Temporal-Spatial Optimal Tracking Control of Microwave Heating Process Based on the Spectral Galerkin Method

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Abstract: Microwave heating, which is a novel and efficient heating method, has been gradually replacing traditional heating method in industrial application. However, the phenomenon of hotspots or thermal runaway is the major drawback for further development. Consequently, it is necessary to regulate the spatial temperature distribution with a reasonable input power. For traditional microwave heating partial differential equation (PDE) model, the inherent infinite-dimensional feature does not allow readily designing corresponding controller. Motivated by this obstruction, a finite-dimensional ordinary differential equation (ODE) model is developed to approximately describe the heating process with the help of spectral Galerkin method. Moreover, an optimal tracking controller with a constrained input is designed to improve the spatial temperature uniformity. Finally, the results of simulation on one-dimensional microwave cavity heating model are provided to demonstrate the effectiveness of proposed controller.

Key–Words: Microwave Heating, Spectral Galerkin Method, Optimal Control, Temperature Tracking

1 Introduction

In modern society, microwave heating has obtained vast applications in domestic and industrial fields, such as drying, cooking, thawing, sintering, defrosting, joining and sterilization [1, 2]. Different with the traditional heating method, microwave heating has incomparable advantages which are beneficial to the energy conservation and emission reduction because of the characteristics of volumetric heating. However, the major drawback associated with microwave heating is the nonhomogeneous temperature distribution which will lead to the hotspots and thermal runaway.

The process of microwave heating is a kind of thermal process which is influenced by multi-physical field, such as, thermodynamics field and electromagnetic field. Moreover, due to the propagation characteristics of electromagnetic field and inherent properties of materials, it is usually appeared a great temperature difference inside the heated material. In theoretical research, Maxwell's equation and Lambert's law are usually described the distribution of electromagnetic energy. Thus, by analyzing the propagation of microwave in waveguide or resonant cavity, we can derive an explicit dissipation power, which will contribute to analyzing the characteristics of microwave heating in numerical simulations.

Mathematically, the process of microwave heating is usually represented by a nonhomogeneous partial differential equation (PDE) with boundary and initial conditions. In terms of parabolic PDEs, some control algorithms have also been proposed, such as, robust control [3], collocated feedback control [4], backstepping control [5], adaptive control [6], sliding mode control [7] and optimal control [8]. Specially, Wei [9] first derives a necessary condition for optimal control solution to reach a relatively uniform heat profile in process of microwave heating. However, due to the infinite-dimensional nature of PDEs, it is difficult to readily design a controller with a reasonable input [10]. From the view of microwave heating applications, a finite number of temperature sensors will restrict the design and implementation of the controller further. Therefore, it is necessary to obtain a finite-dimensional microwave model to lay a foundation for controller designing.

To overcome the inherent problems, Christofides [11] has proposed spectral Galerkin method to partition the eigenspectrum of spatial differential operator into a finite-dimensional slow complement and an infinite-dimensional stable fast one in a catalytic packed-bed reactor. Motivated by this consideration, Zhong [12] proposes a finite-dimensional ordinary differential equation (ODE) model to approximately describe the temperature distribution in waveguide. However, waveguide heating is only a particular case, which cannot be generalized to other prototype heating systems. It is necessary to develop another finite-dimensional ODE model to describe the spatiotemporal evolution. From the view of cybernetics, the open-loop microwave heating system is a single input multiple output (SIMO) system, whose state, input, output, boundary conditions and process parameters may change on spatial domain and time domain. Moreover, power input is also restricted by the true physical nature. Therefore, to improve the uniformity of the temperature profile, one of choices is to select a suitable input, which can restrain hotspots or thermal runaway. Then, with the help of thermodynamics, a relatively uniform temperature distribution can be obtained. Although some traditional control algorithm, which is based on the input and output, can effectively reach above control expectation, the problem of input shock cannot directly be solved in process of heating. Hence, a global optimal control algorithm needs to be developed in order to achieve temperature tracking with a relative smooth input power.

The rest of this paper is organized as follows. For the reader's convenience, a preliminaries microwave heating model needed in the study is presented in Section 2, where the explicit dissipation power is derived by analyzing the Maxwell's equation. In Section 3, a finite-dimensional ODE model is developed via applying spectral Galerkin's method. In Section 4, based on the spatial temperature distribution, a global optimal controller with a constrained input is developed to tracking the control objectives. In Section 5, numerical simulations on one-dimensional cavity heating are demonstrated the effectiveness of the proposed methodology.

2 Preliminaries

2.1 Traditional Microwave Heating Model

During the process of microwave heating, the transient temperature profiles will significantly change due to the coupling of electromagnetic field and thermodynamics field. Hence, it is necessary to predict the temperature distribution by combining the energy equations and the electromagnetic equations. Mathematically, the nonhomogeneous parabolic PDE describing the temperature distribution can be expressed in the following form [13, 14]:

$$\rho\left(T\right)C_p\left(T\right)\frac{\partial T}{\partial t}=\nabla\left(\kappa\left(T\right)\nabla T\right)+Q_{abs} \qquad (1)$$

where $\rho\left(T\right)$, $C_p\left(T\right)$ and $\kappa\left(T\right)$ are material density, specific heat capacity and thermal conductivity; $\partial T/\partial t$ is the time differential operator; $\nabla\left(\kappa\left(T\right)\nabla T\right)$ is the spatial differential operator; Q_{abs} is a transient dissipation power term, which is depended on the electromagnetic field spatial distribution and permittivity.

To reduce complexity of the problem, several assumptions are offered into heat transfer analysis:

Assumption 1: The material is homogeneous and isotropic;

Assumption 2: The mass transfer is negligible;

Assumption 3: One-dimensional electrical field propagation and heat conduction are only considered;

Assumption 4: No phase and volume changes during the heating process;

Assumption 5: Convective boundary conditions are considered;

Assumption 6: Thermal and dielectric properties are temperature-independent;

By considering all the previous six assumptions, (1) can be simplified as follow

$$\rho C_p\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial z^2}+Q_{abs}\left(z,t\right),\ z\in[0,l] \qquad (2)$$

According to Newton's law of cooling, the boundary condition, which is determined by the convective heat transfer between the materials and surroundings, can be expressed as [15]

$$-\kappa\frac{\partial T}{\partial z}=h\left(T-T_\infty\right)+\sigma_h\varepsilon_h\left(T^4-T_\infty^4\right) \qquad (3)$$

where h denotes the effective heat transfer coefficient; T_∞ is the ambient temperature; σ_h is the Stefan Boltzmann constant ($5.67\times10^{-8}\ \mathrm{W/(m^2\cdot K^4)}$); ε_h represents the emissivity of the material; Generally, $\sigma_h\varepsilon_h\left(T^4-T_\infty^4\right)\to 0$, (3) can be simplified as

$$\kappa\frac{\partial T}{\partial z}=h\left(T-T_a\right)\ \mathrm{at\ z=0} \qquad (4)$$

$$-\kappa\frac{\partial T}{\partial z}=h\left(T-T_b\right)\ \mathrm{at\ z=l} \qquad (5)$$

where T_a and T_b are ambient temperatures in different positions.

2.2 Dissipation Power

Dissipation power, which is also called the internal heat source, is used to describe the relationship between the electromagnetic distribution and dielectric constant in heated material. For the nonmagnetic material, the electric field distribution is usually described by Lambert's law and Maxwell's equations. Lambert's law is the easiest way to describe the exponential decay of microwave absorption within the material. Due to the limitation of penetration depth and temperature-dependent permittivity, the law leads to a poor prediction temperature [1, 16]. On the other hand, Maxwell's equation provides an exact solution for the propagation of microwave irradiation within the sample. Thus, it is a common practice to first suppose that the electric and magnetic fields are time harmonic, with a fixed frequency ω. Based on this assumption, the electric and magnetic fields can then be written as

$$\overrightarrow{E}(z,t) = \overrightarrow{E}(z)\, e^{i\omega t} \tag{6}$$

$$\overrightarrow{H}(z,t) = \overrightarrow{H}(z)\, e^{i\omega t} \tag{7}$$

where $\overrightarrow{E}$ and $\overrightarrow{H}$ denote the electric field and magnetic field intensities, respectively; i denotes the unit complex number. On the assumption that the heating material is conductive and the region is source-free, linear, isotropic and homogeneous, the Maxwell's equation [17, 18] can be reduced to the following form

$$\frac{d^2E}{dz^2} + k^2E = 0 \tag{8}$$

For nonmagnetic material, we can define that

$$k^2 = \omega^2\mu_0\varepsilon_0\left(\varepsilon' + i\varepsilon''\right) \tag{9}$$

where μ_0 and ε_0 are the free space permeability and permittivity; ε' denotes the relative dielectric constant which represents the material's ability to store electrical energy; ε'' denotes the relative dielectric loss which accounts for dielectric loss through energy dissipation. For further simplified calculation, the propagation constant k is usually defined as a complex quantity

$$\begin{aligned} k = \alpha + i\beta = \frac{2\pi f}{c}\sqrt{\frac{\varepsilon'\left(\sqrt{1+\tan^2\delta}+1\right)}{2\varepsilon_0}} \\ + i\frac{2\pi f}{c}\sqrt{\frac{\varepsilon'\left(\sqrt{1+\tan^2\delta}-1\right)}{2\varepsilon_0}} \end{aligned} \tag{10}$$

where $\tan\delta = \varepsilon''/\varepsilon'$ is indicated by the ratio of the dielectric loss to the dielectric constant and c is speed of light.

By using substitution method, the solution of E in (8) is obtained as

$$E = E^+e^{-jkz} + E^-e^{jkz} \tag{11}$$

where E^+ and E^- represent the incident and reflection electric field intensity. With a knowledge of incident electric field intensity and phase ($E^+ = |E_0| \cdot e^{i\varphi_1} \cdot e^{-\beta z}$), the reflect electric field can be expressed as

$$E^- = |E_0| \cdot e^{-2\beta l} \cdot e^{i\varphi_2} \cdot e^{\beta z} \tag{12}$$

where $|E_0|$ is the initial electrical intensity. The dissipation power is determined from the following relationship [18]:

$$Q_{abs}(z,t) = \frac{1}{2}\omega\varepsilon_0\varepsilon'' E \cdot E^* \tag{13}$$

where E^* is the complex conjugate of E. On assumption that the reflection coefficient is equal with 1, we could substitute (11) and (12) into (13), the one-dimensional dissipation power in resonant cavity can be expressed as,

$$\begin{aligned} Q_{abs} &= \frac{\omega\varepsilon_0\varepsilon''}{2}\left[|E_0|^2e^{-2\beta z} + \left|E_0e^{-2\beta l}\right|^2e^{2\beta z}\right. \\ &\quad \left. + |E_0|^2e^{-2\beta l}\cos\left(\varphi_A - \varphi_B + 2\alpha z\right)\right] \\ &= \frac{\omega\varepsilon_0\varepsilon''}{2}\left[e^{-2\beta z} + \left|e^{-2\beta l}\right|^2e^{2\beta z}\right. \\ &\quad \left. + e^{-2\beta l}\cos\left(\varphi_A - \varphi_B + 2\alpha z\right)\right]|E_0|^2 \end{aligned} \tag{14}$$

3 Model Reduction

For a typical non-uniform heating model which consists of an nonhomogeneous PDE, boundary conditions and initial condition, it is difficult to design controller based on the aforementioned model to achieve temperature tracking. Mathematically, the characteristic spectrum of spatial differential operator in heat transport equation can be transformed into infinite-dimensional complements and finite-dimensional one [19, 20, 21]. Then, the finite-dimensional ODE model can be developed by applying the Galerkin's method in order to readily analyze and design the controller.

3.1 Infinite-dimensional ODE model

To distinctly discuss the spectrum in (2)-(3), we first introduce the Hilbert space $\mathbf{H}([0,l];\mathbb{R})$ with $\mathbf{H}$ being the space of infinite-dimensional vector functions defined on $[0,l]$ that satisfies the boundary conditions, with inner produce and norm

$$(g_1, g_2) = \int_0^l (g_1(z), g_2(z))_{\mathbb{R}}dz \tag{15}$$

$$\|g_1\|_2 = (g_1, g_1)^{1/2} \tag{16}$$

where g_1 and g_2 are two elements of $\mathbf{H}([0,l];\mathbb{R})$ and the notation $(\cdot,\cdot)_{\mathbb{R}}$ denotes the standard inner product $\mathbb{R}$. Defining the state function $\overline{T}$ on $\mathbf{H}([0,l];\mathbb{R})$ as

$$\overline{T}(t) = T(z,t),\ t > 0,\ z \in [0,l] \tag{17}$$

and the operators as

$$\begin{aligned}\mathbf{A}\overline{T} &= k_1 \cdot \frac{\partial^2 T}{\partial z^2},\\ \overline{T} \in D(A) &= \{z \in \mathbf{H}([0,l];\mathbb{R}):\\ \kappa\frac{\partial T(0)}{\partial z} &= h(T(0)-T_a),\\ \kappa\frac{\partial T(l)}{\partial z} &= -h(T(l)-T_b)\}\end{aligned} \tag{18}$$

where $k_1 = \kappa/(\rho C_p)$. Based on above analysis, the eigenvalue problem can be defined as

$$\mathbf{A}\phi_i(z) = \lambda_i \phi_i(z),\ i = 0,1,\cdots\infty \tag{19}$$

where λ_i denotes an eigenvalue and $\phi_i(z)$ denotes the corresponding eigenfunction. Thus, the solution of (2), (4) and (5) can be approximately expressed in an orthogonally decoupled series

$$T(z,t) = \sum_{i=0}^{\infty} \overline{T}_i(t) \cdot \phi_i(z) \tag{20}$$

For precisely expressing the relationship between the electric field intensity and phase, we formulate input electric power as $u = |E_0|^2$ and the coefficient of dissipation power as $\tilde{Q}_{abs}(z) = 1/2\omega\varepsilon_0\varepsilon''\left[e^{-2\beta z} + \left|e^{-2\beta l}\right|^2 e^{2\beta z} + e^{-2\beta l}\cos(\varphi_A - \varphi_B + 2\alpha z)\right]$. Then, the input operator can be defined as:

$$\mathbf{B} \cdot u(t) = k_2 \cdot \frac{2}{l} \cdot \int_0^l \tilde{Q}_{abs} \cdot \phi_i(z)\, dz \cdot u(t) \tag{21}$$

where $k_2 = 1/(\rho C_p)$.

The traditional microwave heating model can be rewritten as the following infinite-dimensional ODE form:

$$\dot{\overline{T}}(t) = \mathbf{A} \cdot \overline{T}(t) + \mathbf{B} \cdot u(t) \tag{22}$$

$$T(z,t) = [\mathbf{C}(z_1), \mathbf{C}(z_2), \cdots, \mathbf{C}(z_m)]' \cdot \overline{T}(t) \tag{23}$$

where

$$\overline{T}(t) = \left[\overline{T}_0(t), \overline{T}_1(t), \overline{T}_2(t), \cdots, \overline{T}_n(t), \cdots\right]'$$

$$\mathbf{A} = k_1 \cdot diag(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n, \cdots)$$

$$\mathbf{B} = k_2 \cdot \frac{2}{l} \cdot \int_0^l \tilde{Q}_{abs} \cdot \left[\frac{\phi_0}{2}, \phi_1, \phi_2, \cdots \phi_n, \cdots\right]' dz$$

$$\mathbf{C}(z) = [\phi_0(z), \phi_1(z), \phi_2(z), \cdots \phi_n(z), \cdots]$$

Remark 1 *From model (22) and (23), it is easily to find that the model has only a manipulated input $u(t)$, which can directly bring the different temperature output in spatial domain. The output matrix $[\boldsymbol{C}(z_1), \boldsymbol{C}(z_2), \cdots, \boldsymbol{C}(z_m)]'$ indicates that the virtual temperature sensors can detect m points in different positions at the same time. On assumption that $m \to \infty$, the global temperature distribution in material can be approximately obtained.*

3.2 Galerkin's method

Aforementioned analysis mainly focuses on obtaining the spectrum to derive the infinite-dimensional ODE model. However, the infinite-dimensional model is not suitable for simulation in computer due to the overlarge calculated load. In this subsection, we apply spectrum Galerkin's method to obtain a finite-dimensional microwave heating model.

For the eigenspectrum of $\mathbf{A}$, $\sigma(\mathbf{A}) = \{\lambda_0, \lambda_1, \lambda_2, \cdots \lambda_n, \cdots\}$ considers the following assumptions [19]:

Assumption 7: $\mathrm{Re}\{\lambda_0\} \geq \mathrm{Re}\{\lambda_1\} \geq \mathrm{Re}\{\lambda_2\} \geq \cdots \geq \mathrm{Re}\{\lambda_n\} \geq \cdots$, where $\mathrm{Re}\{\lambda_n\}$ represents the real part of λ_n;

Assumption 8: $\sigma(\mathbf{A})$ can be divided as $\sigma(\mathbf{A}) = \sigma_1(\mathbf{A}) + \sigma_2(\mathbf{A})$, where $\sigma_1(\mathbf{A})$ consists of first n (with n finite) eigenvalues, i.e. $\sigma_1(A) = \{\lambda_0, \lambda_1, \cdots \lambda_n\}$, and $|\mathrm{Re}\{\lambda_1\}| / |\mathrm{Re}\{\lambda_n\}| = O(l)$;

Assumption 9: $\mathrm{Re}\{\lambda_{n+1}\} < 0$ and $|\mathrm{Re}\{\lambda_n\}| / |\mathrm{Re}\{\lambda_{n+1}\}| = O(\varepsilon)$, where $\varepsilon := |\mathrm{Re}\lambda_1| / |\mathrm{Re}\lambda_{n+1}| < 1$ is a small positive parameter.

Based on above assumptions, we can first define the orthogonal projection operators P_s and P_f, such that $\overline{T}_s(t) = P_s\overline{T}(t)$ and $\overline{T}_f(t) = P_f\overline{T}(t)$, the state vector $\overline{T}(t)$ can be decomposed as

$$\overline{T} = \overline{T}_s + \overline{T}_f = P_s\overline{T} + P_f\overline{T} \tag{24}$$

Substituting (24) into (22) and (23), the finite-dimensional microwave heating model can be expressed as,

$$\begin{aligned}\dot{\overline{T}}_s(t) &= P_s\mathbf{A}P_s^-\overline{T}_s(t) + P_s\mathbf{B} \cdot u(t)\\ &= \mathbf{A}_s\overline{T}_s(t) + \mathbf{B}_s \cdot u(t)\end{aligned} \tag{25}$$

$$\begin{aligned}T(z,t) &= [\mathbf{C}(z_1), \cdots, \mathbf{C}(z_m)]' P_s^-\overline{T}_s(t)\\ &= [\mathbf{C}_s(z_1), \cdots, \mathbf{C}_s(z_m)]'\overline{T}_s(t)\end{aligned} \tag{26}$$

where

$$\overline{T}_s(t) = \left[\overline{T}_0(t), \overline{T}_1(t), \overline{T}_2(t), \cdots, \overline{T}_n(t)\right]'$$

$$\mathbf{A}_s = k_1 \cdot diag(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n)$$

$$\mathbf{B}_s = k_2 \cdot \frac{2}{l} \cdot \int_0^l \tilde{Q}_{abs} \cdot \left[\frac{\phi_0}{2}, \phi_1, \phi_2, \cdots \phi_n\right]' dz$$

$$\mathbf{C}_s(z) = [\phi_0(z), \phi_1(z), \phi_2(z), \cdots \phi_n(z)]$$

Remark 2 *For finite-dimensional model (25) and (26), the problem of relationship between the order n and the model error needs to be focused on. Though the higher order n will bring more accurate temperature, it is difficult to readily obtain and implement* *an on-line controller with a reasonable input. Therefore, it is better to choose an order to approximately describe global temperature profile and lay a foundation to controller design.*

4 Optimal Tracking Controller Design

In this section, a global optimal control for traditional microwave heating model (2), (4) and (5), based on n-dimensional ODE model, is designed to achieve the global temperature tracking and a relative uniform temperature distribution.

From the view of cybernetics, the open-loop one-dimensional microwave heating model is a single input multiple output (SIMO) model, whose input is the square of incident electric field intensity u and outputs are temperature $T(z,t)$ in different positions. Different from the traditional heating system (i.e. chemical reaction furnace [22]), we can only regulate the incident electric field intensity by designing a temporal-spatial tracking controller. However, the single tracking point may bring hotspots and coldspots, which are the main obstacles for heating to obtain a relative uniform temperature distribution. To this end, an optimal tracking position should be first chosen.

We first assume that m temperature value can be obtained, the average temperature can be obtained

$$\hat{T}(t) = \frac{\mathbf{C}_s(z_1) + \cdots + \mathbf{C}_s(z_m)}{m}\overline{T}_s(t) \tag{27}$$

The optimal tracking position can be obtained from the following equation

$$\tilde{z}_i \in \min_t \left(\left|\hat{T}(t) - T(\tilde{z}_i,t)\right|\right) \quad \text{if} \quad T(z_i,t) < T_{\max} \tag{28}$$

where $T_{\max}$ is the critical temperature. If temperature in some hot spots exceed the critical temperature, thermal runaway will occur.

Remark 3 *In the process of microwave heating, the global temperature should be monitored at real time. When the temperature in any position exceeds the critical temperature $T_{\max}$, the material will be highly sensitive to the incident power. Then, the power source should be turn off. It is nessessary for us to apply some sensors to detect the maximum temperature point. And corresponding control algorithm can be proposed to reach uniform temperature distribution and restrain thermal runaway.*

For a typical tracking control system, there are two main problems needing to consideration, such as, tracking error and input constraint. To this end, a cost function, which is based on the optimal tracking position in spatial domain, is proposed

$$J = \frac{1}{2}\int_0^\infty \Big[\left(y_r(t) - \mathbf{C}_s(\tilde{z}_i)\overline{T}_s(t)\right)^2 + \rho u^2(t)\Big]\,dz \tag{29}$$

where $\rho > 0$ is the penalty factor and y_r is the desired temperature profile.

Remark 4 *For a practical microwave heating system, the amplitude of input power is always constrained in a closed interval. In general, we usually regard the input is limited in $[0, u_{\max}]$. In order to obtain a reasonable computing power, the penalty factor is introduced. For the cost function (29), the bigger ρ will bring smaller $u(t)$.*

Hence, we have the following theorem from above analysis:

Theorem 5 *Consider the finite-dimensional model (25) and (26). For a spatial optimal tracking cost function (29) with an appropriate penalty factor ρ, if there exists a global optimal tracking control u^* satisfying*

$$u^*(t) = -\frac{1}{\rho}\boldsymbol{B}_s'\Big(P\overline{T}_s(t) - \left(\frac{1}{\rho}P\boldsymbol{B}_s\cdot\boldsymbol{B}_s' - \boldsymbol{A}_s'\right)^{-1}\boldsymbol{C}_s'(\tilde{z}_i)\,y_r(t)\Big) \tag{30}$$

where P is positive-definite constant matrix, which satisfies the following Riccati algebraic equations

$$P\boldsymbol{A}_s + \boldsymbol{A}_s'P - \frac{1}{\rho}P\boldsymbol{B}_s\cdot\boldsymbol{B}_s'P + \boldsymbol{C}_s'(\tilde{z}_i)\,\boldsymbol{C}_s(\tilde{z}_i) = 0 \tag{31}$$

then the global optimal closed-loop tracking system

$$\begin{aligned}\dot{\overline{T}}_s(t) &= \boldsymbol{A}_s\overline{T}_s(t) + \boldsymbol{B}_s\cdot -\frac{1}{\rho}\boldsymbol{B}_s'\Big(P\overline{T}_s(t) \\ &- \left(\frac{1}{\rho}P\boldsymbol{B}_s\cdot\boldsymbol{B}_s' - \boldsymbol{A}_s'\right)^{-1}\boldsymbol{C}_s'(\tilde{z}_i)\,y_r(t)\Big) \\ &= \left(\boldsymbol{A}_s - \frac{1}{\rho}\boldsymbol{B}_s\boldsymbol{B}_s'P\right)\overline{T}_s(t) \\ &+ \frac{1}{\rho}\boldsymbol{B}_s\boldsymbol{B}_s'\left(\frac{1}{\rho}P\boldsymbol{B}_s\cdot\boldsymbol{B}_s' - \boldsymbol{A}_s'\right)^{-1}\boldsymbol{C}_s'(\tilde{z}_i)\,y_r(t)\end{aligned} \tag{32}$$

and the initial condition $\overline{T}_s(0) = \overline{T}_0$ can approach to desire temperature profile $y_r(t)$.

Proof: For above optimal tracking position cost function (29), Pontryagin maximum principle can be applied in (25) and (26). First, the Hamiltonian function can be expressed as

$$\begin{aligned} H = &\frac{1}{2}\left[y_r(t) - \mathbf{C}_s(\tilde{z}_i)\overline{T}_s(t)\right]^2 + \frac{1}{2}\rho u^2(t) \\ &+ \overline{T}_s'(t)\mathbf{A}_s'\lambda(t) + u'(t)\mathbf{B}_s'\lambda(t) \end{aligned} \tag{33}$$

On assumption that ρ is an appropriate value for (33), a minimum can exist in a closed interval. According to the extremum condition, we can obtain

$$\frac{\partial H}{\partial u(t)} = \rho u(t) + \mathbf{B}_s'\lambda(t) = 0 \tag{34}$$

Then

$$u^*(t) = -\frac{1}{\rho}\mathbf{B}_s'\lambda(t) \tag{35}$$

The Hamilton canonical equation is

$$\dot{\overline{T}}_s(t) = \frac{\partial H}{\partial \lambda(t)} = \mathbf{A}_s\overline{T}_s(t) - \frac{1}{\rho}\mathbf{B}_s\cdot\mathbf{B}_s'\lambda(t) \tag{36}$$

$$\begin{aligned} \dot{\lambda}(t) = &-\frac{\partial H}{\partial \overline{T}_s(t)} \\ =&\mathbf{C}_s'(\tilde{z}_i)\left[y_r(t) - \mathbf{C}_s(\tilde{z}_i)\overline{T}_s(t)\right] - \mathbf{A}_s'\lambda(t) \end{aligned} \tag{37}$$

According to the linear relation in (37), we can assume that

$$\lambda(t) = P\overline{T}_s(t) - g \tag{38}$$

Substituting (38) into (36), the canonical equation can be transformed as

$$\begin{aligned} \dot{\overline{T}}_s(t) = &\frac{\partial H}{\partial \lambda(t)} \\ =&\mathbf{A}_s\overline{T}_s(t) - \frac{1}{\rho}\mathbf{B}_s\cdot\mathbf{B}_s'\left(P\overline{T}_s(t) - g\right) \\ =&\left(\mathbf{A}_s - \frac{1}{\rho}\mathbf{B}_s\cdot\mathbf{B}_s'P\right)\overline{T}_s(t) + \frac{1}{\rho}\mathbf{B}_s\cdot\mathbf{B}_s'g \end{aligned} \tag{39}$$

Substituting (39) into the differential of (38), we can obtain that

$$\begin{aligned} \dot{\lambda}(t) = &P\dot{\overline{T}}_s(t) \\ =&P\left(\mathbf{A}_s - \frac{1}{\rho}\mathbf{B}_s\cdot\mathbf{B}_s'P\right)\overline{T}_s(t) + \frac{1}{\rho}P\mathbf{B}_s\cdot\mathbf{B}_s'g \end{aligned} \tag{40}$$

With (38), adjoint equation (37) is written as

$$\begin{aligned} \dot{\lambda}(t) = &-\frac{\partial H}{\partial \overline{T}_s(t)} \\ &= \mathbf{C}_s'(\tilde{z}_i)\left[y_r(t) - \mathbf{C}_s(\tilde{z}_i)\overline{T}_s(t)\right] \\ &- \mathbf{A}_s'\left(P\overline{T}_s(t) - g\right) \\ &= \left(-\mathbf{C}_s'(\tilde{z}_i)\mathbf{C}_s(\tilde{z}_i) - \mathbf{A}_s'P\right)\overline{T}_s(t) \\ &+ \mathbf{A}_s'g + C_s'(\tilde{z}_i)y_r(t) \end{aligned} \tag{41}$$

Comparing (40) and (41), the Riccati algebraic equations the adjoint matrix can easily obtain

$$P\mathbf{A}_s + \mathbf{A}_s'P - \frac{1}{\rho}P\mathbf{B}_s\cdot\mathbf{B}_s'P + \mathbf{C}_s'(\tilde{z}_i)\mathbf{C}_s(\tilde{z}_i) = 0 \tag{42}$$

$$g = \left(\frac{1}{\rho}P\mathbf{B}_s\cdot\mathbf{B}_s' - \mathbf{A}_s'\right)^{-1}\mathbf{C}_s'(\tilde{z}_i)y_r(t) \tag{43}$$

This completes the proof. □

5 Simulation and Validation

Consider a long, thin rod in one-dimensional resonant cavity which is filled with deionized water. And the uniform microwave energy with zero phase condition $\varphi_A = 0$ perpendicularly incidents to the left side along the z-axis and and the length of material L is equivalent to the wavelength of microwave with the frequency of 2.45 GHz. Moreover, on assumption that the boundary condition is adiabatic, the spatiotemporal evolution for one-dimensional heating model (14) can be described by the following parabolic PDE:

$$\begin{aligned} \rho C_p\frac{\partial T}{\partial t} = &\kappa\frac{\partial^2 T}{\partial z^2} + \frac{1}{2}\omega\varepsilon_0\varepsilon''\left[e^{-2\beta z}\right. \\ &\left.+\left|e^{-2\beta l}\right|^2 e^{2\beta z} + e^{-2\beta l}\cos(2\alpha z)\right]\cdot u \end{aligned} \tag{44}$$

subject to the boundary conditions:

$$\left.\frac{\partial T}{\partial t}\right|_{z=0} = 0 \text{ and } \left.\frac{\partial \mathrm{T}}{\partial \mathrm{t}}\right|_{z=1} = 0 \tag{45}$$

where $\rho = 1$ g/(cm)^3, $C_p = 4.2$ J/(g $\cdot^\circ$ C), $\omega = 2\pi f = 1.54\times 10^{10}$ rad/s, $\varepsilon_0 = 8.854\times 10^{-12}$ F/m, $\varepsilon'' = 6.5$, $\beta = 0.195$, $\alpha = 4.39$ and $l = 1.43$ cm. For these values, the eigenvalue and characteristic function of the spatial differential operator can be solved analytically and its solution yields

$$\begin{aligned} \lambda_i = -\left(\frac{i\cdot\pi}{l}\right)^2, \phi_i(z) = \cos\left(\frac{i\pi z}{l}\right), \\ i = 0, 1, 2, \cdots \end{aligned} \tag{46}$$

Moreover, according to the Galerkin's method, we choose a 5^{th}-order Galerkin truncation for the finite-dimensional model, whose corresponding basis functions are shown in Fig. 1.

Then, the optimal controller (30) is designed and implemented into the finite-dimensional model (25) and (26). Before we proceed with the implementation and design of the controller, we use penalty factor $\rho = 0.003$ to guarantee the input is always restricted in a region, i.e. $[0, 700]$. However, due to

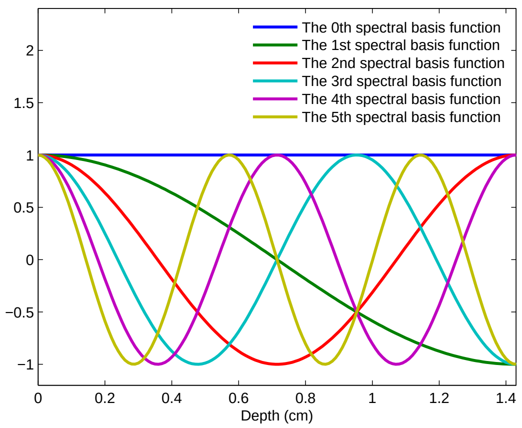

Figure 1: The first six spectral basis function for microwave heating model

the characteristics of time-dependent tracking positions variation, we could adapt the following procedures to achieve optimal tracking trajectories:

Step 1 *Based on the initial condition and expectation temperature rise curve, the initial power* $u^*(t)$ *is first applied in the microwave heating model (25) and (26);*

Step 2 *According to (27) and (28), optimal tracking positions* $\tilde{z}_i$ *and output vector* $\boldsymbol{C}_s(\tilde{z}_i)$ *could also be obtained;*

Step 3 *With global optimal input tracking controller* $u^*(t)$ *in (30), global temperature distribution in the next time could be obtained;*

Step 4 *Validating the global tracking error: if* $y_r(t) - \boldsymbol{C}_s(\tilde{z}_i)\overline{T}_s(t) \rightarrow 0$, $u^* = 0$; *Otherwise, return to the Step 2.*

Based on aforementioned analysis and parameters, the spatiotemporal profile for the global temperature distribution is shown in Fig. 2, which indicates that the closed-loop tracking system can reach expected temperature profile with a steady increase and small amplitude of overshoot. As shown in Fig. 3, although the non-uniform heating is one of the main characteristics for microwave heating, proposed tracking controller can also regulate the input power to reach a relatively uniform temperature profile. Moreover, thermal away or hotspot is successfully restrained by considering the interaction of microwave heating and heat conduction.

In order to further validate the effective of proposed control algorithm, the temperature rise curve in dynamic optimal tracking positions is shown in Fig. 4

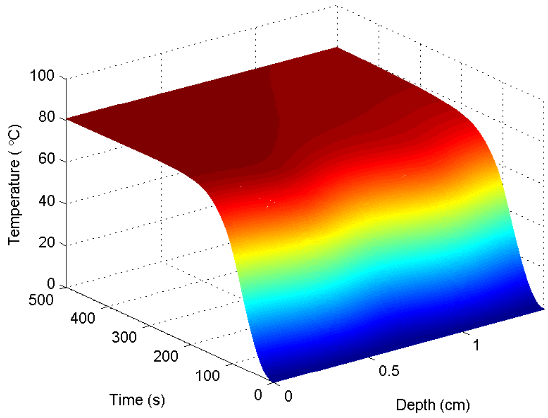

Figure 2: Temperature distribution of one-dimensional heating model with global optimal tracking controller

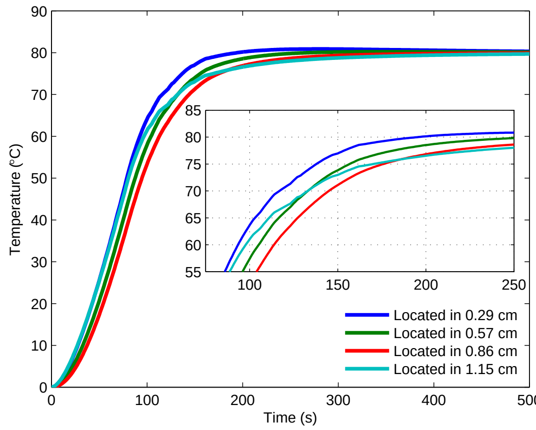

Figure 3: Temperature rise curves in different positions with global optimal tracking controller

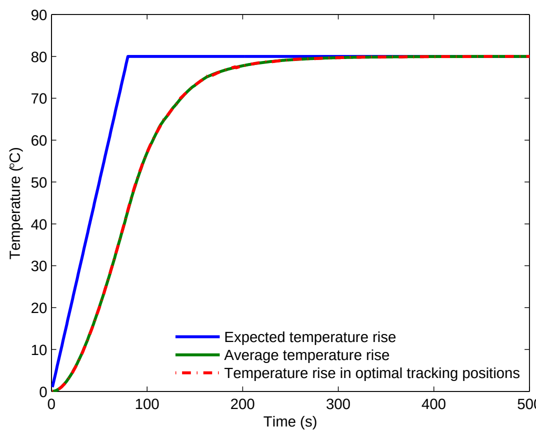

Figure 4: Expected, average temperature rise curve and temperature rise in optimal tracking positions

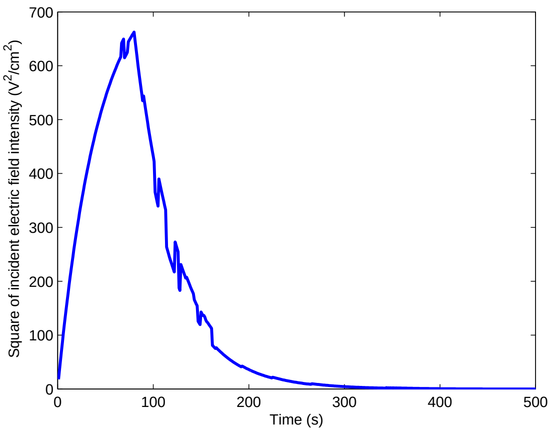

Figure 5: Closed-loop input for global optimal temperature tracking

to compare the expected and average temperature rise curve at the same time. It is seen that the temperature in optimal tracking positions coincides with the average temperature. However, the constrained input, which is shown in Fig. 5, can also affect the tracking performance. We observe that the deviation between the expected and tracking curve is always existed in the stage of temperature rise. Due to the sustained input power, the tracking temperature profile can also reach expected temperature with a relative longer time. It is clearly demonstrated that the synthesized optimal controller could regulate microwave input power with a relative stable change to reach a relatively uniform temperature distribution and guarantee the stability of closed-loop system.

6 Conclusion

In this paper, based on the spectral Galerkin method, an optimal tracking control problem for the microwave heating process has been studied. Initially, a traditional PDE model with an explicit dissipation power is presented by analyzing the thermodynamics equation and Maxwell's equation. Subsequently, the spectral Galerkin method is employed to derive a finite-dimensional ODE model for approximately describing the spatial temperature distribution. Then, based on the proposed model, a global optimal tracking controller is developed in order to reach a relatively uniform heat profile. Finally, the simulation results on one-dimensional cavity heating model indicate that the proposed design methodology is effective. Moreover, this work lay a foundation for solving the problem of hotspots and thermal runaway in microwave heating process.

Acknowledgements: The authors would like to thank the National Basic Research Program of China for financial support (No. 2013CB328903).

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