distribution of $B_k(n,p)$ is given by*

$$P(N_n^{(k)}=r)=\sum_{s=0}^{k-1}\sum_{\substack{m_1,m_2,\cdots,m_k\ni\\ m_1+2m_2+\cdots+km_k=n-s-kr}}$$

$$\binom{m_1+\cdots+m_k+r}{m_1,m_2,\cdots,m_k,r}\left(\frac{q}{p}\right)^{m_1+\cdots+m_k}p^n, \qquad (3)$$

where $r=0,1,\cdots,[\frac{n}{k}]$, and $[x]$ denotes the greatest integer not exceeding $x\in\mathbb{R}$.

Proof. Suppose $X_{(k,r)}$ is a variable distributed as $NB_k(r,p)$, then $\{X_{(k,r+1)}=m\}$ is equivalent to that either the $(m-k)$th trial is a failure and all the subsequent k trials are successes, or all the posterior $2k$ trials are successes in the m trials. Let "$\oplus$" be a success, "$\ominus$"be a failure, "$\odot$"be a trial, for $m=n+k,n+k-1,\cdots,n+1$, we have the figure of the events $\{X_{(k,r+1)}=n+k-s\}$, $s=0,1,\cdots,k-1$ as follows, where $k=4$ for simplicity.

$$\begin{array}{ll}
\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot\ominus}^{n\text{ trials}}\oplus\oplus\oplus\oplus & s=0\\
\odot\cdots\odot\odot\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=0\\
\odot\cdots\odot\odot\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=1\\
\odot\cdots\odot\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=1\\
\odot\cdots\odot\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=2\\
\odot\cdots\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=2\\
\odot\cdots\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=3\\
\odot\cdots\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=3
\end{array}$$

Fig. 2: The events of $\{\xi_{(k,r+1)}=n+k-s\}$

Let $\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}$ denote the event that $(r+1)$ success runs of length k occur in $(n+k-s)$ trials and the last $(k-s)$ successes are deleted, where $s=0,1,\cdots,k-1$. Following Fig. 2, we shall find that $\bigcup_{s=0}^{k-1}\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}$ means all the possible ways the r success runs occur in n trials. Hence we have

$$\{N_n^{(k)}=r\}=\bigcup_{s=0}^{k-1}\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}.$$

Therefore

$$P(N_n^{(k)} = r)$$

$$= \sum_{s=0}^{k-1} P\left(\{X_{(k,r+1)} = n+k-s | (k-s) \oplus\}\right)$$

$$= \sum_{s=0}^{k-1} p^{-(k-s)} \cdot P\left(X_{(k,r+1)} = n+k-s\right)$$

$$= \sum_{s=0}^{k-1} \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=n-s-kr}}$$

$$\binom{m_1+m_2+\cdots+m_k+r}{m_1,m_2,\cdots,m_k,r} \left(\frac{q}{p}\right)^{m_1+m_2+\cdots+m_k} p^n.$$

Theorem 7 has been proven. □

3 Some properties of the binomial distribution of order k

It is well known that when $k = 1$, the usual binomial distribution $B(n,p)$ converges to the usual Poisson distribution $P(\lambda)$ as $n \to \infty, p \to 0$ and $np \to \lambda$. In addition, if Z_1 and Z_2 are independently distributed as $P(\lambda_1)$ and $P(\lambda_2)$, then the conditional random variable $Z_1|Z_1 + Z_2 = n$ is distributed as $B(n,p)$ where $p = \lambda_1/(\lambda_1 + \lambda_2)$. However, for $k > 1$, the above conclusions are not true.

Proposition 8 *$B_k(n,p)$ does not converge to $P_k(\lambda)$ as $n \to \infty, p \to 0$ and $np \to \lambda$ if $k > 1$.*

Proof. To show this, we represent the probability generating function of $P_k(\lambda)$ derived by Shao in [15] as follows

$$G_{Z_{(k)}}(x) = e^{\lambda(x+\cdots+x^k-k)}.$$

Let $G_{N_n^{(k)}}(x)$ be the generating function of $N_n^{(k)}$ distributed as $B_k(n,p)$. It is easy to find that if $P(N_n^{(k)} = 0) \nrightarrow e^{-\lambda k}$ as $n \to \infty, np \to \lambda$, then $G_{N_n^{(k)}}(x) \nrightarrow G_{Z_{(k)}}(x)$, where $P(N_n^{(k)} = 0)$ and $e^{-k\lambda}$ are respectively the constant terms of $G_{N_n^{(k)}}(x)$ and $G_{Z_{(k)}}(x)$.

We only consider $k = 2$. By formula (3), the constant term of the generating function $G_{N_n^{(2)}}(x)$ is

$$P(N_n^{(2)} = 0) = \sum_{s=0}^{1} \sum_{\substack{m_1,m_2 \ni \\ m_1+2m_2=n-s}}$$

$$\binom{m_1+m_2}{m_1,m_2} \left(\frac{q}{p}\right)^{m_1+m_2} p^n$$

$$= \sum_{\substack{m_1,m_2 \ni \\ m_1+2m_2=n}} \binom{m_1+m_2}{m_1,m_2} \left(\frac{q}{p}\right)^{m_1+m_2} p^n$$

$$+ \sum_{\substack{m_1,m_2 \ni \\ m_1+2m_2=n-1}} \binom{m_1+m_2}{m_1,m_2} \left(\frac{q}{p}\right)^{m_1+m_2} p^n$$

$$= q^n + \sum_{l=1}^{[\frac{n+1}{2}]} \binom{n-l+1}{l} q^{n-l} p^l,$$

let $n \to \infty, p \to 0$ and $np \to \lambda$, then

$$P(N_n^{(2)} = 0) \to e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} + \cdots = 1.$$

On the other hand, for the random variable $Z_{(2)}$ that distributed as $P_2(\lambda)$, the constant term of $G_{Z_{(2)}}(x) = e^{\lambda(x+x^2-2)}$ is $e^{-2\lambda}$. By $P(N_n^{(2)} = 0) \nrightarrow e^{-2\lambda}$, we say that $G_{N_n^{(2)}}(x) \nrightarrow e^{\lambda(x+x^2-2)}$. Thus for any positive integer $k > 1$, we shall conclude that

$$G_{N_n^{(k)}}(x) \nrightarrow e^{\lambda(x+\cdots+x^k-k)}.$$

It means that the binomial distribution of order $k(> 1)$ does not converge to the Poisson distribution of order k as $n \to \infty, p \to 0$ and $np \to \lambda$. □

Proposition 9 *If Z_i are independently distributed as $P_k(\lambda_i)$, $i = 1, 2$, then the conditional random variable $Z_1|Z_1 + Z_2 = n$ is not distributed as $B_k(n,p)$ with parameter $p = \lambda_1/(\lambda_1 + \lambda_2)$ when $k > 1$.*

Proof. The probability distribution of $P_k(\lambda)$ can be found in [5], [10] and [15], we represent it as follows

$$P(Z = m) = \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=m}} \frac{\lambda^{m_1+\cdots+m_k}}{m_1! \cdots m_k!} e^{-\lambda k},$$

where $m = 0, 1, 2, \cdots$. Furthermore, we can show that if Z_1 and Z_2 are independently distributed as $P_k(\lambda_1)$ and $P_k(\lambda_2)$ respectively, then $Z_1 + Z_2$ is distributed as $P_k(\lambda_1 + \lambda_2)$.

Assume that $k = 2, \lambda_1 = \lambda_2 = \lambda$, then $Z_1 + Z_2$ is distributed as $P_2(2\lambda)$. Hence we arrive at

$$P(Z_1=1|Z_1+Z_2=2)$$

$$=\frac{P(Z_1=1,Z_1+Z_2=2)}{P(Z_1+Z_2=2)}$$

$$=\frac{P(Z_1=1)P(Z_2=1)}{P(Z_1+Z_2=2)}$$

$$=\left(\sum_{\substack{m_1,m_2\ni\\ m_1+2m_2=1}}\frac{\lambda^{m_1+m_2}e^{-2\lambda}}{m_1!m_2!}\right)^2$$

$$\div\sum_{\substack{m_1,m_2\ni\\ m_1+2m_2=2}}\frac{(2\lambda)^{m_1+m_2}}{m_1!m_2!}e^{-4\lambda}=\frac{\lambda}{2\lambda+2}. \quad (4)$$

However, if $N_2^{(2)}$ has a distribution $B_2(2,p)$, where $p=\lambda_1/(\lambda_1+\lambda_2)=0.5$, by formula (3), we obtain

$$P(N_2^{(2)}=1)=\sum_{s=0}^{1}\sum_{\substack{m_1,m_2\ni\\ m_1+2m_2=2-s-2}}$$

$$\binom{m_1+m_2+1}{m_1,m_2,1}\left(\frac{0.5}{0.5}\right)^{m_1+m_2}(0.5)^2=0.25. \quad (5)$$

Combining (4) with (5), we find that if $k>1$, $Z_1|Z_1+Z_2=n$ is not distributed as $B_k(n,p)$. □

Proposition 10 *Let $P_k^*(n,r)\hat{=}P(N_n^{(k)}=r)$, then we have the recurrence*

$$P_k^*(n,r)=P_k^*(n-1,r)-qp^kP_k^*(n-k-1,r)$$

$$+p^kP_k^*(n-k,r-1)-p^{k+1}P_k^*(n-k-1,r-1),$$

where $n>k$ and $0\le r\le[n/k]$.

Proof. Similar to Fig. 2, we still assume that $k=4$ for simplicity. Considering the events

$B_1=\{$there are r success runs occur in n trials and they occur in the first $(n-1)$ trials too$\}$

and

$B_2=\{$there are r success runs occur in n trials and no r success runs occur in the first $(n-1)$ trials$\}$,

we come to

$$B_1=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\odot}_{r\text{ runs in }n\text{ trials}}\},$$

$$B_2=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{(r-1)\text{ runs in }(n-1)\text{ trials}}\odot}_{r\text{ runs in }n\text{ trials}}\},$$

obviously,

$$\{N_n^{(k)}=r\}=B_1\cup B_2. \quad (6)$$

For the event B_1, regardless of success or failure, the nth trial doesn't change the number of success runs, hence it can be decomposed as

$$B_1=B_{11}\cup B_{12},$$

where

$$B_{11}=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\ominus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\ominus\},$$

$$B_{12}=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\oplus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\oplus\}\backslash\{\overbrace{\underbrace{\odot\cdots\odot\odot}_{n-k-1\text{ trials}}\ominus\oplus\oplus\oplus}^{r\text{ runs in }(n-1)\text{ trials}}\oplus\}.$$

Hence we have

$$P(B_1)=P(B_{11})+P(B_{12})$$

$$=qP_k^*(n-1,r)+pP_k^*(n-1,r)-qp^kP_k^*(n-k-1,r)$$

$$=P_k^*(n-1,r)-qp^kP_k^*(n-k-1,r). \quad (7)$$

On the other hand, for the event B_2, all the last k trials must be success, otherwise, the nth trial, i.e. the last one can't lead to a success run. So

$$B_2=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\oplus\oplus\oplus}^{(r-1)\text{ runs in }(n-1)\text{ trials}}\oplus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot}^{(n-k)\text{ trials}}}_{(r-1)\text{ runs}}\oplus\oplus\oplus\oplus\}\setminus\{\underbrace{\overbrace{\odot\cdots\odot\odot}^{n-k-1\text{ trials}}}_{(r-1)\text{ runs}}\oplus\oplus\oplus\oplus\oplus\},$$

then we get

$$P(B_2)=p^kP_k^*(n-k,r-1)-p^{k+1}P_k^*(n-k-1,r-1). \quad (8)$$

Combining (6), (7) with (8), we can derive the recurrence relation in Proposition 10. □

Remark 11 *Proposition 10 yields*

$$P_k^*(n,0) = P_k^*(n-1,0) - qp^k P_k^*(n-k-1,0),$$

where $n > k$.

Proposition 12 *The mean of the random variable $N_n^{(k)}$ distributed as $B_k(n,p)$ is given by*

$$EN_n^{(k)} = \sum_{m=1}^{[n/k]} \{1 + (n - mk)q\} p^{mk}.$$

Proof. Let $E_n = EN_n^{(k)}$, obviously, we find that

$$E_0 = E_1 = \cdots = E_{k-1} = 0, E_k = p^k.$$

When $n \geq k+1$, following the equation in Proposition 10, we have

$$\sum_{r=1}^{[n/k]} r \cdot P_k^*(n,r) = \sum_{r=1}^{[n/k]} r \cdot P_k^*(n-1,r)$$

$$-qp^k \sum_{r=1}^{[n/k]} r \cdot P_k^*(n-k-1,r)$$

$$+p^k \sum_{r=1}^{[n/k]} (r-1+1) \cdot P_k^*(n-k,r-1)$$

$$-p^{k+1} \sum_{r=1}^{[n/k]} (r-1+1) \cdot P_k^*(n-k-1,r-1),$$

that is

$$E_n = E_{n-1} - qp^k E_{n-k-1}$$
$$+p^k E_{n-k} + p^k - p^{k+1} E_{n-k-1} - p^{k+1},$$

or

$$E_n - p^k E_{n-k} = E_{n-1} - p^k E_{n-k-1} + qp^k.$$

Let $H_n = E_n - p^k E_{n-k}$, then we get

$$H_k = E_k - p^k E_0 = p^k$$

and

$$H_n = H_{n-1} + qp^k, n \geq k+1.$$

By the above recurrence, we come to

$$H_n = H_k + (n-k)qp^k = p^k + (n-k)qp^k,$$

where $n \geq k+1$. That is

$$E_n - p^k E_{n-k} = p^k + (n-k)qp^k, n \geq k+1.$$

So, when $n \geq k$,

$$E_n = p^k E_{n-k} + p^k + (n-k)qp^k$$

$$= p^{2k} E_{n-2k} + p^k + p^{2k} + (n-k)qp^k + (n-2k)qp^{2k}$$

$$= \cdots = p^{[\frac{n}{k}]k} E_{n-[\frac{n}{k}]k} + (p^k + p^{2k} + \cdots + p^{[\frac{n}{k}]k}) +$$

$$(n-k)qp^k + (n-2k)qp^{2k} + \cdots + (n - [\frac{n}{k}]k)qp^{[\frac{n}{k}]k}$$

$$= 0 + \sum_{m=1}^{[n/k]} p^{mk} + \sum_{m=1}^{[n/k]} (n-mk)qp^{mk}$$

$$= \sum_{m=1}^{[n/k]} \{1 + (n-mk)q\} p^{mk}.$$

The proof is complete. □

A Reliability Example. For transmitting signals from site A to site B, n independent microwave transfer stations with the same failure rate p are distributed equidistantly between them. Each station can transfer the signals to the next kth one. We know that the whole microwave transfer system will be in operation until k or more consecutive stations fail at the same time [7]. We consider the reliability of the system R_n^*.

Obviously, $R_n^* = P(N_n^{(k)} = 0) = P_k^*(n,0)$. So, by Remark 11, we have $(k+1)$ initial values

$$\begin{bmatrix} R_k^* \\ R_{k+1}^* \\ R_{k+2}^* \\ \cdots \\ R_{2k}^* \end{bmatrix} = \begin{bmatrix} 1-p^k \\ R_k^* \\ R_{k+1}^* \\ \cdots \\ R_{2k-1}^* \end{bmatrix} - \begin{bmatrix} 0 \\ qp^k \\ qp^k \\ \cdots \\ qp^k \end{bmatrix}$$

or

$$R_i^* = 1 - p^k - (i-k)qp^k, \tag{9}$$

where $k \leq i \leq 2k$.

$$\begin{bmatrix} R_i^* \\ R_{i+1}^* \\ R_{i+2}^* \\ \cdots \\ R_{i+k}^* \end{bmatrix} = \begin{bmatrix} R_{i-1}^* \\ R_i^* \\ R_{i+1}^* \\ \cdots \\ R_{i+k-1}^* \end{bmatrix} - qp^k \begin{bmatrix} R_{i-k-1}^* \\ R_{i-k}^* \\ R_{i-k+1}^* \\ \cdots \\ R_{i-1}^* \end{bmatrix}, \tag{10}$$

where $2k + 1 \le i \le n - k$.

By the iterative formula (10), we can calculate the reliability of the system R_n^*. If the number of the microwave transfer stations n and the failure rate p are fixed, then R_n^* is an increasing function of k; if the number n and k are fixed, then R_n^* is a decreasing function of p; moreover, for a fixed n and any $k > 1$, R_n^* is significantly greater than $R_n = (1 - p)^n$, the series reliability of the system. This is shown clearly in Fig. 3 where the horizontal axis denotes the failure rate p and the vertical axis denotes the values of the reliability.

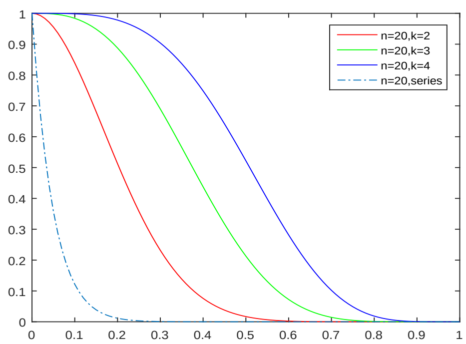

Fig. 3: The curves of $R_n^* - p$ and $R_n - p$

4 Conclusions

We introduce the transition probability flow graphs methods in studying the distributions of order k and derive the probability generating function of $G_k(p)$ and $NB_k(r, p)$, show the correlation between $G_k(p)$ and $NB_k(r, p)$. Following $NB_k(r, p)$, we arrive at the probability distribution of $B_k(n, p)$. Some corresponding properties of $B_k(n, p)$, such as its mean, convergence and recurrence formula etc, are presented. Finally, we discuss a reliability question by the properties.

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References:

[1] S. Aki, K. Hirano, Distributions of numbers of failures and successes until the first consecutive k successes, *Annals of the Institute of Statistical Mathematics*, 46(1), 1994, pp. 193-202.

[2] Y. L. Fan, The probability theoretical basis and application methods for transition probability flow graphs(I), *Application of Statistics and Management*, 17(1), 1998, pp. 45-51.

[3] Y. L. Fan, The probability theoretical basis and application methods for transition probability flow graphs(II), *Application of Statistics and Management*, 17(2), 1998, pp. 53-59.

[4] J. C. Fu, M. V. Koutras, Distributions theory of runs:A Markov chain approach, *Journal of the American Statistical Association*, 89(427), 1994, pp. 1050-1058.

[5] C. Georghiou, A. N. Philippou, A. Saghafi, On the modes of the Poisson distribution of order k, *The Fibonacci Quarterly*, 51(1), 2013, pp. 44-48.

[6] A. P. Godbole, Specific formulae for some success run distributions, *Statist. Probab. Lett.*, 10(2), 1990, pp. 119-124.

[7] Q. Han, Exact distribution theory of runs, *Chinese Journal of Applied Probability and Statistics*, 15(2), 1999, pp. 199-212.

[8] T. Koyama, Switching characteristics under MIL-STD-105D, *Technometrics*, 21(1), 1979, pp. 9-19.

[9] S. J. Mason, Feedback theory: Some properties of signal flow graphs, *Proceedings of the IRE.*, 41(9), 1953, pp. 1144-1156.

[10] A. N. Philippou, C. Georghiou, G. N. Philippou, A generalized geometric distribution and some of its properties, *Statist. Probab. Lett.*, 1(4), 1983, pp. 171-175.

[11] A. N. Philippou, F. S. Makri, Success, runs and longest runs, *Statist. Probab. Lett.*, 4(4), 1986, pp. 211-215.

[12] J. G. Shao, Numerical characteristics for the number of independent trials, *Journal of Beijing Jiaotong University*, 38(6), 2014, pp. 106–111.

[13] J. G. Shao, S. Fu, Some generalizations of geometric distribution in Bernoulli trials by TPFG methods, *WSEAS Transactions on Mathematics*, 14(2), 2015, pp. –.

[14] J. G. Shao, S. Fu, Generalization and research of Pascal distribution by TPFG methods, *Acta Mathematicae Applicatae Sinica*, 38(2), 2015, pp. –.

[15] J. G. Shao, Q. S. Wang, Multidimensional geometric distribution and multidimensional Poisson distribution in independent trials, *Journal of Beijing Jiaotong University*, 39(3), 2015, pp. 122–129.